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# Solution and two types of Ulam-Hyers stability of $n-$ dimensional cubic-quartic functional equation in intuitionistic normed spaces

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**Abstract**
In this paper, the authors investigate the generalized Ulam-Hyers stability of $n-$ dimensional cubic-quartic functional equation

$$f\left(\sum_{b=1}^{n-1} v_b + rv_n\right) + f\left(\sum_{b=1}^{n-1} v_b - rv_n\right) = r^2\left[f\left(\sum_{b=1}^{n} v_b\right) + f\left(\sum_{b=1}^{n-1} v_b - v_n\right)\right] - 2(r^2-1)f\left(\sum_{b=1}^{n-1} v_b\right) + \frac{2(r+1)}{r}\left[f(rv_n) - r^3 f(v_n)\right]$$

where $r$ is a positive integer with $r \neq \pm 0, 1$ in the setting of intuitionistic fuzzy normed spaces using direct and fixed point methods.

**Keywords**
Cubic functional equation, quartic functional equation, generalized Ulam-Hyers stability, fixed point, intuitionistic fuzzy normed spaces.

**AMS Subject Classification**
39B52, 32B72, 32B82.

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## Contents



## 1. Introduction

Stability problem of a functional equation was first posed by S.M. Ulam [46] which was answered by D.H. Hyers [24] and then generalized by T. Aoki [2], Th.M. Rassias [38], J.M. Rassias [36] for additive mappings and linear mappings, respectively. Further generalizations on the above stability results was given in [16, 21, 22, 40]. Since then several stability problems for various functional equations have been investigated in [1, 3–13, 17, 25, 34, 37, 39, 47]; various fuzzy stability results concerning Cauchy, Jensen, quadratic and cubic functional equations were discussed in [19, 20, 29–32, 43–45].

Jun and Kim [26] considered the following functional equation

$$f(2x+y)+f(2x-y) = 2f(x+y)+2f(x-y)+12f(x) \quad (1.1)$$

which is called a cubic functional equation and every solution of the cubic functional equation is said to be a cubic mapping.

W.G Park and J.H Bae considered the following functional equation

$$f(x+2y)+f(x-2y)=4\left[f(x+y)+f(x-y)\right]-6f(x)+24(y) \tag{1.2}$$

It is easy to show that the function $f(x)=x^4$ satisfies the functional equation (1.2), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function.

In this paper, the authors investigate the generalized Ulam-Hyers stability of a $n$ dimensional cubic-quartic functional equation

$$\begin{aligned} f\left(\sum_{b=1}^{n-1} v_b + rv_n\right) + f\left(\sum_{b=1}^{n-1} v_b - rv_n\right) &= r^2\left[f\left(\sum_{b=1}^{n} v_b\right) + f\left(\sum_{b=1}^{n-1} v_b - v_n\right)\right] \\ &-2(r^2-1)f\left(\sum_{b=1}^{n-1} v_b\right) + \frac{2(r+1)}{r}\left[f(rv_n) - r^3 f(v_n)\right] \end{aligned} \tag{1.3}$$

where $r$ is a positive integer with $r \pm 0,1$ in the setting of intuitionistic fuzzy normed spaces using direct and fixed point methods.

In Section 2, the general solution of the functional equation (1.3) is given, In Section 3, basic definition and preliminaries of intuitionistic fuzzy normed space is present, In Section 4 and 5, the generalized Ulam - Hyers stability of the functional equation (1.3) is proved via Hyers method and fixed point Method.

## 2. The General solution of the Functional Equation

In this section, we present the general solution of the functional equation (1.3). Throughout this section let $X$ and $Y$ be real vector spaces.

**Lemma 2.1.** *An odd function $f: X \to Y$ satisfies the cubic functional equation (1.3) if $f: X \to Y$ satisfies the functional equation (1.1) for all $v_1, v_2 \dots, v_{n-1}, v_n \in X$.*

*Proof.* Assume $f: X \to Y$ satisfies the functional equation (1.3). Substituting $(v_1, v_2 \dots, v_{n-1}, v_n)$ by $(0,\dots,0,0)$ in (1.2), we get $f(0)=0$. Replacing $(v_2, v_3, \dots, v_{n-1})$ by $(0,0\dots,0)$ in (1.2),we have

$$f(v_1+rv_n)+f(v_1-rv_n)=r^2[f(v_1+v_n)+f(v_1-v_n)] -2(r^2-1)f(v_1)+\frac{2(r+1)}{r}[f(rv_n)-r^3f(v_n)] \tag{2.1}$$

for all $v_1, v_n \in X$. Setting $v_1=0$ and using oddness of $f$ in (2.1), we obtain

$$f(rv_n)=r^3f(v_n) \tag{2.2}$$

for all $v_n \in X$. Using (2.2) in (2.1), we get

$$f(v_1+rv_n)+f(v_1-rv_n)=r^2[f(v_1+v_n)+f(v_1-v_n)] -2(r^2-1)f(v_1) \tag{2.3}$$

for all $v_1, v_n \in X$. Replacing $v_1$ by $rv_1$ in (2.3), we have

$$\begin{aligned} &f(r(v_1+v_n))+f(r(v_1-v_n)) \\ &= r^2[f(rv_1+v_n)+f(rv_1-v_n)]-2(r^2-1)f(rv_1) \end{aligned} \tag{2.4}$$

for all $v_1, v_n \in X$. Using (2.2) in (2.4), we obtain

$$\begin{aligned} &f(rv_1+v_n)+f(rv_1-v_n) \\ &= r[f(v_1+v_n)+f(v_1-v_n)]+2r(r^2-1)f(v_1) \end{aligned} \tag{2.5}$$

for all $v_1, v_n \in X$. Replacing $v_1$ by $(v_1+v_n)$ in (2.3), we get

$$\begin{aligned} &f(v_1+(r+1)v_n)+f(v_1-(r-1)v_n) \\ &= r^2[f(v_1+2v_n)+f(v_1)]-2(r^2-1)f(v_1+2v_n) \end{aligned} \tag{2.6}$$

for all $v_1, v_n \in X$. Replacing $v_1$ by $(v_1-v_n)$ in (2.3), we obtain

$$\begin{aligned} &f(v_1-(r+1)v_n)+f(v_1+(r-1)v_n) \\ &= r^2[f(v_1-2v_n)+f(v_1)]-2(r^2-1)f(v_1-2v_n) \end{aligned} \tag{2.7}$$

for all $v_1, v_n \in X$. Adding (2.6) and (2.7), we arrive

$$\begin{aligned} &f(v_1+(r+1)v_n)+f(v_1-(r-1)v_n) \\ &+f(v_1-(r+1)v_n)+f(v_1+(r-1)v_n) \\ &= r^2[f(v_1+2v_n)+f(v_1-2v_n)+f(v_1)] \\ &+2r^2f(x)-2(r^2-1)[f(v_1+v_n)+f(v_1-v_n)] \end{aligned} \tag{2.8}$$

for all $v_1, v_n \in X$. Further replacing $v_n$ by $(v_1+v_n)$ in (2.3), we have

$$\begin{aligned} &f((r+1)v_1+rv_n)+f((1-r)v_1-rv_n) \\ &= r^2[f(2v_1+v_n)+f(v_n)]-2(r^2-1)f(v_1) \end{aligned} \tag{2.9}$$

for all $v_1, v_n \in X$. Replacing $v_n$ by $(-v_1+v_n)$ in (2.3), we get

$$\begin{aligned} &f((1-r)v_1-rv_n)+f((1+r)v_1+rv_n) \\ &= r^2[f(2v_1-v_n)-f(v_n)]-2(r^2-1)f(v_1) \end{aligned} \tag{2.10}$$

for all $v_1, v_n \in X$. Adding (2.9) and (2.10), we arrive

$$\begin{aligned} &f((r+1)v_1+rv_n)+f((1-r)v_1-rv_n) \\ &+f((1-r)v_1-rv_n)+f((1+r)v_1+rv_n) \\ &= r^2[f(2v_1+v_n)+f(2v_1-v_n)]-4(r^2-1)f(v_1) \end{aligned} \tag{2.11}$$

for all $v_1, v_n \in X$. Interchanging $v_1$ and $v_n$ in (2.11), we get

$$\begin{aligned} &f(rv_1+(r+1)v_n)+f(-rv_1-(r+1)v_n) \\ &+f(rv_1-(1-r)v_n)+f(-rv_1+(1+r)v_n) \\ &=r^2[f(v_1+2v_n)-f(v_1-2v_n)]-4(r^2-1)f(v_n) \end{aligned} \tag{2.12}$$

for all $v_1, v_n \in X$. Using oddness of $f$ in (2.12), we have

$$\begin{aligned} &f(rv_1+(r+1)v_n)-f(rv_1-(r+1)v_n) \\ &+f(rv_1-(r-1)v_n)-f(rv_1+(r-1)v_n) \\ &=r^2[f(v_1+2v_n)-f(v_1-2v_n)]-4(r^2-1)f(v_n) \end{aligned} \tag{2.13}$$

for all $v_1, v_n \in X$. Subtracting (2.6) and (2.7), we obtain

$$\begin{aligned} &f(v_1+(r+1)v_n)-f(v_1-(r-1)v_n) \\ &+f(v_1-(r-1)v_n)-f(v_1+(r-1)v_n) \\ &=r^2[f(v_1+2v_n)-f(v_1-2v_n)] \\ &-2(r^2-1)[f(v_1+v_n)-f(v_1-v_n)] \end{aligned} \tag{2.14}$$

for all $v_1, v_n \in X$. Replacing $v_1$ by $rv_1$ in (2.14), we get

$$\begin{aligned} &f(rv_1+(r+1)v_n)-f(rv_1-(r-1)v_n) \\ &+f(rv_1-(r-1)v_n)-f(rv_1+(r-1)v_n) \\ &=r^2[f(rv_1+2v_n)-f(rv_1-2v_n)] \\ &-2(r^2-1)[f(rv_1+v_n)-f(rv_1-v_n)] \end{aligned} \tag{2.15}$$

for all $v_1, v_n \in X$. By Comparing (2.13) and (2.15), we have

$$\begin{aligned} &r^2[f(v_1+2v_n)-f(v_1-2v_n)]-4(r^2-1)f(v_n) \\ &=r^2[f(rv_1+2v_n)-f(rv_1-2v_n)] \\ &-2(r^2-1)[f(rv_1+v_n)-f(rv_1-v_n)] \end{aligned} \tag{2.16}$$

for all $v_1, v_n \in X$. Interchanging $v_1$ and $v_n$ in (2.16), we arrive

$$\begin{aligned} &r^2[f(2v_1+v_n)+f(2v_1-v_n)]-4(r^2-1)f(v_1) \\ &=r^2[f(2v_1+rv_n)+f(2v_1-rv_n)] \\ &-2(r^2-1)[f(v_1+rv_n)+f(v_1-rv_n)] \end{aligned} \tag{2.17}$$

for all $v_1, v_n \in X$. Substituting (2.3) in (2.17), we obtain

$$\begin{aligned} &f(2v_1+v_n)+f(2v_1-v_n) \\ &=[f(2v_1+rv_n)+f(2v_1-rv_n)] \\ &-2(r^2-1)[f(v_1+v_n)+f(v_1-v_n)] \\ &+4(r^2-1)f(v_1) \end{aligned} \tag{2.18}$$

for all $v_1, v_n \in X$. Remodify in (2.18), we get

$$\begin{aligned} &f(2v_1+rv_n)+f(2v_1-rv_n) \\ &=r^2[f(2v_1+v_n)+f(2v_1-v_n)] \\ &+2(r^2-1)[f(v_1+v_n)+f(v_1-v_n)] \\ &-4(r^2-1)f(v_1) \end{aligned} \tag{2.19}$$

for all $v_1, v_n \in X$. Replacing $v_1$ by $2v_1$ in (2.3), we have

$$\begin{aligned} &f(2v_1+rv_n)+f(2v_1-rv_n) \\ &=r^2[f(2v_1+v_n)+f(2v_1-v_n)] \\ &-16(r^2-1)f(v_1) \end{aligned} \tag{2.20}$$

for all $v_1, v_n \in X$. By comparing (2.19) and (2.20), we arrive

$$\begin{aligned} &r^2[f(2v_1+v_n)+f(2v_1-v_n)] \\ &-16(r^2-1)f(v_1)=r^2[f(2v_1+v_n)+f(2v_1-v_n)] \\ &+2(r^2-1)[f(v_1+v_n)+f(v_1-v_n)]-4(r^2-1)f(v_1) \end{aligned} \tag{2.21}$$

for all $v_1, v_n \in X$. Replacing $(v_1, v_n)$ by $(x, y)$ in (2.21) and Simplify the equation, we desired our result. Hence the lemma is proved. □

**Lemma 2.2.** *An even function $f : X \to Y$ satisfies the quartic functional equation (1.3) if $f : X \to Y$ satisfies the functional equation (1.2) for all $v_1, v_2 \ldots, v_{n-1}, v_n \in X$.*

*Proof.* Assume $f : X \to Y$ satisfies the functional equation (1.3). Substituting $(v_1, v_2 \ldots, v_{n-1}, v_n)$ by $(0,\ldots,0,0)$ in (1.2), we get $f(0) = 0$. Replacing $(v_2, v_3, \ldots, v_{n-1})$ by $(0,0\ldots,0)$ in (1.2),we have

$$\begin{aligned} &f(v_1+rv_n)+f(v_1-rv_n) \\ &=r^2[f(v_1+v_n)+f(v_1-v_n)]-2(r^2-1)f(v_1) \\ &+\frac{2(r+1)}{r}[f(rv_n)-r^3f(v_n)] \end{aligned} \tag{2.22}$$

for all $v_1, v_n \in X$. Setting $v_1 = 0$ and using evenness of $f$ in (2.22), we obtain

$$f(rv_n)=r^4f(v_n) \tag{2.23}$$

for all $v_n \in X$. Using (2.23) in (2.22), we get

$$\begin{aligned} &f(v_1+rv_n)+f(v_1-rv_n) \\ &=r^2[f(v_1+v_n)+f(v_1-v_n)] \\ &-2(r^2-1)f(v_1)+2r^2(r^2-1)f(v_n) \end{aligned} \tag{2.24}$$

for all $v_1, v_n \in X$. Replacing $v_1$ by $2v_1$ in (2.24),we have

$$\begin{aligned} &f(2v_1+rv_n)+f(2v_1-rv_n) \\ &=r^2[f(2v_1+v_n)+f(2v_1-v_n)] \\ &-32(r^2-1)f(v_1)+2r^2(r^2-1)f(v_n) \end{aligned} \tag{2.25}$$

for all $v_1, v_n \in X$. Replacing $v_1$ by $(v_1 + v_n)$ in (2.24), we reach

$$\begin{aligned} &f(v_1+(r+1)v_n)+f(v_1+(1-r)v_n) \\ &=r^2[f(v_1+2v_n)+f(v_1)] \\ &-2(r^2-1)f(v_1+2v_n)+2r^2(r^2-1)f(v_n) \end{aligned} \tag{2.26}$$

for all $v_1, v_n \in X$. Replacing $v_1$ by $(v_1 - v_n)$ in (2.3), we get

$$\begin{aligned} &f(v_1-(r+1)v_n)+f(v_1-(1-r)v_n)\\ &=r^2[f(v_1-2v_n)+f(v_1)]\\ &-2(r^2-1)f(v_1-2v_n)+2r^2(r^2-1)f(v_n) \end{aligned} \tag{2.27}$$

for all $v_1, v_n \in X$. Adding (2.26) and (2.27), we arrive

$$\begin{aligned} &f(v_1+(r+1)v_n)+f(v_1+(1-r)v_n)\\ &+f(v_1-(r+1)v_n)+f(v_1-(1-r)v_n)\\ &=r^2[f(v_1+2v_n)+f(v_1-2v_n)]+2r^2f(x)\\ &-2(r^2-1)[f(v_1+v_n)+f(v_1-v_n)]\\ &+4r^2(r^2-1)f(v_n) \end{aligned} \tag{2.28}$$

for all $v_1, v_n \in X$. Replacing $v_1$ by $sv_1$ in (2.28), we get

$$\begin{aligned} &f(rv_1+(r+1)v_n)+f(rv_1+(1-r)v_n)\\ &+f(rv_1-(r+1)v_n)+f(rv_1-(1-r)v_n)\\ &=r^2[f(rv_1+2v_n)+f(rv_1-2v_n)]+2r^6f(x)\\ &-2(r^2-1)[f(rv_1+v_n)+f(rv_1-v_n)]\\ &+4r^2(r^2-1)f(v_n) \end{aligned} \tag{2.29}$$

for all $v_1, v_n \in X$. Further replacing $v_n$ by $(v_1 + v_n)$ in (2.24), we have

$$\begin{aligned} &f(v_1+r(v_1+v_n))+f(v_1-r(v_1+v_n))\\ &=r^2[f(2v_1+v_n)+f(v_n)]\\ &-2(r^2-1)f(v_1)+2r^2(r^2-1)f(v_1+v_n) \end{aligned} \tag{2.30}$$

for all $v_1, v_n \in X$. Replacing $v_n$ by $(v_1 - v_n)$ in (2.24), we get

$$\begin{aligned} &f((1+r)v_1-r(v_n))+f((1-r)v_1+r(v_n))\\ &=r^2[f(2v_1-v_n)+f(v_n)]\\ &-2(r^2-1)f(v_1)+2r^2(r^2-1)f(v_1-v_n) \end{aligned} \tag{2.31}$$

for all $v_1, v_n \in X$. Adding (2.30) and (2.31), we obtain

$$\begin{aligned} &f((r+1)v_1+rv_n)+f((1-r)v_1-rv_n)\\ &+f((1+r)v_1+rv_n)+f((1-r)v_1+rv_n)\\ &=r^2[f(2v_1+v_n)+f(2v_1-v_n)]\\ &+2r^2f(v_n)-4(r^2-1)f(v_1)\\ &+2r^2(r^2-1)[f(v_1+v_n)+f(v_1-v_n)] \end{aligned} \tag{2.32}$$

for all $v_1, v_n \in X$. Interchanging $v_1$ and $v_n$ in (2.11), we arrive

$$\begin{aligned} &f(rv_1+(r+1)v_n)+f(rv_1-(1-r)v_n)\\ &+f(rv_1-(1+r)v_n)+f(rv_1+(1-r)v_n)\\ &=r^2[f(v_1+2v_n)+f(v_1-2v_n)]\\ &+2r^2f(v_1)-4(r^2-1)f(v_n)\\ &+2r^2(r^2-1)[f(v_1+v_n)+f(v_1-v_n)] \end{aligned} \tag{2.33}$$

for all $v_1, v_n \in X$. Comparing (2.29) and (2.33), we have

$$\begin{aligned} &r^2[f(rv_1+2v_n)+f(rv_1-2v_n)]+2r^6f(x)\\ &-2(r^2-1)[f(rv_1+v_n)+f(rv_1-v_n)]\\ &+4r^2(r^2-1)f(v_n)=r^2[f(v_1+2v_n)+f(v_1-2v_n)]\\ &+2r^2f(v_1)-4(r^2-1)f(v_n)\\ &+2r^2(r^2-1)[f(v_1+v_n)+f(v_1-v_n)] \end{aligned} \tag{2.34}$$

for all $v_1, v_n \in X$. Simplifying (2.34), we obtain

$$\begin{aligned} &f(rv_1+2v_n)+f(rv_1-2v_n)\\ &-4(r^2-1)[f(v_1+v_n)+f(v_1-v_n)]\\ &-[f(v_1+2v_n)+f(v_1-2v_n)]\\ &=[2(1-r^4)+4(r^2-1)^2]f(v_1)-8(r^2-1)f(v_n) \end{aligned} \tag{2.35}$$

for all $v_1, v_n \in X$. Interchanging $v_1$ and $v_n$ in (2.25), we get

$$\begin{aligned} &f(rv_1+2v_n)+f(rv_1-2v_n)\\ &=r^2[f(v_1+2v_n)+f(v_1-2v_n)]\\ &-32(r^2-1)f(v_n)+2r^2(r^2-1)f(v_1) \end{aligned} \tag{2.36}$$

for all $v_1, v_n \in X$. Substituting (2.36) and (2.35), we have

$$\begin{aligned} &r^2[f(v_1+2v_n)+f(v_1-2v_n)]\\ &-32(r^2-1)f(v_n)+2r^2(r^2-1)f(v_1)\\ &-4(r^2-1)[f(v_1+v_n)+f(v_1-v_n)]\\ &-[f(v_1+2v_n)+f(v_1-2v_n)]\\ &=[2(1-r^4)+4(r^2-1)^2]f(v_1)-8(r^2-1)f(v_n) \end{aligned} \tag{2.37}$$

for all $v_1, v_n \in X$. Simplifying (2.37), we arrive

$$\begin{aligned} &(r^2-1)[f(v_1+2v_n)+f(v_1-2v_n)]\\ &-4(r^2-1)[f(v_1+rv_n)+f(v_1-rv_n)]\\ &=-6(r^2-1)f(v_1)+24(r^2-1)f(v_n) \end{aligned} \tag{2.38}$$

for all $v_1, v_n \in X$. Dividing the equation (2.38) by $(r^2-1)$, we have

$$\begin{aligned} &f(v_1+2v_n)+f(v_1-2v_n)\\ &=4[f(v_1+v_n)+f(v_1-v_n)]\\ &-6f(v_1)+24(v_n) \end{aligned} \tag{2.39}$$

for all $v_1, v_n \in X$. Replacing $(v_1, v_n)$ by $(x, y)$ in (2.39), we desired our result. Hence the lemma is proved. □

## 3. Preliminaries Of Intuitionistic Fuzzy Normed Spaces

In this section, some preliminaries about intuitionistic fuzzy normed space is given.

**Lemma 3.1.** *[18] Consider the set $L^*$ and the order relation $\leq_{L^*}$ defined by:*

$$L^* = \left\{ (x_1, x_2) : (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1 \right\},$$

$$(x_1,x_2) \leq_{L^*} (y_1,y_2) \Leftrightarrow x_1 \leq y_1, x_2 \geq y_2, \quad \forall (x_1,x_2), (y_1,y_2) \in L^*$$

*Then $(L^*, \leq_{L^*})$ is a complete lattice.*

**Definition 3.2.** *[15] An intuitionistic fuzzy set $A_{\zeta,\eta}$ in a universal set $U$ is an object*

$$A_{\zeta,\eta} = \{ (\zeta_A(u), \eta_A(u)) | u \in U \}$$

*for all $u \in U$, $\zeta_A(u) \in [0,1]$ and $\eta_A(u) \in [0,1]$ are called the membership degree and the non-membership degree, respectively, of $u$ in $A_{\zeta,\eta}$ and, furthermore, they satisfy $\zeta_A(u) + \eta_A(u) \leq 1$.*

We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. Classically, a *triangular norm* $* = T$ on $[0,1]$ is defined as an increasing, commutative, associative mapping $T : [0,1]^2 \to [0,1]$ satisfying $T(1,x) = 1 * x = x$ for all $x \in [0,1]$. A *triangular conorm* $S = \diamondsuit$ is defined as an increasing, commutative, associative mapping $S : [0,1]^2 \to [0,1]$ satisfying $S(0,x) = 0 \diamondsuit x = x$ for all $x \in [0,1]$.

Using the lattice $(L^*, \leq_{L^*})$, these definitions can be straightforwardly extended.

**Definition 3.3.** *[15] A triangular norm ($t-$norm) on $L^*$ is a mapping $T : (L^*)^2 \to L^*$ satisfying the following conditions:*

*(i)* $(\forall \in L^*)(T(x, 1_{L^*}) = x)$
*(boundary condition);*

*(ii)* $\left(\forall (x,y) \in (L^*)^2\right)(T(x,y) = T(y,x))$
*(commutativity);*

*(iii)* $\left(\forall (x,y,z) \in (L^*)^3\right)(T(x, T(y,z)) = T(T(x,y),z))$
*(associativity);*

*(iv)* $\left(\forall (x,x',y,y') \in (L^*)^4\right)$
$(x \leq_{L^*} x' \text{ and } y \leq_{L^*} y' \Rightarrow T(x,y) \leq_{L^*} T(x',y'))$
*(monotonicity).*

If $(L^*, \leq_{L^*}, T)$ is an Abelian topological monoid with unit $1_{L^*}$, then $L^*$ is said to be a *continuous $t-$norm.*

**Definition 3.4.** *[15] A continuous $t-$norms $T$ on $L^*$ is said to be continuous $t-$representable if there exist a continuous $t-$norm $*$ and a continuous $t-$conorm $\diamondsuit$ on $[0,1]$ such that, for all $x = (x_1,x_2), y = (y_1,y_2) \in L^*$,*

$$T(x,y) = (x_1 * y_1, x_2 \diamondsuit y_2).$$

For example,

$$T(a,b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$$

and

$$M(a,b) = (\min\{a_1,b_1\}, \max\{a_2,b_2\})$$

for all $a = (a_1,a_2)$, $b = (b_1,b_2) \in L^*$ are continuous $t-$representable.

Now, we define a sequence $T^n$ recursively by $T^1 = T$ and

$$T^n\left(x^{(1)}, \ldots, x^{(n+1)}\right) = T\left(T^{n-1}\left(x^{(1)}, \ldots, x^{(n)}\right), x^{(n+1)}\right),$$

$$\forall \; n \geq 2, \, x^{(i)} \in L^*.$$

**Definition 3.5.** *[43] A negator on $L^*$ is any decreasing mapping $N : L^* \to L^*$ satisfying $N : (0_{L^*}) = 1_{L^*}$ and $N(1_{L^*}) = 0_{L^*}$. If $N(N(x)) = x$ for all $x \in L^*$, then $N$ is called an involutive negator. A negator on $[0,1]$ is a decreasing mapping $N : [0,1] \to [0,1]$ satisfying $P_{\mu,\nu}(0) = 1$ and $P_{\mu,\nu}(1) = 0$. $N_s$ denotes the standard negator on $[0,1]$ defined by*

$$N_s(x) = 1 - x, \quad \forall x \in [0,1].$$

**Definition 3.6.** *[43] Let $\mu$ and $\nu$ be membership and non-membership degree of an intuitionistic fuzzy set from $X \times (0,+\infty)$ to $[0,1]$ such that $\mu_x(t) + \nu_x(t) \leq 1$ for all $x \in X$ and all $t > 0$. The triple $(X, P_{\mu,\nu}, T)$ is said to be an intuitionistic fuzzy normed space (briefly IFN-space) if $X$ is a vector space, $T$ is a continuous $t-$representable and $P_{\mu,\nu}$ is a mapping $X \times (0,+\infty) \to L^*$ satisfying the following conditions: for all $x, y \in X$ and $t, s > 0$,*

*(IFN1)* $P_{\mu,\nu}(x,0) = 0_{L^*}$;

*(IFN2)* $P_{\mu,\nu}(x,t) = 1_{L^*}$ if and only if $x = 0$;

*(IFN3)* $P_{\mu,\nu}(\alpha x, t) = P_{\mu,\nu}\left(x, \frac{t}{|\alpha|}\right)$ for all $\alpha \neq 0$;

*(IFN4)* $P_{\mu,\nu}(x+y, t+s) \geq_{L^*} T\left(P_{\mu,\nu}(x,t), P_{\mu,\nu}(y,s)\right)$.

*In this case, $P_{\mu,\nu}$ is called an intuitionistic fuzzy norm. Here, $P_{\mu,\nu}(x,t) = (\mu_x(t), \nu_x(t))$.*

**Example 3.7.** *[43] Let $(X, \|\cdot\|)$ be a normed space. Let $T(a,b) = (a,b, \min(a_2+b_2,1))$ for all $a = (a_1,a_2), b = (b_1,b_2) \in L^*$ and $\mu, \nu$ be membership and non-membership degree of an intuitionistic fuzzy set defined by*

$$P_{\mu,\nu}(x,t) = (\mu_x(t), \nu_x(t)) = \left(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|}\right), \; \forall t \in R^+.$$

*Then $(X, P_{\mu,\nu}, T)$ is an IFN-sapce.*

**Definition 3.8.** *[43] A sequence $\{x_n\}$ in an IFN-space $(X, P_{\mu,\nu}, T)$ is called a Cauchy sequence if, for any $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in N$ such that*

$$P_{\mu,\nu}(x_n - x_m, t) > L^* \; (N_s(\varepsilon), \varepsilon), \quad \forall n, \, m \geq n_0,$$

*where $N_s$ is the standard negator.*

**Definition 3.9.** *[43] The sequence $\{x_n\}$ is said to be convergent to a point $x \in X$ (denoted by $x_n \xrightarrow{P_{\mu,\nu}} x$) if $P_{\mu,\nu}(x_n - x, t) \to 1_{L^*}$ as $n \to \infty$ for every $t > 0$.*

**Definition 3.10.** *[43] An IFN-space $\left(X,P_{\mu,\nu},T\right)$ is said to be complete if every Cauchy sequence in $X$ is convergent to a point $x\in X$.*

Here after, throughout this paper, assume that $X$ be a linear space, $\left(Z,P'_{\mu,\nu},T\right)$ be an IFN-space and $\left(Y,P'_{\mu,\nu},T\right)$ be a complete IFN-space.

## 4. Stability Results: Direct Method

In this section, the authors present the generalized Ulam-Hyers stability of the cubic-quartic functional equation (1.3) in intuitionistic fuzzy normed spaces. Now we use the following notation for a given mapping $Df:X\to Y$ such that

$$\begin{aligned}Df(v_1,v_2,\cdots v_{n-1},v_n)&=f\left(\sum_{b=1}^{n-1}v_b+rv_n\right)\\&+f\left(\sum_{b=1}^{n-1}v_b-rv_n\right)-r^2\left[f\left(\sum_{b=1}^{n}v_b\right)-f\left(\sum_{b=1}^{n-1}v_b-v_n\right)\right]\\&+2(r^2-1)f\left(\sum_{b=1}^{n-1}v_b\right)-\frac{2(r+1)}{r}\left[f(rv_n)-r^3f(v_n)\right]\end{aligned}$$

for all $v_1,v_2,\cdots v_{n-1},v_n\in X$.

**Theorem 4.1.** *Let $\tau\in\{1,-1\}$. Let $\sigma:X^n\to Z$ be a function such that for some $0<\left(\frac{a}{r^3}\right)^{\tau}<1$,*

$$P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},r^{\tau}v\right),s\right)\geq_{L^*}P'_{\mu,\nu}\left(a^{\tau}\ \sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),s\right)\tag{4.1}$$

*for all $v\in X$ and all $s>0$ and*

$$\lim_{n\to\infty}P'_{\mu,\nu}\left(\sigma\left(r^{\tau n}v_1,r^{\tau n}v_2,r^{\tau n}v_3,\cdots,r^{\tau n}v_{n-1},r^{\tau n}v_n\right),r^{\tau n}s\right)=1_{L^*}\tag{4.2}$$

*for all $v_1,v_2,\cdots v_{n-1},v_n\in X$ and all $s>0$. Let $f_o:X\to Y$ be an odd function satisfies the inequality*

$$P_{\mu,\nu}\left(Df_o(v_1,v_2,\cdots v_{n-1},v_n),s\right)\geq_{L^*}P'_{\mu,\nu}\left(\sigma\left(v_1,v_2,\cdots v_{n-1},v_n\right),s\right)\tag{4.3}$$

*for all $v_1,v_2,\cdots v_{n-1},v_n\in X$ and all $s>0$. Then the limit*

$$P_{\mu,\nu}\left(C(v)-\frac{f_o(r^nv)}{r^{3n}},s\right)\to 1_L^*,\qquad as\ \ n\ \to\ \infty,\ \ s>0\tag{4.4}$$

*exists for all $v\in X$ and the mapping $C:X\to Y$ is a unique cubic mapping satisfying (1.3) and*

$$P_{\mu,\nu}\left(f_o\left(x\right)-C\left(x\right),r\right)\geq_{L^*}P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),\frac{2(r+1)}{r}|r^3-d|s\right)\tag{4.5}$$

*for all $v\in X$ and all $s>0$.*

*Proof.* Let $\tau=1$. Since $f_o$ is an odd function, replacing $(v_1,v_2,\cdots v_{n-1},v_n)$ by $\left(\underbrace{0,\cdots,0}_{n-1 times},v\right)$ in (4.3), we get

$$P_{\mu,\nu}\left(\frac{2(r+1)}{r}[f_o(rv)-r^3f_o(v)],s\right)\geq_{L^*}P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),s\right)\tag{4.6}$$

for all $v\in X$ and all $s>0$. Using $(IFN3)$ in (4.6), we obtain

$$P_{\mu,\nu}\left(\frac{f_o(rv)}{r^3}-f_o(v),\frac{s}{2r^2(r+1)}\right)\geq_{L^*}P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),s\right)\tag{4.7}$$

for all $v\in X$ and all $s>0$. Replacing $v$ by $r^kv$ in (4.7), we have

$$P_{\mu,\nu}\left(\frac{f_o(r^{k+1}v)}{r^3}-f_o(r^kv),\frac{s}{2r^2(r+1)}\right)\geq_{L^*}P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},r^nv\right),s\right)\tag{4.8}$$

for all $x\in X$ and all $r>0$. Using (4.1), $(IFN3)$ in (4.8), we arrive

$$P_{\mu,\nu}\left(\frac{f_o(r^{k+1}v)}{r^3}-f_o(r^kv),\frac{s}{2r^2(r+1)}\right)\geq_{L^*}P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),\frac{s}{a^k}\right)\tag{4.9}$$

for all $x\in X$ and all $r>0$. It is easy to verify from (4.9), that

$$P_{\mu,\nu}\left(\frac{f_o(r^{k+1}v)}{r^{3(k+1)}}-\frac{f_o(r^kv)}{r^{3k}},\frac{s}{2r^2(r+1)\cdot r^{3k}}\right)\geq_{L^*}P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),\frac{s}{a^k}\right)\tag{4.10}$$

holds for all $v \in X$ and all $s > 0$. Replacing $s$ by $a^k s$ in (4.10), we get

$$P_{\mu,\nu}\left(\frac{f_o(r^{k+1}v)}{r^{k+1}} - \frac{f_o(r^k v)}{r^{3k}}, \frac{a^k\ s}{2r^2(r+1)\cdot r^{3k}}\right) \geq_{L^*} P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),s\right) \tag{4.11}$$

for all $v \in X$ and all $s > 0$. It is easy to see that

$$\frac{f_o(r^k v)}{r^{3k}} - f_o(v) = \sum_{i=0}^{n-1} \frac{f_o(r^{i+1}v)}{r^{3(i+1)}} - \frac{f_o(r^i v)}{r^{3i}} \tag{4.12}$$

for all $v \in X$. From equations (4.11) and (4.12), we have

$$\begin{aligned} &P_{\mu,\nu}\left(\frac{f_o(r^k v)}{r^{3k}} - f_o(v), \sum_{i=0}^{n-1} \frac{a^i\ s}{2r^2(r+1)\cdot r^{3n}}\right) \\ &\geq_{L^*} T_{i=0}^{n-1}\left(P_{\mu,\nu}\left(\sum_{i=0}^{n-1} \frac{f_o(r^{i+1}x)}{r^{3(i+1)}} - \frac{f_o(3^i v)}{r^{3i}}, \sum_{i=0}^{n-1} \frac{a^i\ s}{2r^2(r+1)\cdot r^{3i}}\right)\right) \\ &\geq_{L^*} T_{i=0}^{n-1}\left\{P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),s\right)\right\} \\ &\geq_{L^*} P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),s\right) \end{aligned} \tag{4.13}$$

for all $v \in X$ and all $s > 0$. Replacing $v$ by $r^m v$ in (4.13) and using (4.1), $(IFN3)$, we obtain

$$P_{\mu,\nu}\left(\frac{f_o(r^{k+m}v)}{r^{3(k+m)}} - \frac{f_o(r^m v)}{r^{3m}}, \sum_{i=0}^{n-1} \frac{a^i\ s}{2r^2(r+1)\cdot r^{3(i+m)}}\right) \geq_{L^*} P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),\frac{s}{a^m}\right) \tag{4.14}$$

for all $v \in X$ and all $s > 0$ and all $m,k \geq 0$. Replacing $s$ by $a^m s$ in (4.14), we get

$$P_{\mu,\nu}\left(\frac{f_o(r^{k+m}v)}{r^{3(k+m)}} - \frac{f_o(r^m v)}{r^{3m}}, \sum_{i=m}^{m+k-1} \frac{a^i\ r}{2r^2(r+1)\cdot r^{3i}}\right) \geq_{L^*} P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),s\right) \tag{4.15}$$

for all $v \in X$ and all $s > 0$ and all $m,k \geq 0$. It follows from (4.15), that

$$P_{\mu,\nu}\left(\frac{f_o(r^{k+m}v)}{r^{3(k+m)}} - \frac{f_o(r^m v)}{r^{3m}}, s\right) \geq_{L^*} P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),\frac{r}{\sum_{i=m}^{m+n-1}\frac{a^i}{2r^2(r+1)\cdot r^{3i}}}\right) \tag{4.16}$$

holds for all $v \in X$ and all $s > 0$ and all $m,n \geq 0$. Since $0 < a < r^3$ and $\sum_{i=0}^{n}\left(\frac{a}{r^3}\right)^i < \infty$. Thus $\left\{\frac{f_o(r^k v)}{r^{3k}}\right\}$ is a Cauchy sequence in $\left(Y, P_{\mu,\nu}, T\right)$. Since $\left(Y, P_{\mu,\nu}, T\right)$ is a complete IFN-space this sequence convergent to some point $C(v) \in Y$. So, one can define the mapping $C : X \to Y$ by

$$P_{\mu,\nu}\left(Q(v) - \frac{f_o(r^k v)}{r^{3k}}, s\right) \to 1_L^*, \qquad as \quad n \to \infty, \quad s > 0 \tag{4.17}$$

for all $v \in X$. Letting $m = 0$ in (4.16), we get

$$P_{\mu,\nu}\left(\frac{f_o(r^k v)}{r^{3k}} - f_o(v), s\right) \geq_{L^*} P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),\frac{r}{\sum_{i=0}^{n-1}\frac{a^i}{2r^2(r+1)\cdot r^{3i}}}\right) \tag{4.18}$$

for all $x \in X$ and all $r > 0$. Now for every $\delta > 0$ and from (4.18), we have

$$\begin{aligned} &P_{\mu,\nu}\left(C(v) - f_o(v), s+\delta\right) \\ &\geq_{L^*} T\left(P'_{\mu,\nu}\left(C(v) - \frac{f_o\left(r^k v\right)}{r^{3k}}, \delta\right), P'_{\mu,\nu}\left(f_o(v) - \frac{f_o\left(r^k v\right)}{r^{3k}}, s\right)\right) \\ &\geq_{L^*} T\left(P'_{\mu,\nu}\left(C(v) - \frac{f_o\left(r^n v\right)}{r^{3k}}, \delta\right), P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),\frac{r}{\sum_{i=0}^{n-1}\frac{a^i}{2r^2(r+1)\cdot r^{3i}}}\right)\right) \end{aligned} \tag{4.19}$$

for all $v \in X$ and all $r > 0$. Taking the limit as $n \to \infty$ in (4.19), we get

$$\begin{aligned} &P_{\mu,\nu}\left(C(v) - f_o(v), s+\delta\right) \\ &\geq_{L^*} T\left(1_{L^*}, P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),\frac{2(r+1)}{r}(r^3-a)s\right)\right) \\ &\geq_{L^*} P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),\frac{2(r+1)}{r}(r^3-a)s\right) \end{aligned} \tag{4.20}$$

for all $v \in X$ and all $s > 0$ and $\delta > 0$. Since $\delta$ is arbitrary, by taking $\delta \to 0$ in (4.20), we obtain

$$P_{\mu,\nu}\left(C(v) - f_o(v), s\right) \geq_{L^*} P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),\frac{2(r+1)}{r}(r^3-a)s\right) \tag{4.21}$$

for all $v \in X$ and all $r > 0$. To prove $C$ satisfies (1.3), replacing $(v_1, v_2, \cdots v_{n-1}, v_n)$ by $(r^n v_1, r^n v_2, \cdots r^n v_{n-1}, r^n v_n)$ in (4.3) respectively, we obtain

$$P_{\mu,\nu}\left(\frac{1}{r^{3n}}Df_o(r^n v_1, r^n v_2, \cdots r^n v_{n-1}, r^n v_n), s\right) \geq_{L^*} P'_{\mu,\nu}\left(\sigma(r^n v_1, r^n v_2, \cdots r^n v_{n-1}, r^n v_n), r^{3n}s\right) \tag{4.22}$$

for all $v_1, v_2, \cdots v_{n-1}, v_n \in X$ and all $s > 0$. Now,

$$\begin{aligned} &P_{\mu,\nu}\left(C\left(\sum_{b=1}^{n-1} v_b + rv_n\right) + C\left(\sum_{b=1}^{n-1} v_r - rv_n\right)\right. \\ &-r^2\left[C\left(\sum_{b=1}^{n} v_b\right) + C\left(\sum_{b=1}^{n-1} v_b - rv_n\right)\right] \\ &+2(r^2-1)C\left(\sum_{b=1}^{n-1} v_b\right) \\ &\left.-\frac{2}{r}(r+1)\left[C(rv_n) - r^3C(v_n)\right], s\right) \\ &\geq_{L^*} T^5\left\{P'_{\mu,\nu}\left(C\left(\sum_{b=1}^{n-1} r^n(v_r + rv_n)\right)\right.\right. \\ &\qquad \left.-\frac{1}{r^{3n}}f_o\left(\sum_{b=1}^{n-1} r^n(v_b + rv_n)\right), \frac{r}{6}\right), \\ &P'_{\mu,\nu}\left(C\left(\sum_{b=1}^{n-1} r^n(v_b - rv_n)\right)\right. \\ &\qquad \left.-\frac{1}{r^{3n}}f_o\left(\sum_{b=1}^{n-1} r^n(v_b - rv_n)\right), \frac{r}{6}\right), \\ &P'_{\mu,\nu}\left(-r^2\left[C\left(\sum_{b=1}^{n} v_b\right) + C\left(\sum_{b=1}^{n-1} v_b - v_n\right)\right]\right. \\ &\left.+\frac{r^2}{r^{3n}}\left[f\left(r^n\sum_{b=1}^{n} v_b\right) + f\left(r^n\sum_{b=1}^{n-1} v_b - v_n\right)\right], \frac{r}{6}\right) \\ &P'_{\mu,\nu}\left(2(r^2-1)C\left(\sum_{b=1}^{n-1} v_b\right)\right. \\ &\qquad \left.+\frac{2(r^2-1)}{r^{3n}}f\left(r^3\sum_{b=1}^{n-1} v_b\right), \frac{r}{6}\right), \\ &P'_{\mu,\nu}\left(-\frac{2}{r}(r+1)\left[C(rv_n) - r^3C(v_n)\right] +\right. \\ &\left.\frac{2}{r.r^{3n}}(r+1)\left[f(r^n rv_n) - r^3 f(r^n v_n)\right], \frac{r}{6}\right), \\ &P'_{\mu,\nu}\left(\frac{1}{r^{3n}}f\left(r^n\sum_{b=1}^{n-1} v_b + rv_n\right)\right. \\ &+\frac{1}{r^{3n}}f\left(r^n\sum_{b=1}^{n-1} v_b - rv_n\right) \\ &-\frac{r^2}{r^{3n}}\left[f\left(r^n\sum_{b=1}^{n} v_b\right) + f\left(r^n\sum_{b=1}^{n-1} v_b - v_n\right)\right] \\ &+\frac{2(r^2-1)}{r^{3n}}f\left(r^n\sum_{b=1}^{n-1} v_b\right) \\ &\left.\left.-\frac{2}{r.r^{3n}}(r+1)\left[f(r.r^n v_n) - r^3 f(r^n v_n)\right], \frac{r}{6}\right)\right\} \end{aligned} \tag{4.23}$$

for all $v_1, v_2, \cdots v_{n-1}, v_n \in X$ and all $s > 0$. Letting $n \to \infty$ in (4.23) and using (4.22),(4.2), we arrive

$$\begin{aligned} &P_{\mu,\nu}\left(C\left(\sum_{b=1}^{n-1} v_b + rv_n\right) + C\left(\sum_{b=1}^{n-1} v_b - rv_n\right)\right. \\ &-r^2\left[C\left(\sum_{b=1}^{n} v_b\right) + C\left(\sum_{b=1}^{n-1} v_b - v_n\right)\right] \\ &\left.+2(r^2-1)C\left(\sum_{b=1}^{n-1} v_b\right) - \frac{2}{r}(r+1)\left[C(rv_n) - r^3C(v_n)\right], s\right) \\ &\geq_{L^*} T^5\left(1_{L^*}, 1_{L^*}, 1_{L^*}, 1_{L^*}, 1_{L^*}\right. \\ &\qquad \left., P'_{\mu,\nu}\left(\sigma(r^n v_1, r^n v_2, \cdots r^n v_{n-1}, r^n v_n), r^{3n}s\right)\right) \\ &\geq_{L^*} P'_{\mu,\nu}\left(\sigma(r^n v_1, r^n v_2, \cdots r^n v_{n-1}, r^n v_n), r^{3n}s\right) \end{aligned} \tag{4.24}$$

for all $v_1, v_2, \cdots v_{n-1}, v_n \in X$ and all $s > 0$. Letting $n \to \infty$ in (4.24) and using (4.2), $(IFN2)$, we arrive

$$\begin{aligned} &C\left(\sum_{b=1}^{n-1} v_b + sv_n\right) + C\left(\sum_{b=1}^{n-1} v_b - sv_n\right) \\ &\quad = r^2\left[C\left(\sum_{b=1}^{n} v_b\right) + C\left(\sum_{b=1}^{n-1} v_b - v_n\right)\right] \\ &\quad -2(s^2-1)C\left(\sum_{b=1}^{n-1} v_b\right) + \frac{2(s+1)}{s}\left[C(sv_n) - s^3C(v_n)\right] \end{aligned}$$

for all $v_1, v_2, \cdots v_{n-1}, v_n \in X$. Hence $C$ satisfies the functional equation (2.1). In order to prove $C(x)$ is unique, let $C'(x)$ be another cubic functional equation satisfying (2.1) and (4.5).

Hence,

$$\begin{aligned} & P'_{\mu,\nu}(C(v)-C'(v),r) \\ &= P'_{\mu,\nu}\left(\frac{C(r^n v)}{r^{3n}}-\frac{C'(r^n v)}{r^{3n}},s\right) \\ &\geq_{L^*} T\left(P'_{\mu,\nu}\left(C(r^n v)-\frac{f_o(r^n v)}{r^{3n}},\frac{r^{3n}s}{2}\right)\right. \\ &\qquad \left.,P'_{\mu,\nu}\left(\frac{f_o(r^n v)}{r^{3n}}-C'(r^n v),\frac{r^{3n}s}{2}\right)\right) \\ &\geq_{L^*} P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},r^n v\right),\frac{r^{3n}(r+1)}{r}(r^3-a)s\right) \\ &\geq_{L^*} P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),\frac{r^{3n}(r+1)}{r.a^n}(r^3-a)s\right) \end{aligned}$$

for all $x \in X$ and all $r > 0$. Since $\lim_{n\to\infty}\frac{r^{3n}(r+1)}{r.a^n}(r^3-a)=\infty$, we obtain

$$\lim_{n\to\infty} P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),\frac{r^{3n}(r+1)}{r.a^n}(r^3-a)s\right)=1_{L^*}.$$

Thus

$$P'_{\mu,\nu}(C(x)-C'(x),s)=1_{L^*}$$

for all $v \in X$ and all $r > 0$, hence $C(x) = C'(x)$. Therefore $C(x)$ is unique.

For $\tau = -1$, we can prove the similar stability result. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 4.1, regarding the stability of(1.3)

**Corollary 4.2.** *Suppose that an odd function $f_o : X \to Y$ satisfies the inequality*

$$P_{\mu,\nu}\left(Df_o(v_1,v_2,\ldots,v_{n-1},v_n),s\right) \geq_{L^*} \begin{cases} P'_{\mu,\nu}(\lambda,s), \\ P'_{\mu,\nu}\left(\lambda\sum_{i=1}^n ||v_i||^s,s\right), \\ P'_{\mu,\nu}\left(\lambda\left(\prod_{i=1}^n ||v_i||^s+\sum_{i=1}^n ||v_i||^{ns}\right),s\right), \end{cases} \tag{4.25}$$

*for all $v_1, v_2, \ldots, v_{n-1}, v_n \in X$ and all $s > 0$ , where $\lambda, s$ are constants with $\lambda > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that*

$$P_{\mu,\nu}\left(f_o(v)-C(v),s\right) \geq_{L^*} \begin{cases} P'_{\mu,\nu}\left(\lambda,\frac{2(r+1)}{r}|r^3-1|s\right), \\ P'_{\mu,\nu}\left(\lambda||v||^s,\frac{2(r+1)}{r}|r^3-r^s|s\right), & s\neq 3; \\ P'_{\mu,\nu}\left(\lambda||v||^{ns},\frac{2(r+1)}{r}|r^3-r^{ns}|s\right), & s\neq \frac{3}{n}; \end{cases} \tag{4.26}$$

*for all $v \in X$ and all $s > 0$.*

*Proof.* Replacing

$$\sigma(v_1,v_2,\ldots,v_{n-1},v_n) = \begin{cases} \lambda, \\ \lambda\left(||v_1||^s+||v_2||^s+\cdots+||v_{n-1}||^s+||v_n||^s\right), \\ \lambda\left\{||v_1||^s\,||v_2||^s\cdots||v_{n-1}||^s\,||v_n||^s \right.\\ \qquad \left.+\left(||v_1||^{ns}+||v_1||^{ns}+\cdots+||v_{n-1}||^{ns}+||v_n||^{ns}\right)\right\}, \end{cases}$$

we arrive (5.22) by defining

$$a = \begin{cases} r^0, \\ r^s, \\ r^{ns}. \end{cases}$$

in Theorem 4.1 □

The proof of the following Theorem and Corollary is similar tracing to that of Theorem 4.1 and Corollary 4.2, when $f_e$ is even. Hence the details of the proof is omitted.

**Theorem 4.3.** *Let $\tau \in \{1,-1\}$. Let $\sigma : X^n \to Z$ be a function such that for some $0 < \left(\frac{a}{r^4}\right)^\tau < 1$,*

$$P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},r^\tau v\right),s\right) \geq_{L^*} P'_{\mu,\nu}\left(a^\tau\ \sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),s\right) \tag{4.27}$$

*for all $v \in X$ and all $s > 0$ and*

$$\lim_{n\to\infty} P'_{\mu,\nu}\left(\sigma\left(r^{\tau n}v_1,r^{\tau n}v_2,r^{\tau n}v_3,\cdots,r^{\tau n}v_{n-1},r^{\tau n}v_n\right),r^{\tau n}s\right)=1_{L^*} \tag{4.28}$$

*for all $v_1, v_2, \cdots v_{n-1}, v_n \in X$ and all $s > 0$. Let $f_e : X \to Y$ be an even function satisfies the inequality*

$$P_{\mu,\nu}\left(Df_e(v_1,v_2,\cdots v_{n-1},v_n),s\right) \geq_{L^*} P'_{\mu,\nu}\left(\sigma\left(v_1,v_2,\cdots v_{n-1},v_n\right),s\right) \tag{4.29}$$

*for all $v_1, v_2, \cdots v_{n-1}, v_n \in X$ and all $s > 0$. Then the limit*

$$P_{\mu,\nu}\left(Q(v)-\frac{f_e(r^n v)}{r^{4n}},s\right)\to 1_L^*, \quad as\ n\to\infty,\ s>0 \tag{4.30}$$

*exists for all $v \in X$ and the mapping $Q : X \to Y$ is a unique quartic mapping satisfying (2.1) and*

$$P_{\mu,\nu}\left(f_e(x)-Q(x),r\right) \geq_{L^*} P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),\frac{2}{r}|r^4-d|s\right) \tag{4.31}$$

*for all $v \in X$ and all $s > 0$.*

**Corollary 4.4.** *Suppose that an even function $f_e : X \to Y$ satisfies the inequality*

$$P_{\mu,\nu}\left(Df_e(v_1,v_2,\ldots,v_{n-1},v_n),s\right) \geq_{L^*} \begin{cases} P'_{\mu,\nu}\left(\lambda,s\right), \\ P'_{\mu,\nu}\left(\lambda\sum_{i=1}^{n}||v_i||^s,s\right), \\ P'_{\mu,\nu}\left(\lambda\left(\prod_{i=1}^{n}||v_i||^s+\sum_{i=1}^{n}||v_i||^{ns}\right),s\right), \end{cases} \tag{4.32}$$

*for all $v_1,v_2,\ldots,v_{n-1},v_n \in X$ and all $r>0$, where $\lambda, s$ are constants with $\lambda>0$. Then there exists a unique quartic mapping $Q : X \to Y$ such that*

$$P_{\mu,\nu}\left(f_e(v)-Q(v),s\right) \geq_{L^*} \begin{cases} P'_{\mu,\nu}\left(\lambda,\frac{2}{r}|r^4-1|s\right), \\ P'_{\mu,\nu}\left(\lambda||v||^s,\frac{2}{r}|r^4-r^s|s\right), & s\neq 4; \\ P'_{\mu,\nu}\left(\lambda||v||^{ns},\frac{2}{r}|r^4-r^{ns}|s\right), & s\neq \frac{4}{n}; \end{cases} \tag{4.33}$$

*for all $v \in X$ and all $s>0$.*

**Theorem 4.5.** *Let $\tau=\pm 1$ be fixed and let $\sigma : X^n \to Z$ be a mapping such that for some $d$ with $0<\left(\frac{a}{r^3}\right)^\tau<1, 0<\left(\frac{a}{r^4}\right)^\tau<1$ and satisfying (4.1), (4.2), (4.27) and (4.28). Suppose that a function $f : X \to Y$ satisfies the inequality*

$$P_{\mu,\nu}\left(D\,f(v_1,v_2,\cdots v_{n-1},v_n),s\right) \geq_{L^*} P'_{\mu,\nu}\left(\sigma(v_1,v_2,\cdots v_{n-1},v_n),r\right) \tag{4.34}$$

*for all $v_1,v_2,\cdots v_{n-1},v_n \in X$ and all $s>0$. Then there exists a unique cubic mapping $C : X \to Y$ and unique quartic mapping $Q : X \to Y$ satisfying (1.3) and*

$$P_{\mu,\nu}\left(f(x)-C(x)-Q(x),r\right) \geq_{L^*} P^3_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),s\right) \tag{4.35}$$

*where*

$$P^3_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),s\right) = T\left\{P^1_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),\frac{2(r+1)}{r}|r^3-d|s\right),\right. \left. P^2_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),\frac{2}{r}|r^4-d|s\right)\right\} \tag{4.36}$$

*for all $v \in X$ and all $s>0$.*

*Proof.* Clearly

$$|r^4| \leq |r^3| \leq a.$$

Let $f_c(v)=\dfrac{f_o(v)-f_o(-v)}{2}$ for all $v \in X$. Then $f_c(0)=0$ and $f_c(-v)=-f_c(v)$ for all $v \in X$. Hence

$$\begin{aligned} &P_{\mu,\nu}\left(D\,f_c(v_1,v_2,\cdots v_{n-1},v_n),s\right) \\ &\geq_{L^*} T\left\{P_{\mu,\nu}\left(D\,f_o(v_1,v_2,\cdots v_{n-1},v_n),s\right),\right. \left. P_{\mu,\nu}\left(D\,f_o(-v_1,-v_2,\cdots -v_{n-1},-v_n),s\right)\right\} \\ &\geq_{L^*} T\left\{P_{\mu,\nu}\left(\sigma(v_1,v_2,\cdots v_{n-1},v_n),s\right),\right. \left. P_{\mu,\nu}\left(\sigma(-v_1,-v_2,\cdots -v_{n-1},-v_n),s\right)\right\} \end{aligned} \tag{4.37}$$

for all $v_1,v_2,\cdots v_{n-1},v_n \in X$ and all $s>0$. By Theorem 4.1 there exists a unique cubic mapping $C : X \to Y$ such that

$$P_{\mu,\nu}\left(f_o(v)-C(v),r\right) \geq_{L^*} P^1_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),\frac{2(r+1)}{r}|r^3-d|s\right) \tag{4.38}$$

for all $v \in X$ and all $s>0$, where

$$P^1_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),s\right) = T\left\{P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),s\right),\right. \left. P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},-v\right),s\right)\right\} \tag{4.39}$$

for all $v_1,v_2,\cdots v_{n-1},v_n \in X$ and all $s>0$.

Also, let $f_q(v)=\dfrac{f_e(v)+f_e(-v)}{2}$ for all $v \in X$. Then $f_q(0)=0$ and $f_q(-v)=f_q(v)$ for all $v \in X$. Hence

$$\begin{aligned} &P_{\mu,\nu}\left(D\,f_q(v_1,v_2,\cdots v_{n-1},v_n),s\right) \\ &\geq_{L^*} T\left\{P_{\mu,\nu}\left(D\,f_e(v_1,v_2,\cdots v_{n-1},v_n),s\right),\right. \left. P_{\mu,\nu}\left(D\,f_e(-v_1,-v_2,\cdots -v_{n-1},-v_n),s\right)\right\} \\ &\geq_{L^*} T\left\{P_{\mu,\nu}\left(\sigma(v_1,v_2,\cdots v_{n-1},v_n),s\right),\right. \left. P_{\mu,\nu}\left(\sigma(-v_1,-v_2,\cdots -v_{n-1},-v_n),s\right)\right\} \end{aligned} \tag{4.40}$$

for all $v_1,v_2,\cdots v_{n-1},v_n \in X$ and all $s>0$. By Theorem 4.3, there exists a unique quartic mapping $Q : X \to Y$ such that

$$P_{\mu,\nu}\left(f_e(v)-Q(v),s\right) \geq_{L^*} P^2_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),\frac{2}{r}|r^4-d|s\right) \tag{4.41}$$

for all $v \in X$ and all $s > 0$, where

$$P^2_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),s\right) = T\left\{P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),s\right), P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},-v\right),s\right)\right\} \quad (4.42)$$

for all $v_1, v_2, \cdots v_{n-1}, v_n \in X$ and all $s > 0$. Define

$$f(v) = f_c(v) + f_q(v) \quad (4.43)$$

for all $v \in X$. From (4.35),(4.38) and (4.39), we arrive

$$\begin{aligned} & P_{\mu,\nu}(f(v) - C(v) - Q(x), s) \\ &= P_{\mu,\nu}(f_c(v) + f_q(v) - C(v) - Q(v), s) \\ &\geq_{L^*} T\left\{P_{\mu,\nu}\left(f_c(v) - C(v), \frac{s}{2}\right), P_{\mu,\nu}\left(f_q(v) - Q(v), \frac{s}{2}\right)\right\} \\ &\geq_{L^*} T\left\{P^1_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right), \frac{(r+1)}{r}|r^3 - d|s\right), P^2_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right), \frac{1}{r}|r^4 - d|s\right)\right\} \\ &= P^3_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),s\right) \end{aligned}$$

where

$$P^3_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right),s\right) = T\left\{P^1_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right), \frac{2(r+1)}{r}|r^3 - d|s\right), P^2_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times},v\right), \frac{2}{r}|r^4 - d|s\right)\right\} \quad (4.44)$$

for all $v \in X$ and all $s > 0$. Hence the theorem is proved. □

The following corollary is the immediate consequence of corollaries 4.2, 4.4 and Theorem 4.5 concerning the stability for the functional equation (2.1).

**Corollary 4.6.** *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$P_{\mu,\nu}(Df(v_1, v_2, \ldots, v_{n-1}, v_n), s) \geq_{L^*} \begin{cases} P'_{\mu,\nu}(\lambda, s), \\ P'_{\mu,\nu}(\lambda \sum_{i=1}^n ||v_i||^s, s), \\ P'_{\mu,\nu}(\lambda (\prod_{i=1}^n ||v_i||^s + \sum_{i=1}^n ||v_i||^{ns}), s), \end{cases} \quad (4.45)$$

*for all $v_1, v_2, \ldots, v_{n-1}, v_n \in X$ and all $s > 0$, where $\lambda, s$ are constants with $\lambda > 0$. Then there exists a unique cubic mapping $C : X \to Y$ and a unique quartic mapping $Q : X \to Y$ such that*

$$P_{\mu,\nu}(f(x) - C(x) - Q(x), r) \geq_{L^*} \begin{cases} T\left\{P'_{\mu,\nu}\left(\lambda, \frac{2(r+1)}{r}|r^3 - 1|s\right), P'_{\mu,\nu}\left(\lambda, \frac{2}{r}|r^4 - 1|s\right)\right\} \\ T\left\{P'_{\mu,\nu}\left(\lambda||v||^s, \frac{2(r+1)}{r}|r^3 - r^s|s\right), P'_{\mu,\nu}\left(\lambda||v||^s, \frac{2}{r}|r^4 - r^s|s\right)\right\}, & s \neq 3,4; \\ T\left\{P'_{\mu,\nu}\left(\lambda||v||^{ns}, \frac{2(r+1)}{r}|r^3 - r^s|s\right), P'_{\mu,\nu}\left(\lambda||v||^{ns}, \frac{2(r+1)}{r}|r^- r^s|s\right)\right\}, & s \neq \frac{3}{n}, \frac{4}{n}; \end{cases} \quad (4.46)$$

*for all $v \in X$ and all $r > 0$.*

## 5. Stability Results: Fixed Point Method

In this section, the authors discuss the generalized Ulam - Hyers stability of the functional equation (2.1) in intuitionistic fuzzy normed space using fixed point method.

Now we will recall the fundamental results in fixed point theory.

**Theorem 5.1.** *(Banach's contraction principle) Let $(X,d)$ be a complete metric space and consider a mapping $T : X \to X$ which is strictly contractive mapping, that is*

*(A1) $d(Tx,Ty) \leq Ld(x,y)$ for some (Lipschitz constant) $L < 1$. Then,*
*(i) The mapping $T$ has one and only fixed point $x^* = T(x^*)$;*
*(ii)The fixed point for each given element $x^*$ is globally attractive, that is*

*(A2) $lim_{n\to\infty} T^n x = x^*$, for any starting point $x \in X$;*
*(iii) One has the following estimation inequalities:*

*(A3) $d(T^n x, x^*) \leq \frac{1}{1-L}\ d(T^n x, T^{n+1}x), \forall\ n \geq 0, \forall\ x \in X$;*

*(A4) $d(x,x^*) \leq \frac{1}{1-L}\ d(x,x^*), \forall\ x \in X$.*

**Theorem 5.2.** *[28](The alternative of fixed point) Suppose that for a complete generalized metric space $(X,d)$ and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*
$(B1)$ $\qquad d(T^n x, T^{n+1}x) = \infty \quad \forall\ n \geq 0,$
*or*
$(B2)$ *there exists a natural number $n_0$ such that:*
*(i) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$ ;*
*(ii)The sequence $(T^n x)$ is convergent to a fixed point $y^*$ of $T$*
*(iii) $y^*$ is the unique fixed point of $T$ in the set $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;*
*(iv) $d(y^*, y) \leq \frac{1}{1-L}\ d(y, Ty)$ for all $y \in Y$.*

For to prove the fixed point stability result, we define a constant $\chi_i$ such that:

$$\chi_i = \begin{cases} r^3 & if \quad i=0, \\ \frac{1}{r^3} & if \quad i=1, \end{cases}$$

and $\Omega$ is the set such that

$$\Omega = \{g \mid g : X \to Y, g(0)=0\}.$$

**Theorem 5.3.** *Let $f_o : X \to Y$ be an odd mapping for which there exist a function $\sigma : X^n \to Z$ with the condition*

$$\lim_{n\to\infty} P'_{\mu,\nu}\left(\sigma\left(\chi_i^n v_1, \chi_i^n v_2, \cdots \chi_i^n v_{n-1}, \chi_i^n v_n\right), \chi_i^{3n} s\right) = 1_{L^*} \tag{5.1}$$

*for all $v_1, v_2, \dots, v_{n-1}, v_n \in X$ and all $s>0$ and satisfying the functional inequality*

$$P_{\mu,\nu}\left(D\, f_o(v_1,v_2,\dots,v_{n-1},v_n), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\sigma(v_1,v_2,\dots,v_{n-1},v_n), r\right) \tag{5.2}$$

*for all $v_1, v_2, \dots, v_{n-1}, v_n \in X$ and all $s>0$. If there exists $L = L(i) > 0$ such that the function*

$$x \to \rho(v) = \frac{r}{2(r+1)}\sigma\left(\underbrace{0,\cdots,0}_{n-1 times}, \frac{v}{r}\right),$$

*has the property*

$$P'_{\mu,\nu}\left(L\frac{\rho(\chi_i v)}{\chi_i^3}, r\right) = P'_{\mu,\nu}\left(\rho(v), s\right), \ \forall\, v \in X, s>0. \tag{5.3}$$

*Then there exists a unique cubic function $C : X \to Y$ satisfying the functional equation (1.3) and*

$$P_{\mu,\nu}\left(f_o(v) - C(v), s\right) \geq_{L^*} P'_{\mu,\nu}\left(\rho(v), \frac{L^{1-i}}{1-L}s\right), \ \forall\, v \in X, s>0. \tag{5.4}$$

*Proof.* Let $d$ be a general metric on $\Omega$, such that

$$d(g,h) = inf\left\{K \in (0,\infty) | P_{\mu,\nu}\left(g(v)-h(v), s\right) \geq_{L^*} P'_{\mu,\nu}\left(\rho(v), Ks\right), v \in \overrightarrow{X}\right\}.$$

It is easy to see that $(\Omega, d)$ is complete. Define $T : \Omega \to \Omega$ by $Tg(v) = \frac{1}{\chi_i^3} g(\chi_i v)$, for all $v \in X$. For $g, h \in \Omega$, we have

$$\begin{aligned}
& d(g,h) \leq K \\
\Rightarrow \quad & P_{\mu,\nu}\left(g(v)-h(v), s\right) \geq_{L^*} P'_{\mu,\nu}\left(\rho(v), Ks\right) \\
\Rightarrow \quad & P_{\mu,\nu}\left(\frac{g(\chi_i v)}{\chi_i} - \frac{h(\chi_i v)}{\chi_i}, s\right) \geq_{L^*} P'_{\mu,\nu}\left(\rho(\chi_i v), K\chi_i s\right) \\
\Rightarrow \quad & P_{\mu,\nu}\left(Tg(v)-Th(v), s\right) \geq_{L^*} P'_{\mu,\nu}\left(\rho(v), KLs\right) \\
\Rightarrow \quad & d\left(Tg(v), Th(v)\right) \leq KL \qquad (5.5) \\
\Rightarrow \quad & d\left(Tg, Th\right) \leq Ld(g,h) \qquad (5.6)
\end{aligned}$$

for all $g, h \in \Omega$. Therefore $T$ is strictly contractive mapping on $\Omega$ with Lipschitz constant $L$. Replacing $(v_1, v_2, \dots, v_{n-1}, v_n)$ by $\left(\underbrace{0,\cdots,0}_{n-1 times}, v\right)$ in (5.2) and using oddness, we get

$$P_{\mu,\nu}\left(\frac{2(r+1)}{r}[r^3 f_o(v) - f_o(rv)], s\right) \geq_{L^*} P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times}, v\right), s\right). \tag{5.7}$$

for all $v \in X, s>0$. Using (IFN2) in (5.7), we arrive

$$P_{\mu,\nu}\left([r^3 f_o(v) - f_o(rv)], s\right) \geq_{L^*} P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times}, v\right), s\right). \tag{5.8}$$

for all $v \in X, s>0$. With the help of (5.3), when $i=0$, it follows from (5.8), we get

$$\begin{aligned}
& P_{\mu,\nu}\left(\frac{f_o(rv)}{r^3} - f_o(v), s\right) \geq_{L^*} P'_{\mu,\nu}\left(\rho(v), Ls\right), \ \forall\, v \in X, s>0 \\
\Rightarrow \quad & d(Tf_o, f_o) \leq L = L^{1-0}. \qquad (5.9)
\end{aligned}$$

Replacing $v$ by $\frac{v}{r}$ in (5.8), we obtain

$$P_{\mu,\nu}\left(f_o(v) - r^3 f_o\left(\frac{v}{r}\right), s\right) \geq_{L^*} P'_{\mu,\nu}\left(\sigma\left(\underbrace{0,\cdots,0}_{n-1 times}, \frac{v}{r}\right), s\right). \tag{5.10}$$

for all $v \in X, s>0$. With the help of (5.3), when $i=1$, it follows from (5.10), we get

$$\begin{aligned}
& P_{\mu,\nu}\left(f_o(v) - r^3 f_o\left(\frac{v}{r}\right), s\right) \geq_{L^*} P'_{\mu,\nu}\left(\rho(v), s\right), \ \forall\, v \in X, s>0 \\
\Rightarrow \quad & d(f_o, Tf_o) \leq 1 = L^0 = L^{1-i}. \qquad (5.11)
\end{aligned}$$

One can conclude from (5.9) and (5.11) that

$$d(f_o, Tf_o) \leq L^{1-i} < \infty$$

Now, using fixed point alternative in both cases, it follows that there exists a fixed point $C$ of $T$ in $\Omega$ such that

$$\lim_{n\to\infty} P_{\mu,\nu}\left(\frac{f_o\left(\chi_i^n v\right)}{\chi_i^n} - C(v), s\right) \to 1_{L^*} \qquad \forall v \in X, s>0. \tag{5.12}$$

By proceeding the same procedure as in the Theorem 4.1, we see that the function $C : X \to Y$ is cubic and it satisfies the functional equation (2.1).

By fixed point alternative, since $C$ is unique fixed point of $T$ in the set

$$\Delta = \{f_o \in \Omega | d(f_o, C) < \infty\},$$

therefore $C$ is a uniqe function such that

$$P_{\mu,\nu}(f_o(v) - C(v), s) \geq_{L^*} P'_{\mu,\nu}(\rho(v), Ks) \tag{5.13}$$

for all $v \in X$, $s > 0$ and $K > 0$. Again, using the fixed point alternative, we reach

$$\begin{aligned} & d(f_o, C) \leq \frac{1}{1-L} d(f_o, Tf_o) \\ \Rightarrow \quad & d(f_o, C) \leq \frac{L^{1-i}}{1-L} \\ \Rightarrow \quad & P_{\mu,\nu}(f_o(v) - C(v), s) \geq_{L^*} P'_{\mu,\nu}\left(\rho(v), \frac{L^{1-i}}{1-L} s\right) \end{aligned} \tag{5.14}$$

for all $v \in X$ and all $s > 0$. This completes the proof of the theorem. □

From Theorem 5.3, we obtain the following corollary concerning the stability for the functional equation (2.1).

**Corollary 5.4.** *Suppose that an odd function $f_o : X \to Y$ satisfies the inequality*

$$P_{\mu,\nu}(Df_o(v_1, v_2, \ldots, v_{n-1}, v_n), s) \geq_{L^*} \begin{cases} P'_{\mu,\nu}(\lambda, s), \\ P'_{\mu,\nu}(\lambda \sum_{i=1}^n ||v_i||^s, s), \\ P'_{\mu,\nu}(\lambda (\prod_{i=1}^n ||v_i||^s + \sum_{i=1}^n ||v_i||^{ns}), s), \end{cases} \tag{5.15}$$

*for all $v_1, v_2, \ldots, v_{n-1}, v_n \in X$ and all $s > 0$ , where $\lambda, s$ are constants with $\lambda > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that*

$$P_{\mu,\nu}(f_o(v) - C(v), s) \geq_{L^*} \begin{cases} P'_{\mu,\nu}\left(\lambda, \frac{2r^2(r+1)}{|r^3-1|} s\right), \\ P'_{\mu,\nu}\left(\lambda ||v||^s, \frac{2r^2(r+1)}{|r^3-r^s|} s\right), & s \neq 3; \\ P'_{\mu,\nu}\left(\lambda ||v||^{ns}, \frac{2r^2(r+1)}{|r^3-r^{ns}|} s\right), & s \neq \frac{3}{n}; \end{cases} \tag{5.16}$$

*for all $v \in X$ and all $s > 0$.*

*Proof.* Setting

$$\sigma(v_1, v_2, \ldots, v_{n-1}, v_n) = \begin{cases} \lambda, \\ \lambda \sum_{i=1}^n ||v_i||^s, \\ \lambda \left(\prod_{i=1}^n ||v_i||^s + \sum_{i=1}^n ||v_i||^{ns}\right). \end{cases}$$

for all $v_1, v_2, \ldots, v_{n-1}, v_n \in X$. Then,

$$\begin{aligned} & P'_{\mu,\nu}(\sigma(\chi_i^n v_1, \chi_i^n v_2, \cdots, \chi_i^n v_n), \chi_i^{3n} s) \\ & = \begin{cases} P'_{\mu,\nu}(\lambda, \chi_i^{3n} s) \\ P'_{\mu,\nu}\left(\lambda \sum_{i=1}^n ||v_i||^s, \chi_i^{(3-s)n} s\right) \\ P'_{\mu,\nu}\left(\lambda \left(\prod_{i=1}^n ||v_i||^s + \sum_{i=1}^n ||v_i||^{ns}\right), \chi_i^{(3-ns)n} s\right) \end{cases} \\ & = \begin{cases} \to 1_{L^*} \text{ as } n \to \infty, \\ \to 1_{L^*} \text{ as } n \to \infty, \\ \to 1_{L^*} \text{ as } n \to \infty. \end{cases} \end{aligned}$$

Thus, (5.1) is holds. But, we have $\rho(v) = \sigma\left(\underbrace{0, \cdots, 0}_{n-1 times}, \frac{v}{r}\right)$ has the property

$$P'_{\mu,\nu}\left(L \frac{1}{\chi_i^3} \rho(\chi_i v), r\right) \geq_{L^*} P'_{\mu,\nu}(\rho(v), s), \ \forall\, v \in X, s > 0.$$

Hence

$$\begin{aligned} & P'_{\mu,\nu}(\rho(v), s) \\ & = P'_{\mu,\nu}\left(\sigma\left(\underbrace{0, \cdots, 0}_{n-1 times}, \frac{v}{r}\right), \frac{2(r+1)}{r} s\right), \\ & = \begin{cases} P'_{\mu,\nu}\left(\lambda, \frac{2(r+1)}{r} s\right), \\ P'_{\mu,\nu}\left(\lambda \frac{1}{r^s} ||v||^s, \frac{2(r+1)}{r} s\right), \\ P'_{\mu,\nu}\left(\lambda \frac{1}{r^{ns}} ||v||^{ns}, \frac{2(r+1)}{r} s\right). \end{cases} \end{aligned}$$

Now,

$$\begin{aligned} & P'_{\mu,\nu}\left(\frac{1}{\chi_i^3} \rho(\chi_i v), s\right) \\ & = \begin{cases} P'_{\mu,\nu}\left(\frac{\lambda}{\chi_i^3}, \frac{2(r+1)}{r} s\right), \\ P'_{\mu,\nu}\left(\frac{\lambda}{\chi_i^3} \frac{1}{r^s} ||\chi_i v||^s, \frac{2(r+1)}{r} s\right), \\ P'_{\mu,\nu}\left(\frac{\lambda}{\chi_i^3} \frac{1}{r^{ns}} ||\chi_i v||^{ns}, \frac{2(r+1)}{r} s\right). \end{cases} \\ & = \begin{cases} P'_{\mu,\nu}(\lambda, \chi_i^3 s), \\ P'_{\mu,\nu}(\rho(v), \chi_i^{3-s} s), \\ P'_{\mu,\nu}(\rho(v), \chi_i^{3-ns} s). \end{cases} \end{aligned}$$

Hence the inequality (5.3) holds for the following cases. Now from (5.4), we prove the following cases.

**Case:1** $L = r^3$ ,if $i = 0$

$$\begin{aligned} P_{\mu,\nu}\left(f_o(v) - C(v), s\right) &\geq_{L^*} P'_{\mu,\nu}\left(\rho(v), \frac{L^{1-0}}{1-L}s\right) \\ &= P'_{\mu,\nu}\left(\lambda, \frac{2r^2(r+1)}{[1-r^3]}s\right). \end{aligned}$$

**Case:2** $L = r^{-3}$,if $i = 1$

$$\begin{aligned} P_{\mu,\nu}\left(f_o(v) - C(v), s\right) &\geq_{L^*} P'_{\mu,\nu}\left(\rho(v), \frac{L^{1-1}}{1-L}s\right) \\ &= P'_{\mu,\nu}\left(\lambda, \frac{2r^2(r+1)}{[r^3-1]}s\right). \end{aligned}$$

**Case:3** $L = r^{s-3}$ for $s < 3$ if $i = 0$

$$\begin{aligned} P_{\mu,\nu}\left(f_o(v) - C(v), s\right) &\geq_{L^*} P'_{\mu,\nu}\left(\rho(v), \frac{L^{1-0}}{1-L}s\right) \\ &= P'_{\mu,\nu}\left(\lambda|v||^s, \frac{2r^2(r+1)}{|r^3-r^s|}s\right). \end{aligned}$$

**Case:4** $L = r^{3-s}$ for $s > 3$ if $i = 1$

$$\begin{aligned} P_{\mu,\nu}\left(f_o(v) - C(v), s\right) &\geq_{L^*} P'_{\mu,\nu}\left(\rho(v), \frac{L^{1-1}}{1-L}s\right) \\ &= P'_{\mu,\nu}\left(\lambda|v||^s, \frac{2r^2(r+1)}{|r^s-r^3|}s\right). \end{aligned}$$

**Case:5** $L = r^{ns-3}$ for $ns < 3$ if $i = 0$

$$\begin{aligned} P_{\mu,\nu}\left(f_o(v) - C(v), s\right) &\geq_{L^*} P'_{\mu,\nu}\left(\rho(v), \frac{L^{1-0}}{1-L}s\right) \\ &= P'_{\mu,\nu}\left(\lambda|v||^s, \frac{2r^2(r+1)}{|r^3-r^{ns}|}s\right). \end{aligned}$$

**Case:6** $L = r^{3-ns}$ for $ns > 3$ if $i = 1$

$$\begin{aligned} P_{\mu,\nu}\left(f_o(v) - C(v), s\right) &\geq_{L^*} P'_{\mu,\nu}\left(\rho(v), \frac{L^{1-1}}{1-L}s\right) \\ &= P'_{\mu,\nu}\left(\lambda|v||^s, \frac{2r^2(r+1)}{|r^{ns}-r^3|}s\right). \end{aligned}$$

Hence the proof is complete. □

The proof of the following Theorem and Corollary is similar tracing to that of Theorem 5.3 and corollary 5.4, when $f$ is even. Hence we omit the proof.

**Theorem 5.5.** *Let $f_e : X \to Y$ be an even mapping for which there exist a function $\sigma : X^n \to Z$ with the condition*

$$\lim_{n\to\infty} P'_{\mu,\nu}\left(\sigma\left(\chi_i^n v_1, \chi_i^n v_2, \cdots, \chi_i^n v_{n-1}, \chi_i^n v_n\right), \chi_i^{4n} s\right) = 1_{L^*} \tag{5.17}$$

*for all $v_1, v_2, \ldots, v_{n-1}, v_n \in X$ and all $s > 0$ and satisfying the functional inequality*

$$\begin{aligned} &P_{\mu,\nu}\left(D\, f_e(v_1, v_2, \ldots, v_{n-1}, v_n), s\right) \\ &\geq_{L^*} P'_{\mu,\nu}\left(\sigma(v_1, v_2, \ldots, v_{n-1}, v_n), r\right) \end{aligned} \tag{5.18}$$

*for all $v_1, v_2, \ldots, v_{n-1}, v_n \in X$ and all $s > 0$. If there exists $L = L(i)$ such that the function*

$$v \to \rho(v) = \frac{2}{r}\sigma\left(\underbrace{0, \cdots, 0}_{n-1 times}, \frac{v}{r}\right),$$

*has the property*

$$P'_{\mu,\nu}\left(L\frac{1}{\chi_i^4}\rho(\chi_i v), s\right) = P'_{\mu,\nu}\left(\rho(v), s\right), \ \forall\, v \in X, s > 0. \tag{5.19}$$

*Then there exists a unique quartic function $Q : X \to Y$ satisfying the functional equation (1.3) and*

$$P_{\mu,\nu}\left(f_e(v) - Q(v), s\right) \geq_{L^*} P'_{\mu,\nu}\left(\rho(v), \frac{L^{1-i}}{1-L}s\right) \ \forall\, v \in X, s > 0. \tag{5.20}$$

**Corollary 5.6.** *Suppose that an even function $f_e : X \to Y$ satisfies the inequality*

$$\begin{aligned} &P_{\mu,\nu}\left(Df_e(v_1, v_2, \ldots, v_{n-1}, v_n), s\right) \\ &\geq_{L^*} \begin{cases} P'_{\mu,\nu}\left(\lambda, s\right), \\ P'_{\mu,\nu}\left(\lambda \sum_{i=1}^n ||v_i||^s, s\right), \\ P'_{\mu,\nu}\left(\lambda\left(\prod_{i=1}^n ||v_i||^s + \sum_{i=1}^n ||v_i||^{ns}\right), s\right), \end{cases} \end{aligned} \tag{5.21}$$

*for all $v_1, v_2, \ldots, v_{n-1}, v_n \in X$ and all $s > 0$, where $\lambda, s$ are constants with $\lambda > 0$. Then there exists a unique quartic mapping $Q : X \to Y$ such that*

$$\begin{aligned} &P_{\mu,\nu}\left(f_e(v) - Q(v), s\right) \\ &\geq_{L^*} \begin{cases} P'_{\mu,\nu}\left(\lambda, \frac{2r^3}{|r^4-1|}s\right), \\ P'_{\mu,\nu}\left(\lambda||v||^s, \frac{2r^3}{|r^4-r^s|}s\right), & s \neq 4; \\ P'_{\mu,\nu}\left(\lambda||v||^{ns}, \frac{2r^3}{|r^4-r^{ns}|}s\right), & s \neq \frac{4}{n}; \end{cases} \end{aligned} \tag{5.22}$$

*for all $v \in X$ and all $s > 0$.*

**Theorem 5.7.** *Let $f : X \to Y$ be a mapping for which there exist a function $\sigma : X^n \to Z$ with the condition (5.1) and (5.17) satisfying the functional inequality*

$$\begin{aligned} &P_{\mu,\nu}\left(D\, f(v_1, v_2, \ldots, v_{n-1}, v_n), r\right) \\ &\geq_{L^*} P'_{\mu,\nu}\left(\sigma(v_1, v_2, \ldots, v_{n-1}, v_n), s\right) \end{aligned} \tag{5.23}$$

*for all $v_1, v_2, \ldots, v_{n-1}, v_n \in X$ and all $s > 0$. If there exists $L = L(i)$ such that the function*

$$x \to \rho(v) = \sigma\left(\underbrace{0, \cdots, 0}_{n-1 times}, \frac{v}{r}\right),$$

*has the properties (5.3) and (5.19) for all $v \in X$. Then there exists a unique cubic function $C : X \to Y$ and a unique quartic function $Q : X \to Y$ satisfying the functional equation (1.3) and*

$$P_{\mu,v}(f(v) - C(v) - Q(v), s) \geq_{L^*} P^3_{\mu,v}(\rho(v), s), \forall\, v \in X, s > 0. \tag{5.24}$$

*Proof.* By Theorem 5.3 in (4.37), there exists a unique Cubic mapping $C : X \to Y$ such that

$$P_{\mu,v}(f_c(v) - C(v), s) \geq_{L^*} P^1_{\mu,v}\left(\rho(v), \frac{L^{1-i}}{1-L}s\right) \tag{5.25}$$

for all $v \in X$ and all $s > 0$. Using Theorem 5.5, in (4.37) there exists a unique quartic mapping $Q : X \to Y$ such that

$$P_{\mu,v}(f_q(v) - Q(v), s) \geq_{L^*} P^2_{\mu,v}\left(\rho(v), \frac{L^{1-i}}{1-L}s\right) \tag{5.26}$$

for all $v \in X$ and all $s > 0$. Define

$$f(v) = f_c(v) + f_q(v) \tag{5.27}$$

for all $v \in x$. From (5.24),(5.25) and (5.26), we arrive

$$\begin{aligned} &P_{\mu,v}(f(v) - A(v) - Q(v), s) \\ &= P_{\mu,v}(f_q(v) + f_c(v) - C(v) - Q(v), s) \\ &\geq_{L^*} T\left\{P'_{\mu,v}\left(f_c(v) - C(v), \frac{s}{2}\right), P'_{\mu,v}\left(f_q(v) - Q(v), \frac{s}{2}\right)\right\} \\ &\geq_{L^*} T\left\{P^1_{\mu,v}\left(\rho(v), \frac{L^{1-i}}{1-L}r\right), P^2_{\mu,v}\left(\rho(v), \frac{L^{1-i}}{1-L}r\right)\right\} \\ &= P^3_{\mu,v}(\rho(v), s), \end{aligned}$$

where

$$P^3_{\mu,v}(\rho(v), s) = T\left\{P^1_{\mu,v}\left(\rho(v), \frac{L^{1-i}}{1-L}r\right), P^2_{\mu,v}\left(\rho(v), \frac{L^{1-i}}{1-L}s\right)\right\} \tag{5.28}$$

for all $v \in X$ and all $r > 0$. Hence the theorem is proved. □

The following corollary is the immediate consequence of Corollaries 5.4, 5.6 and Theorem 5.7 concerning the stability for the functional equation (1.3) using fixed point method.

**Corollary 5.8.** *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$\begin{aligned} &P_{\mu,v}(Df(v_1, v_2, \ldots, v_{n-1}, v_n), s) \\ &\geq_{L^*} \begin{cases} P'_{\mu,v}(\lambda, s), \\ P'_{\mu,v}(\lambda \sum_{i=1}^{n} ||v_i||^s, s), \\ P'_{\mu,v}(\lambda (\prod_{i=1}^{n} ||v_i||^s + \sum_{i=1}^{n} ||v_i||^{ns}), s), \end{cases} \end{aligned} \tag{5.29}$$

*for all $v_1, v_2, \ldots, v_{n-1}, v_n \in X$ and all $s > 0$, where $\lambda, s$ are constants with $\lambda > 0$. Then there exists a unique cubic mapping $C : X \to Y$ and a unique quartic mapping $Q : X \to Y$ such that*

$$\begin{aligned} &P_{\mu,v}(f(x) - C(x) - Q(x), r) \\ &\geq_{L^*} \begin{cases} T\left\{P'_{\mu,v}\left(\lambda, \frac{2r^2(r+1)}{r}|r^3 - 1|s\right),\ P'_{\mu,v}\left(\lambda, \frac{2r^3}{|r^4-1|}s\right)\right\} \\ T\left\{P'_{\mu,v}\left(\lambda||v||^s, \frac{2r^2(r+1)}{r}|r^3 - r^s|s\right),\ P'_{\mu,v}\left(\lambda||v||^s, \frac{2r^3}{|r^4-r^s|}s\right)\right\}, & s \neq 3, 4; \\ T\left\{P'_{\mu,v}\left(\lambda||v||^{ns}, \frac{2r^2(r+1)}{r}|r^3 - r^s|s\right),\ P'_{\mu,v}\left(\lambda||v||^{ns}, \frac{2r^3}{|r^4-r^s|}s\right)\right\}, & s \neq \frac{3}{n}, \frac{4}{n}; \end{cases} \end{aligned} \tag{5.30}$$

*for all $v \in X$ and all $s > 0$.*

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