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Generalized Hyers-Ulam stability of functional equation deriving from additive and quadratic functions in fuzzy Banach space via two different techniques

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Abstract

In this paper, authors given the generalized Hyers - Ulam stability of the functional equation deriving from additive and quadratic functions

$$\sum_{i=1}^{n} f\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right) = \sum_{i=1}^{n} f(x_i) - nf\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right)$$

where n is a positive integer with $n \geq 2$ in Fuzzy Banach space via two different techniques.

Keywords

Additive, Quadratic, mixed additive-quadratic functional equations, Generalized Ulam - Hyers stability, Fuzzy Banach space, fixed point.

AMS Subject Classification

39B52, 32B72, 32B82.

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Contents

1. Introduction

S.M. Ulam, in his famous lecture in 1940 to the Mathematics Club of the University of Wisconsin, presented a number of unsolved problems. This is the starting point of the theory of the stability of functional equations. One of the questions led to a new line of investigation, nowadays known as the stability problems. Ulam [62] discusses:

. . . the notion of stability of mathematical theorems considered from a rather general point of view: When is it true that by changing a little the hypothesis of a theorem one can still assert that the thesis of the theorem remains true or approximately true? . . .

For very general functional equations one can ask the following question. When is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation? Similarly, if we replace a given functional equation by a functional inequality, when can one assert that the solutions of the inequality lie near to the solutions of the strict equation?

Suppose G is a group, $H(d)$ is a metric group, and $f : G \to H$. For any $\varepsilon > 0$, does there exist a $\delta > 0$ such that

$$d(f(xy), f(x)f(y)) < \delta$$

holds for all $x,y \in G$ and implies there is a homomorphism $M : G \to H$ such that

$$d(f(x), M(x)) < \varepsilon$$

for all $x \in G$?

If the answer is affirmative, then we say that the Cauchy functional equation is stable. These kinds of questions form the basics of stability theory, and D.H. Hyers [35] obtained the first important result in this field. Many examples of this have been solved and many variations have been studied since (one can refer [2, 32, 48, 54, 60]). Several investigations followed, and almost all functional equations are stabilized.

The solution and stability of following additive - quadratic functional equations

$$f(x+y)+f(x-y)=2f(x)+f(y)+f(-y) \tag{1.1}$$

$$f\left(\sum_{i=1}^{n} x_i\right)+(n-2)\sum_{i=1}^{n} f(x_i) = \sum_{1\le i<j\le n} f(x_i+x_j) \tag{1.2}$$

$$\begin{aligned} &f(-x_1)+f\left(2x_1-\sum_{i=2}^{n} x_i\right)+f\left(2\sum_{i=2}^{n} x_i\right)\\ &+f\left(x_1+\sum_{i=2}^{n} x_i\right)-f\left(-x_1-\sum_{i=2}^{n} x_i\right)\\ &-f\left(x_1-\sum_{i=2}^{n} x_i\right)-f\left(-x_1+\sum_{i=2}^{n} x_i\right)\\ &=3f(x_1)+3f\left(\sum_{i=2}^{n} x_i\right) \end{aligned} \tag{1.3}$$

$$\begin{aligned} &\sum_{i=0}^{n}\left[f(x_{2i}+x_{2i+1})+f(x_{2i}-x_{2i+1})\right]\\ &=\sum_{i=0}^{n}\left[2f(x_{2i})+f(x_{2i+1})+f(-x_{2i+1})\right] \end{aligned} \tag{1.4}$$

where introduced and discussed in [4, 5, 9, 37].

A. Najati, Th.M. Rassias [45], introduced and investigate the general solution the generalized Hyers - Ulam stability of the functional equation deriving from additive and quadratic functions

$$\sum_{i=1}^{n} f\left(x_i-\frac{1}{n}\sum_{j=1}^{n} x_j\right)=\sum_{i=1}^{n} f(x_i)-nf\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right) \tag{1.5}$$

where n is a positive integer with $n \ge 2$ in Banach modules. It is easy to see that the function $f(x) = ax + bx^2$ is the solution of the functional equation (1.5). Also, S. Zolfaghari [66] establish the generalized Hyers-Ulam stability of the functional equation (1.5) in $p-$ Banach space. The general solution and generalized Ulam - Hyers stability of various mixed type functional equations were discussed in [7, 8, 11–13, 15, 16, 33, 46, 47, 51, 52, 60].

In this paper, authors proved the generalized Ulam - Hyers stability of the additive quadratic functional equation (1.5) in Fuzzy Banach space via two different techniques.

2. Definitions on Fuzzy Banach Spaces

In this section, we present the definitions and notations on fuzzy normed spaces. We use the definition of fuzzy normed spaces given in [18] and [41–44].

Definition 2.1. *Let X be a real linear space. A function $N : X\times\mathbb{R} \longrightarrow [0,1]$(the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x,y \in X$ and all $s,t \in \mathbb{R}$,*
$(FNS1)$ $N(x,c) = 0$ *for* $c \le 0$;
$(FNS2)$ $x = 0$ *if and only if* $N(x,c) = 1$ *for all* $c > 0$;
$(FNS3)$ $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ *if* $c \ne 0$;
$(FNS4)$ $N(x+y, s+t) \ge min\{N(x,s), N(y,t)\}$;
$(FNS5)$ $N(x,\cdot)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $lim_{t\to\infty} N(x,t) = 1$;
$(FNS6)$ *for* $x \ne 0, N(x,\cdot)$ *is (upper semi) continuous on* $\mathbb{R}$.
The pair (X,N) is called a fuzzy normed linear space. One may regard $N(X,t)$ as the truth-value of the statement the norm of x is less than or equal to the real number t'.

Example 2.2. *Let $(X, ||\cdot||)$ be a normed linear space. Then*

$$N(x,t) = \begin{cases} \dfrac{t}{t+\|x\|}, & t>0,\ x\in X,\\ 0, & t\le 0,\ x\in X \end{cases}$$

is a fuzzy norm on X.

Definition 2.3. *Let (X,N) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N - \lim_{n\to\infty} x_n = x$.*

Definition 2.4. *A sequence x_n in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \ge n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1-\varepsilon$.*

Definition 2.5. *Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.*

Definition 2.6. *A mapping $f : X \longrightarrow Y$ between fuzzy normed spaces X and Y is continuous at a point x_0 if for each sequence $\{x_n\}$ covering to x_0 in X, the sequence $f\{x_n\}$ converges to $f(x_0)$. If f is continuous at each point of $x_0 \in X$ then f is said to be continuous on X.*

The stability of a quiet number of functional equations in Fuzzy normed spaces was given in [3, 20, 21, 41–44]

Hereafter, full of the paper, we consider $\mathscr{S}_3, (\mathscr{S}_1, N)$ and $(\mathscr{S}_2, N')$ are linear space, fuzzy normed space and fuzzy Banach space. Define a mapping $f : \mathscr{S}_1 \longrightarrow \mathscr{S}_2$ by

$$F_{AQ}(x_1,x_2,x_3,\cdots,x_n) = \sum_{i=1}^{n} f\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right) - \sum_{i=1}^{n} f(x_i) - nf\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right)$$

where $n \geq 2$ for all $x_1,x_2,x_3,\cdots,x_n \in \mathscr{S}_1$.

3. Fuzzy Stability Results: Direct Method

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.5) in Fuzzy normed space using direct method.

Theorem 3.1. *Let $p = \pm 1$ and $\lambda, \Lambda : \mathscr{S}_1^2 \longrightarrow \mathscr{S}_3$ be a function such that*

$$\lim_{q\to\infty} N'\left(\lambda\left(2^{pq}x_1, 2^{pq}x_2, 2^{pq}x_3, \cdots, 2^{pq}x_n\right), 2^{pq}s\right) = 1 \quad (3.1)$$

for all $x_1,x_2,x_3,\cdots,x_n \in \mathscr{S}_1$ and all $s > 0$, for some $t > 0$ with $0 < \left(\frac{t}{2}\right)^p < 1$ and

$$N'\left(\Lambda_A\left(2^p x, 2^p x, 2^p x, \cdots, 2^p x\right), s\right) \geq N'\left(t^p \Lambda_A\left(2^p x, 2^p x, 2^p x, \cdots, 2^p x\right), s\right) \quad (3.2)$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Let $f : \mathscr{S}_1 \longrightarrow \mathscr{S}_2$ be an odd mapping fulfilling the inequality

$$N\left(F_{AQ}(x_1,x_2,x_3,\cdots,x_n), s\right) \geq N'\left(\lambda\left(x_1,x_2,x_3,\cdots,x_n\right), s\right) \quad (3.3)$$

for all $x_1,x_2,x_3,\cdots,x_n \in \mathscr{S}_1$ and all $s > 0$. Then there exists a unique Additive mapping $\mathscr{A} : \mathscr{S}_1 \longrightarrow \mathscr{S}_2$ which satisfies (1.5) and

$$N\left(f(x) - \mathscr{A}(x), s\right) \geq N'\left(\Lambda_A\left(x,x,\cdots,x\right), \frac{s|2-t|}{a}\right) \quad (3.4)$$

where a, $\Lambda_A\left(x,x,\cdots,x\right)$ and $\mathscr{A}(x)$ are defined by

$$a = \left[\frac{4+n}{2}\right] \quad (3.5)$$

$$N'\left(\Lambda_A(x,x,x,\cdots,x), s\right) = \min\left\{ N'\left(\lambda\left(-x,\underbrace{x,x\cdots,x}_{n-1\ \ times}\right), s\right), N'\left(\lambda\left(x,\underbrace{-x,-x\cdots,-x}_{n-1\ \ times}\right), s\right), N'\left(\lambda\left(x,x,\underbrace{0,\cdots,0}_{n-2\ \ times}\right), ns\right), N'\left(\lambda\left(2x,\underbrace{0,0,\cdots,0}_{n-1\ \ times}\right), s\right)\right\} \quad (3.6)$$

and

$$\lim_{q\to\infty} N\left(\mathscr{A}(x) - \frac{f(2^{pq}x)}{2^{pq}}, s\right) = 1 \quad (3.7)$$

for all $x \in \mathscr{S}_1$ and all $s > 0$, respectively.

Proof. Setting $(x_1,x_2,x_3,\cdots,x_n)$ by $(nx, -ny, \underbrace{0,\cdots,0}_{n-2\ \ times})$ in (3.3), we get

$$N\left(f\left(nx - \frac{1}{n}(nx-ny)\right) + f\left(-ny - \frac{1}{n}(nx-ny)\right) + (n-2)f(-(x-y)) - f(nx) - f(-ny) + nf(x-y), s\right) \quad (3.8)$$

$$\geq N'\left(\lambda\left(nx, -ny, \underbrace{0,\cdots,0}_{n-2\ \ times}\right), s\right) \quad (3.9)$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Using oddness of f in the above inequality, we obtain

$$N\left(f((n-1)x+y) - f(x+(n-1)y) - f(nx) + f(ny) + 2f(x-y), s\right) \geq N'\left(\lambda\left(nx, -ny, \underbrace{0,\cdots,0}_{n-2\ \ times}\right), s\right) \quad (3.10)$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Substitute y by 0 in (3.10), we arrive

$$N\left(f(nx) - f((n-1)x) - f(x), s\right) \geq N'\left(\lambda\left(nx, \underbrace{0,0,\cdots,0}_{n-1\ \ times}\right), s\right) \quad (3.11)$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Again substitute x by $x-y$ in (3.11), we have

$$N\left(f(n(x-y)) - f((n-1)(x-y)) - f(x-y), s\right) \geq N'\left(\lambda\left(n(x-y), \underbrace{0,0,\cdots,0}_{n-1\ \ times}\right), s\right) \quad (3.12)$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Putting $(x_1,x_2,x_3,\cdots,x_n)$ by $(ny, \underbrace{nx,nx\cdots,nx}_{n-1\ \ times})$ in (3.3), we get

$$N\left(f((n-1)(y-x)) + (n-1)f(x-y) - f(ny) - (n-1)f(nx) + nf((n-1)x+y), s\right) \geq N'\left(\lambda\left(ny, \underbrace{nx,nx\cdots,nx}_{n-1\ \ times}\right), s\right) \quad (3.13)$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Using oddness of f in the above inequality, we obtain

$$N\left((n-1)f(x-y) - f((n-1)(x-y)) - f(ny) - (n-1)f(nx) + nf\left((n-1)x+y\right), s\right) \geq N'\left(\lambda\left(ny, \underbrace{nx, nx\cdots, nx}_{n-1\ \ times}\right), s\right) \tag{3.14}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Interchanging x and y in the above inequality and using oddness of f, we have

$$N\left(f((n-1)(x-y)) - (n-1)f(x-y) - f(nx) - (n-1)f(ny) + nf\left(x+(n-1)y\right), s\right) \geq N'\left(\lambda\left(nx, \underbrace{ny, ny\cdots, ny}_{n-1\ \ times}\right), s\right) \tag{3.15}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. It follows from (3.10), (3.14), (3.15) and $(FNS4)$, we arrive

$$\begin{aligned} &N\left(2f((n-1)(x-y)) + 2f(x-y) - 2f(nx) + 2f(ny), s+s+ns\right) \\ &\geq \min\Big\{N\left(f\left((n-1)x+y\right) - f\left(x+(n-1)y\right) - f\left(nx\right),\right. \\ &\qquad \left. + f\left(ny\right) + 2f\left(x-y\right), s\right) \\ &\quad N\left((n-1)f(x-y) - f((n-1)(x-y)) - f(ny)\right. \\ &, \qquad \left. -(n-1)f(nx) + nf\left((n-1)x+y\right), s\right) \\ &\quad N\left(f((n-1)(x-y)) - (n-1)f(x-y) - f(nx)\right. \\ &\qquad \left. -(n-1)f(ny) + nf\left(x+(n-1)y\right), ns\right)\Big\} \\ &\geq \min\Bigg\{N'\left(\lambda\left(ny, \underbrace{nx, nx\cdots, nx}_{n-1\ \ times}\right), s\right), \\ &\qquad N'\left(\lambda\left(nx, \underbrace{ny, ny\cdots, ny}_{n-1\ \ times}\right), s\right), \\ &\qquad N'\left(\lambda\left(nx, -ny, \underbrace{0, \cdots, 0}_{n-2\ \ times}\right), ns\right)\Bigg\} \end{aligned} \tag{3.16}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Using $(FNS3)$ in above inequality, we get

$$\begin{aligned} &N\left(f((n-1)(x-y)) + f(x-y) - f(nx) + f(ny), \frac{s+s+ns}{2}\right) \\ &\geq \min\Bigg\{N'\left(\lambda\left(ny, \underbrace{nx, nx\cdots, nx}_{n-1\ \ times}\right), s\right), \\ &\qquad N'\left(\lambda\left(nx, \underbrace{ny, ny\cdots, ny}_{n-1\ \ times}\right), s\right), \\ &\qquad N'\left(\lambda\left(nx, -ny, \underbrace{0, \cdots, 0}_{n-2\ \ times}\right), ns\right)\Bigg\} \end{aligned} \tag{3.17}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. From (3.12), (3.17) and $(FNS4)$, we obtain

$$\begin{aligned} &N\left(f(n(x-y)) - f(nx) + f(ny), \frac{s+s+ns}{2} + s\right) \\ &\geq \min\Big\{N\left(f((n-1)(x-y)) + f(x-y) - f(nx) + f(ny), \frac{s+s+ns}{2}\right) \\ &\qquad N\left(f\left(n(x-y)\right) - f\left((n-1)(x-y)\right) - f\left(x-y\right), s\right)\Big\} \\ &\geq \min\Bigg\{N'\left(\lambda\left(ny, \underbrace{nx, nx\cdots, nx}_{n-1\ \ times}\right), s\right), \\ &\qquad N'\left(\lambda\left(nx, \underbrace{ny, ny\cdots, ny}_{n-1\ \ times}\right), s\right), \\ &\qquad N'\left(\lambda\left(nx, -ny, \underbrace{0, \cdots, 0}_{n-2\ \ times}\right), ns\right) \\ &\qquad N'\left(\lambda\left(n(x-y), \underbrace{0, 0, \cdots, 0}_{n-1\ \ times}\right), s\right)\Bigg\} \end{aligned} \tag{3.18}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Replacing (x,y) by $\left(\frac{x}{n}, \frac{-x}{n}\right)$ in

(3.18) and using oddness of f, we have

$$N\left(f(2x)-f(x)-f(x),\left[\frac{4+n}{2}\right]s\right) \geq \min\left\{N'\left(\lambda\left(ny,\underbrace{nx,nx\cdots,nx}_{n-1\ \ times}\right),s\right), N'\left(\lambda\left(nx,\underbrace{ny,ny\cdots,ny}_{n-1\ \ times}\right),s\right), N'\left(\lambda\left(nx,-ny,\underbrace{0,\cdots,0}_{n-2\ \ times}\right),ns\right) N'\left(\lambda\left(n(x-y),\underbrace{0,0,\cdots,0}_{n-1\ \ times}\right),s\right)\right\} \tag{3.19}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Define

$$a = \left[\frac{4+n}{2}\right] \tag{3.20}$$

$$N'(\Lambda_A(x,x,x,\cdots,x),s) = \min\left\{N'\left(\lambda\left(ny,\underbrace{nx,nx\cdots,nx}_{n-1\ \ times}\right),s\right), N'\left(\lambda\left(nx,\underbrace{ny,ny\cdots,ny}_{n-1\ \ times}\right),s\right), N'\left(\lambda\left(nx,-ny,\underbrace{0,\cdots,0}_{n-2\ \ times}\right),ns\right) N'\left(\lambda\left(n(x-y),\underbrace{0,0,\cdots,0}_{n-1\ \ times}\right),s\right)\right\} \tag{3.21}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Using (3.20) and (3.21) in (3.19), we arrive the inequality

$$N(f(2x)-2f(x),a\,s) \geq N'(\Lambda_A(x,x,x,\cdots,x),s) \tag{3.22}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. It follows from (3.22) and $(FNS3)$ that

$$N\left(\frac{f(2x)}{2}-f(x),\frac{a}{2}s\right) \geq N'(\Lambda_A(x,x,x,\cdots,x),s) \tag{3.23}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Replacing x by $2^q x$ in (3.23), we obtain

$$N\left(\frac{f(2^{q+1}x)}{2}-f(2^q x),\frac{a}{2}s\right) \geq N'(\Lambda_A(2^q x,2^q x,2^q x,\cdots,2^q x),s) \tag{3.24}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Using (3.2), $(FNS3)$ in (3.24), we arrive

$$N\left(\frac{f(2^{q+1}x)}{2}-f(2^q x),\frac{a}{2}s\right) \geq N'\left(\Lambda_A(x,x,x,\cdots,x),\frac{s}{t^q}\right) \tag{3.25}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. It is easy to verify from (3.25), that

$$N\left(\frac{f(2^{q+1}x)}{2^{(q+1)}}-\frac{f(2^q x)}{2^q},\frac{a}{2^{q+1}}s\right) \geq N'\left(\Lambda_A(x,x,x,\cdots,x),\frac{s}{t^q}\right) \tag{3.26}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Switching s by $t^q s$ in (3.26), we get

$$N\left(\frac{f(2^{q+1}x)}{2^{(q+1)}}-\frac{f(2^q x)}{2^q},\frac{a}{2}\cdot\left(\frac{t}{2}\right)^q s\right) \geq N'(\Lambda_A(x,x,x,\cdots,x),s) \tag{3.27}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. It is easy to see that

$$\frac{f(2^q x)}{2^q}-f(x) = \sum_{r=0}^{q-1}\left[\frac{f(2^{r+1}x)}{2^{(r+1)}}-\frac{f(2^r x)}{2^r}\right] \tag{3.28}$$

for all $x \in \mathscr{S}_1$. From equations (3.27) and (3.28), we have

$$\begin{aligned} N\left(\frac{f(2^q x)}{2^q}-f(x),\frac{a}{2}\cdot\sum_{r=0}^{q-1}\left(\frac{t}{2}\right)^r s\right) &\geq \min\bigcup_{r=0}^{q-1}\left\{N\left(\frac{f(2^{r+1}x)}{2^{(r+1)}}-\frac{f(2^r x)}{2^r},\frac{a}{2}\cdot\left(\frac{t}{2}\right)^r s\right)\right\} \\ &\geq \min\bigcup_{r=0}^{q-1}\left\{N'(\Lambda_A(x,x,x,\cdots,x),s)\right\} \\ &= N'(\Lambda_A(x,x,x,\cdots,x),s) \end{aligned} \tag{3.29}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Replacing x by $2^m x$ in (3.29) and using (3.2), $(FNS3)$, and substituting s by $t^m s$, we obtain

$$N\left(\frac{f(2^{q+m}x)}{2^{(q+m)}}-\frac{f(2^m x)}{2^m},\frac{a}{2}\cdot\sum_{r=m}^{q+m-1}\left(\frac{t}{2}\right)^r s\right) \geq N'(\Lambda_A(x,x,x,\cdots,x),s) \tag{3.30}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$ and all $m > q \geq 0$. Using $(FNS3)$ in (3.30), we obtain

$$N\left(\frac{f(2^{q+m}x)}{2^{16(q+m)}}-\frac{f(2^m x)}{2^{16m}},s\right) \geq N'\left(\Lambda_A(x,x,x,\cdots,x),\frac{s}{\frac{a}{2}\cdot\sum_{r=m}^{q+m-1}\left(\frac{t}{2}\right)^r}\right) \tag{3.31}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Since $0 < t < 2$ and $\sum_{r=0}^{q} \left(\frac{t}{2}\right)^r < \infty$, the Cauchy criterion for convergence and $(FNS5)$ implies that $\left\{\frac{f(2^q x)}{2^q}\right\}$ is a Cauchy sequence in $(\mathscr{S}_2, N')$. Since $(\mathscr{S}_2, N')$ is a fuzzy Banach space, this sequence converges to some point $\mathscr{A} \in \mathscr{S}_2$. So one can define the mapping $\mathscr{A} : \mathscr{S}_1 \longrightarrow \mathscr{S}_2$ by

$$\lim_{q\to\infty} N(\mathscr{A}(x) - \frac{f(2^s x)}{2^q}, s) = 1 \tag{3.32}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Letting $m = 0$ and $q \to \infty$in (3.31), we get

$$N(\mathscr{A}(x) - f(x), s) \geq N'\left(\Lambda_A(x,x,x,\cdots,x), \frac{s(2\ -t)}{a}\right)$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. To prove $\mathscr{A}$ satisfies the (1.5), replacing $(x_1,x_2,x_3,\cdots,x_n)$ by $(2^q x_1, 2^q x_2, 2^q x_3, \cdots, 2^q x_n)$ in (3.2), we obtain

$$\begin{aligned} &N(F_{AQ}(x_1,x_2,x_3,\cdots,x_n), s) \\ &= N\left(\frac{1}{2^q} F_{AQ}(2^q x_1, 2^q x_2, 2^q x_3, \cdots, 2^q x_n), s\right) \\ &\geq N'(\lambda(2^q x_1, 2^q x_2, 2^q x_3, \cdots, 2^q x_n), 2^q s) \end{aligned} \tag{3.33}$$

for all $x_1,x_2,x_3,\cdots,x_n \in \mathscr{S}_1$ and all $s > 0$. Now,

$$\begin{aligned} &N\left(\sum_{i=1}^{n} \mathscr{A}\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right) - \sum_{i=1}^{n} \mathscr{A}(x_i) + n\mathscr{A}\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right), s\right) \\ &\geq min\left\{N\left(\sum_{i=1}^{n} \mathscr{A}\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right) - \frac{1}{2^q} f\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right), \frac{s}{4}\right),\right. \\ &N\left(-\sum_{i=1}^{n} \mathscr{A}(x_i) + \frac{1}{2^q}\sum_{i=1}^{n} f(x_i), \frac{s}{4}\right), \\ &N\left(n\mathscr{A}\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right) - nf\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right), \frac{s}{4}\right), \\ &\left. N\left(\frac{1}{2^q} f\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right) - \frac{1}{2^q}\sum_{i=1}^{n} f(x_i) + nf\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right), \frac{s}{4}\right)\right\} \end{aligned} \tag{3.34}$$

for all $x_1,x_2,x_3,\cdots,x_n \in \mathscr{S}_1$ and all $s > 0$. Using (3.32), (3.33), $(FNS5)$ in and (3.34), we reach

$$\begin{aligned} &N\left(\sum_{i=1}^{n} \mathscr{A}\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right) - \sum_{i=1}^{n} \mathscr{A}(x_i) + n\mathscr{A}\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right), s\right) \\ &\geq \min\left\{1,1,1,N'(\lambda(2^q x_1, 2^q x_2, 2^q x_3, \cdots, 2^q x_n), 2^q s)\right\} \end{aligned} \tag{3.35}$$

for all $x_1,x_2,x_3,\cdots,x_n \in \mathscr{S}_1$ and all $s > 0$. Approaching q tends to infinity in (3.35) and applying (3.2), we get

$$N\left(\sum_{i=1}^{n} \mathscr{A}\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right) - \sum_{i=1}^{n} \mathscr{A}(x_i) + n\mathscr{A}\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right), s\right) = 1 \tag{3.36}$$

for all $x_1,x_2,x_3,\cdots,x_n \in \mathscr{S}_1$ and all $s > 0$. Using $(FNS2)$ in (3.36), it gives

$$\sum_{i=1}^{n} \mathscr{A}\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right) = \sum_{i=1}^{n} \mathscr{A}(x_i) - n\mathscr{A}\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right)$$

for all $x_1,x_2,x_3,\cdots,x_n \in \mathscr{S}_1$. Hence $\mathscr{A}$ satisfies the Additive functional equation (1.5). The existence of $\mathscr{A}(x)$ is unique. Indeed, if $\mathscr{A}'(x)$ be another Additive functional equation satisfying (1.5) and (3.7). So,

$$\begin{aligned} &N(\mathscr{A}(x) - \mathscr{A}'(x), s) \\ &= N\left(\frac{\mathscr{A}(2^q x)}{2^q} - \frac{\mathscr{A}'(2^q x)}{2^q}, s\right) \\ &\geq \min\left\{N\left(\frac{\mathscr{A}(2^q x)}{2^q} - \frac{f(2^q x)}{2^q}, \frac{s}{2}\right), N\left(\frac{\mathscr{A}'(2^q x)}{2^q} - \frac{f(2^q x)}{2^q}, \frac{s}{2}\right)\right\} \\ &\geq N'\left(\Lambda_A(2^q x, 2^q x, 2^q x, \cdots, 2^q x), \frac{s(2-t)2^q}{2\ a}\right) \\ &= N'\left(\Lambda_A(x,x,x,\cdots,x), \frac{s(2-t)2^q}{2\ t^q\ a}\right) \end{aligned}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Since

$$\lim_{q\to\infty} \frac{s(2-t)2^q}{2\ t^q\ a} = \infty,$$

we obtain

$$\lim_{q\to\infty} N'\left(\Lambda_A(x,x,x,\cdots,x), \frac{s(2^{16}-t)2^q}{2\ t^q\ a}\right) = 1$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Thus

$$N(\mathscr{A}(x) - \mathscr{A}'(x), s) = 1$$

for all $x \in \mathscr{S}_1$ and all $s>0$, hence $\mathscr{A}(x)=\mathscr{A}'(x)$. Therefore $\mathscr{A}(x)-\mathscr{A}'(x)$ is unique. Hence for $p=1$ the theorem holds.

Replacing x by $\frac{x}{2}$ in (3.22), we arrive

$$N\left(f(x)-2f\left(\frac{x}{2}\right),a\,s\right) \geq N'\left(\Lambda_A\left(\frac{x}{2},\frac{x}{2},\frac{x}{2},\cdots,\frac{x}{2}\right),s\right) \tag{3.37}$$

for all $x \in \mathscr{S}_1$ and all $s>0$. The rest of the proof is similar ideas to that of case $p=1$ Hence the theorem holds for the case $p=-1$. This completes the proof of the theorem. □

The following corollary is the immediate consequence of Theorem 3.1 concerning the stabilities of (1.5).

Corollary 3.2. *Let $f:\mathscr{S}_1 \longrightarrow \mathscr{S}_2$ be an odd mapping. If there exist real numbers d and b such that*

$$N\left(F_{AQ}(x_1,x_2,x_3,\cdots,x_n),s\right) \geq \begin{cases} N(d,s) \\ N\left(d\sum_{i=1}^n ||x_i||^b,s\right), & b\neq 1; \\ N\left(d\prod_{i=1}^n ||x_i||^b,s\right), & nb\neq 1; \\ N\left(d\sum_{i=1}^n ||x_i||^{b_i},s\right), & b_i\neq 1; \\ N\left(d\prod_{i=1}^n ||x_i||^{b_i},s\right), & \sum_{i=1}^n b_i\neq 1; \end{cases} \tag{3.38}$$

for all $x_1,x_2,x_3,\cdots,x_n \in \mathscr{S}_1$ and all $s>0$, then there exists a unique Additive mapping $\mathscr{A}:\mathscr{S}_1 \longrightarrow \mathscr{S}_2$ such that

$$N\left(f(x)-\mathscr{A}(x),s\right) \geq \begin{cases} N'\left(d,\frac{s|2-1|}{2\,a}\right), \\ N'\left(nd||x||^b,\frac{s|2-2^b|}{2\,a}\right), \\ N'\left(||x||^{nb},\frac{s|2-2^{nb}|}{2\,a}\right), \\ N'\left(\sum_{i=1}^n d||x||^{b_i},\sum_{i=1}^n\frac{s|2-2^{b_i}|}{2\,a}\right), \\ N'\left(d||x||^{\sum_{i=1}^n b_i},\frac{s|2-2^{\sum_{i=1}^n b_i}|}{2\,a}\right), \end{cases} \tag{3.39}$$

for all $x \in \mathscr{S}_1$ and all $s>0$.

Theorem 3.3. *Let $p=\pm1$ and $\lambda,\Lambda_Q:\mathscr{S}_1^2 \longrightarrow \mathscr{S}_3$ be a function such that*

$$\lim_{q\to\infty} N'\left(\lambda\left(2^{pq}x_1,2^{pq}x_2,2^{pq}x_3,\cdots,2^{pq}x_n\right),4^{pq}s\right)=1 \tag{3.40}$$

for all $x_1,x_2,x_3,\cdots,x_n \in \mathscr{S}_1$ and all $s>0$, for some $t>0$ with $0<\left(\frac{t}{2}\right)^p<1$ and

$$N'\left(\Lambda(2^px,2^px,2^px,\cdots,2^px),s\right) \geq N'\left(t^p\Lambda(2^px,2^px,2^px,\cdots,2^px),s\right) \tag{3.41}$$

for all $x \in \mathscr{S}_1$ and all $s>0$. Let $f:\mathscr{S}_1 \longrightarrow \mathscr{S}_2$ be an even mapping fulfilling the inequality

$$N\left(F_{AQ}(x_1,x_2,x_3,\cdots,x_n),s\right) \geq N'\left(\lambda\left(x_1,x_2,x_3,\cdots,x_n\right),s\right) \tag{3.42}$$

for all $x_1,x_2,x_3,\cdots,x_n \in \mathscr{S}_1$ and all $s>0$. Then there exists a unique quadratic mapping $\mathscr{Q}:\mathscr{S}_1 \longrightarrow \mathscr{S}_2$ which satisfies (1.5) and

$$N\left(f(x)-\mathscr{Q}(x),s\right) \geq N'\left(\Lambda_Q\left(x,x,\cdots,x\right),\frac{s|4-t|}{e}\right) \tag{3.43}$$

where e, $\Lambda_Q(x,x,\cdots,x)$ and $\mathscr{Q}(x)$ are defined by

$$e=\left[\frac{(2n+7)}{(2n-2)}\right] \tag{3.44}$$

$$\begin{aligned} &N'\left(\Lambda_Q(x,x,x,\cdots,x),s\right) \\ &=\min\Bigg\{N'\Bigg(\lambda\Bigg(nx,nx,\underbrace{0,\cdots,0}_{n-2\ \ times}\Bigg),s\Bigg), \\ &N'\Bigg(\lambda\Bigg(0,\underbrace{nx,nx\cdots,nx}_{n-1\ \ times}\Bigg),s\Bigg), \\ &N'\Bigg(\lambda\Bigg(nx,\underbrace{0,0\cdots,0}_{n-1\ \ times}\Bigg),s\Bigg), \\ &N'\Bigg(\lambda\Bigg(nx,\underbrace{0,0,\cdots,0}_{n-1\ \ times}\Bigg),ns\Bigg), \\ &N'\Bigg(\lambda\Bigg(nx,\underbrace{0,0,\cdots,0}_{n-1\ \ times}\Bigg),s\Bigg), \\ &N'\Bigg(\lambda\Bigg(0,\underbrace{nx,nx\cdots,nx}_{n-1\ \ times}\Bigg),s\Bigg), \\ &N'\Bigg(\lambda\Bigg(nx,\underbrace{0,0\cdots,0}_{n-1\ \ times}\Bigg),s\Bigg), \\ &N'\Bigg(\lambda\Bigg(nx,\underbrace{0,0,\cdots,0}_{n-1\ \ times}\Bigg),ns\Bigg), \\ &N'\Bigg(\lambda\Bigg(x,(n-1)x,\underbrace{0,0,\cdots,0}_{n-2\ \ times}\Bigg),s\Bigg)\Bigg\} \end{aligned} \tag{3.45}$$

and

$$\lim_{q\to\infty} N\left(\mathscr{Q}(x)-\frac{f(2^{pq}x)}{4^{pq}},s\right)=1 \tag{3.46}$$

for all $x \in \mathscr{S}_1$ and all $s>0$, respectively.

Proof. Setting $(x_1,x_2,x_3,\cdots,x_n)$ by $(nx,-ny,\underbrace{0,\cdots,0}_{n-2\ \ times})$ in (3.42), we get

$$N\left(f\left(nx-\frac{1}{n}(nx-ny)\right)+f\left(-ny-\frac{1}{n}(nx-ny)\right)+(n-2)f(-(x-y))-f(nx)-f(-ny)+nf(x-y),s\right)$$
$$\geq N'\left(\lambda\left(nx,-ny,\underbrace{0,\cdots,0}_{n-2\ \ times}\right),s\right) \tag{3.47}$$

for all $x\in\mathscr{S}_1$ and all $s>0$. Using evenness of f in the above inequality, we obtain

$$N(f((n-1)x+y)+f(x+(n-1)y)-f(nx)-f(ny)+(2n-2)f(x-y),s)$$
$$\geq N'\left(\lambda\left(nx,-ny,\underbrace{0,\cdots,0}_{n-2\ \ times}\right),s\right) \tag{3.48}$$

for all $x\in\mathscr{S}_1$ and all $s>0$. Substitute y by 0 in (3.48), we arrive

$$N(f(nx)-f((n-1)x)-(2n-1)f(x),s)$$
$$\geq N'\left(\lambda\left(nx,\underbrace{0,0,\cdots,0}_{n-1\ \ times}\right),s\right) \tag{3.49}$$

for all $x\in\mathscr{S}_1$ and all $s>0$. Again substitute (x,y) by $\left(\frac{x}{n},(1-n)x\right)$ in (3.48), we arrive

$$N(f((n-1)x)-f((n-2)x)-(2n-3)f(x),s)$$
$$\geq N'\left(\lambda\left(x,(n-1)x,\underbrace{0,0,\cdots,0}_{n-2\ \ times}\right),s\right) \tag{3.50}$$

for all $x\in\mathscr{S}_1$ and all $s>0$. Putting $(x_1,x_2,x_3,\cdots,x_n)$ by $(nx,\underbrace{ny,ny\cdots,ny}_{n-1\ \ times})$ in (3.42) and using evenness of f, we get

$$N(f((n-1)(x-y))+(n-1)f(x-y)-(n-1)f(ny)-f(nx)+nf(x+(n-1)y),s)$$
$$\geq N'\left(\lambda\left(nx,\underbrace{ny,ny\cdots,ny}_{n-1\ \ times}\right),s\right) \tag{3.51}$$

for all $x\in\mathscr{S}_1$ and all $s>0$. Interchanging x and y in the above inequality and using evenness of f, we have

$$N(f((n-1)(x-y))+(n-1)f(x-y)-(n-1)f(nx)-f(ny)+nf((n-1)x+y),s)$$
$$\geq N'\left(\lambda\left(ny,\underbrace{nx,nx\cdots,nx}_{n-1\ \ times}\right),s\right) \tag{3.52}$$

for all $x\in\mathscr{S}_1$ and all $s>0$. It follows from (3.48), (3.51), (3.52) and $(FNS4)$, we arrive

$$N\left(2f((n-1)(x-y))-2(n-1)^2f(x-y),s+s+ns\right)$$
$$\geq \min\Big\{N(f((n-1)(x-y))+(n-1)f(x-y)-(n-1)f(nx)-f(ny)+nf((n-1)x+y),s),$$
$$N(f((n-1)(x-y))+(n-1)f(x-y)-(n-1)f(ny)-f(nx)+nf(x+(n-1)y),s),$$
$$N(f((n-1)x+y)+f(x+(n-1)y)-f(nx)-f(ny)+(2n-2)f(x-y),ns)\Big\}$$
$$\geq \min\left\{N'\left(\lambda\left(ny,\underbrace{nx,nx\cdots,nx}_{n-1\ \ times}\right),s\right),\ N'\left(\lambda\left(nx,\underbrace{ny,ny\cdots,ny}_{n-1\ \ times}\right),s\right),\ N'\left(\lambda\left(nx,-ny,\underbrace{0,\cdots,0}_{n-2\ \ times}\right),ns\right)\right\} \tag{3.53}$$

for all $x\in\mathscr{S}_1$ and all $s>0$. Replace y by 0 in (3.53) and using $(FNS3)$, we get

$$N\left(f((n-1)(x)-(n-1)^2f(x),\frac{s+s+ns}{2}\right)$$
$$\geq \min\left\{N'\left(\lambda\left(ny,\underbrace{nx,nx\cdots,nx}_{n-1\ \ times}\right),s\right),\ N'\left(\lambda\left(nx,\underbrace{ny,ny\cdots,ny}_{n-1\ \ times}\right),s\right),\ N'\left(\lambda\left(nx,-ny,\underbrace{0,\cdots,0}_{n-2\ \ times}\right),ns\right)\right\} \tag{3.54}$$

for all $x\in\mathscr{S}_1$ and all $s>0$. From (3.49) and (3.54), we obtain

$$N\left(f(nx)-n^2f(x),\frac{s+s+ns}{2}+s\right)$$
$$\geq \min\Big\{N(f(nx)-f((n-1)x)-(2n-1)f(x),s),.\ N\left(f((n-1)(x))-(n-1)^2f(x),\frac{s+s+ns}{2}\right)\Big\}$$

$$\geq \min\left\{N'\left(\lambda\left(0,\underbrace{nx,nx\cdots,nx}_{n-1\ \ times}\right),s\right),\right.$$
$$N'\left(\lambda\left(nx,\underbrace{0,0\cdots,0}_{n-1\ \ times}\right),s\right),$$
$$N'\left(\lambda\left(nx,\underbrace{0,0,\cdots,0}_{n-1\ \ times}\right),ns\right),$$
$$\left.N'\left(\lambda\left(nx,\underbrace{0,0,\cdots,0}_{n-1\ \ times}\right),s\right)\right\} \tag{3.55}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Also, From (3.50) and (3.54), we obtain

$$N\left(f((n-2)x)-(n-2)^2 f(x),\frac{s+s+ns}{2}+s\right)$$
$$\geq \min\left\{N\left(f((n-1)x)-f((n-2)x)-(2n-3)f(x),s\right),\right.$$
$$\left.N\left(f((n-1)(x))-(n-1)^2 f(x),\frac{s+s+ns}{2}\right)\right\}$$
$$\geq \min\left\{N'\left(\lambda\left(0,\underbrace{nx,nx\cdots,nx}_{n-1\ \ times}\right),s\right),\right.$$
$$N'\left(\lambda\left(nx,\underbrace{0,0\cdots,0}_{n-1\ \ times}\right),s\right),$$
$$N'\left(\lambda\left(nx,\underbrace{0,0,\cdots,0}_{n-1\ \ times}\right),ns\right),$$
$$\left.N'\left(\lambda\left(x,(n-1)x,\underbrace{0,0,\cdots,0}_{n-2\ \ times}\right),s\right)\right\} \tag{3.56}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Substitute y by $-x$ in (3.48), we have

$$N\left(2f\left((n-2)x\right)\right)-2f\left(nx\right)+(2n-2)f\left(2x\right),s)$$
$$\geq N'\left(\lambda\left(nx,nx,\underbrace{0,\cdots,0}_{n-2\ \ times}\right),s\right) \tag{3.57}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. It follows from (3.55), (3.56) and (3.57), we arrive

$$N((2n-2)f(2x)-4(2n-2)f(x),$$
$$s+s+s+ns+2s+s+s+ns+2s)$$
$$\geq \min\left\{N\left(2f\left((n-2)x\right)-2f\left(nx\right)+(2n-2)f\left(2x\right),s\right),\right.$$
$$N\left(2f(nx)-2n^2 f(x),s+s+ns+2s\right),$$
$$\left.N\left(2f((n-2)x)-2(n-2)^2 f(x),s+s+ns+2s\right)\right\}$$
$$\geq \min\left\{N'\left(\lambda\left(nx,nx,\underbrace{0,\cdots,0}_{n-2\ \ times}\right),s\right),\right.$$
$$N'\left(\lambda\left(0,\underbrace{nx,nx\cdots,nx}_{n-1\ \ times}\right),s\right),$$
$$N'\left(\lambda\left(nx,\underbrace{0,0\cdots,0}_{n-1\ \ times}\right),s\right),$$
$$N'\left(\lambda\left(nx,\underbrace{0,0,\cdots,0}_{n-1\ \ times}\right),ns\right),$$
$$N'\left(\lambda\left(nx,\underbrace{0,0,\cdots,0}_{n-1\ \ times}\right),s\right),$$
$$N'\left(\lambda\left(0,\underbrace{nx,nx\cdots,nx}_{n-1\ \ times}\right),s\right),$$
$$N'\left(\lambda\left(nx,\underbrace{0,0\cdots,0}_{n-1\ \ times}\right),s\right),$$
$$N'\left(\lambda\left(nx,\underbrace{0,0,\cdots,0}_{n-1\ \ times}\right),ns\right),$$
$$\left.N'\left(\lambda\left(x,(n-1)x,\underbrace{0,0,\cdots,0}_{n-2\ \ times}\right),s\right)\right\} \tag{3.58}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Define

$$N'\left(\Lambda_Q(x,x,x,\cdots,x),s\right)$$
$$= \min\left\{N'\left(\lambda\left(nx,nx,\underbrace{0,\cdots,0}_{n-2\ \ times}\right),s\right),\right.$$
$$N'\left(\lambda\left(0,\underbrace{nx,nx\cdots,nx}_{n-1\ \ times}\right),s\right),$$
$$N'\left(\lambda\left(nx,\underbrace{0,0\cdots,0}_{n-1\ \ times}\right),s\right),$$
$$N'\left(\lambda\left(nx,\underbrace{0,0,\cdots,0}_{n-1\ \ times}\right),ns\right),$$
$$N'\left(\lambda\left(nx,\underbrace{0,0,\cdots,0}_{n-1\ \ times}\right),s\right),$$
$$N'\left(\lambda\left(0,\underbrace{nx,nx\cdots,nx}_{n-1\ \ times}\right),s\right),$$

$$N'\left(\lambda\left(nx,\underbrace{0,0\cdots,0}_{n-1\ \ times}\right),s\right),$$
$$N'\left(\lambda\left(nx,\underbrace{0,0,\cdots,0}_{n-1\ \ times}\right),ns\right),$$
$$N'\left(\lambda\left(x,(n-1)x,\underbrace{0,0,\cdots,0}_{n-2\ \ times}\right),s\right)\Bigg\} \tag{3.59}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Using (3.59) in (3.58), we arrive the inequality

$$N\left((2n-2)f(2x)-4(2n-2)f(x),(2n+9)s\right) \geq N'\left(\Lambda_Q\left(x,x,x,\cdots,x\right),s\right) \tag{3.60}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. It follows from (3.60) and $(FNS3)$ that

$$N\left(f(2x)-4f(x),\frac{(2n+9)}{(2n-2)}s\right) \geq N'\left(\Lambda_Q\left(x,x,x,\cdots,x\right),s\right) \tag{3.61}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Again define

$$e=\frac{(2n+9)}{(2n-2)}. \tag{3.62}$$

Finally, it follows from (3.62) and (3.61)

$$N\left(f(2x)-4f(x),e\,s\right) \geq N'\left(\Lambda_Q\left(x,x,x,\cdots,x\right),s\right) \tag{3.63}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Using $(FNS3)$ in the above inequality, we arrive

$$N\left(\frac{f(2x)}{4}-f(x),\frac{e}{4}s\right) \geq N'\left(\Lambda_Q\left(x,x,x,\cdots,x\right),s\right) \tag{3.64}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Replacing x by $2^q x$ in (3.64), we obtain

$$N\left(\frac{f(2^{q+1}x)}{4}-f(2^q x),\frac{e}{4}s\right) \geq N'\left(\Lambda_Q\left(2^q x,2^q x,2^q x,\cdots,2^q x\right),s\right) \tag{3.65}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Using (3.41), $(FNS3)$ in (3.65), we arrive

$$N\left(\frac{f(2^{q+1}x)}{4}-f(2^q x),\frac{e}{4}s\right) \geq N'\left(\Lambda_Q\left(x,x,x,\cdots,x\right),\frac{s}{t^q}\right) \tag{3.66}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. It is easy to verify from (3.66), that

$$N\left(\frac{f(2^{q+1}x)}{4^{(q+1)}}-\frac{f(2^q x)}{4^q},\frac{e}{4^{q+1}}s\right) \geq N'\left(\Lambda_Q\left(x,x,x,\cdots,x\right),\frac{s}{t^q}\right) \tag{3.67}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Switching s by $t^q s$ in (3.67), we get

$$N\left(\frac{f(2^{q+1}x)}{4^{(q+1)}}-\frac{f(2^q x)}{4^q},\frac{e}{4}\cdot\left(\frac{t}{4}\right)^q s\right) \geq N'\left(\Lambda_Q\left(x,x,x,\cdots,x\right),s\right) \tag{3.68}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. It is easy to see that

$$\frac{f(2^q x)}{4^q}-f(x)=\sum_{r=0}^{q-1}\left[\frac{f(2^{r+1}x)}{4^{(r+1)}}-\frac{f(2^r x)}{4^r}\right] \tag{3.69}$$

for all $x \in \mathscr{S}_1$. From equations (3.68) and (3.69), we have

$$\begin{aligned} &N\left(\frac{f(2^q x)}{4^q}-f(x),\frac{e}{4}\cdot\sum_{r=0}^{q-1}\left(\frac{t}{4}\right)^r s\right)\\ &\geq \min\bigcup_{r=0}^{q-1}\left\{N\left(\frac{f(2^{r+1}x)}{4^{(r+1)}}-\frac{f(2^r x)}{4^r},\frac{e}{4}\cdot\left(\frac{t}{4}\right)^r s\right)\right\}\\ &\geq \min\bigcup_{r=0}^{q-1}\left\{N'\left(\Lambda_Q\left(x,x,x,\cdots,x\right),s\right)\right\}\\ &= N'\left(\Lambda_Q\left(x,x,x,\cdots,x\right),s\right) \end{aligned} \tag{3.70}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Replacing x by $2^m x$ in (3.70) and using (3.41), $(FNS3)$, and substituting s by $t^m s$, we obtain

$$N\left(\frac{f(2^{q+m}x)}{4^{(q+m)}}-\frac{f(2^m x)}{4^m},\frac{e}{4}\cdot\sum_{r=m}^{q+m-1}\left(\frac{t}{4}\right)^r s\right) \geq N'\left(\Lambda_Q\left(x,x,x,\cdots,x\right),s\right) \tag{3.71}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$ and all $m > q \geq 0$. Using $(FNS3)$ in (3.71), we obtain

$$N\left(\frac{f(2^{q+m}x)}{4^{16(q+m)}}-\frac{f(2^m x)}{4^{16m}},s\right) \geq N'\left(\Lambda_Q\left(x,x,x,\cdots,x\right),\frac{s}{\frac{e}{4}\cdot\sum_{r=m}^{q+m-1}\left(\frac{t}{4}\right)^r}\right) \tag{3.72}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Since $0 < t < 2$ and $\sum_{r=0}^{q}\left(\frac{t}{4}\right)^r < \infty$, the Cauchy criterion for convergence and $(FNS5)$ implies that $\left\{\frac{f(2^q x)}{4^q}\right\}$ is a Cauchy sequence in $(\mathscr{S}_2,N')$. Since $(\mathscr{S}_2,N')$ is a fuzzy Banach space, this sequence converges to some point $\mathscr{Q} \in \mathscr{S}_2$. So one can define the mapping $\mathscr{Q}:\mathscr{S}_1 \longrightarrow \mathscr{S}_2$ by

$$\lim_{q\to\infty} N(\mathscr{Q}(x)-\frac{f(2^s x)}{4^q},s)=1 \tag{3.73}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Letting $m = 0$ and $q \to \infty$in (3.72), we get

$$N(\mathscr{Q}(x) - f(x), s) \geq N'\left(\Lambda_Q(x,x,x,\cdots,x), \frac{s(4-t)}{e}\right)$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. The rest of the proof is similar ideas to that of Theorem 3.1. □

The following corollary is the immediate consequence of Theorem 3.3 concerning the stabilities of (1.5).

Corollary 3.4. *Let $f : \mathscr{S}_1 \longrightarrow \mathscr{S}_2$ be an even mapping. If there exist real numbers d and b such that*

$$N(F_{AQ}(x_1,x_2,x_3,\cdots,x_n),s) \geq \begin{cases} N(d,s) \\ N\left(d\sum_{i=1}^n ||x_i||^b, s\right), & b \neq 2; \\ N\left(d\prod_{i=1}^n ||x_i||^b, s\right), & nb \neq 2; \\ N\left(d\sum_{i=1}^n ||x_i||^{b_i}, s\right), & b_i \neq 2; \\ N\left(d\prod_{i=1}^n ||x_i||^{b_i}, s\right), & \sum_{i=1}^n b_i \neq 2; \end{cases} \tag{3.74}$$

for all $x_1,x_2,x_3,\cdots,x_n \in \mathscr{S}_1$ and all $s > 0$, then there exists a unique quadratic mapping $\mathscr{Q} : \mathscr{S}_1 \longrightarrow \mathscr{S}_2$ such that

$$N(f(x) - \mathscr{Q}(x), s) \geq \begin{cases} N'\left(d, \frac{s|4-1|}{4\ e}\right), \\ N'\left(nd||x||^b, \frac{s|4-2^b|}{4\ e}\right), \\ N'\left(||x||^{nb}, \frac{s|4-2^{nb}|}{4\ e}\right), \\ N'\left(\sum_{i=1}^n d||x||^{b_i}, \sum_{i=1}^n \frac{s|4-2^{b_i}|}{4\ e}\right), \\ N'\left(d||x||^{\sum_{i=1}^n b_i}, \frac{s|4-2^{\sum_{i=1}^n b_i}|}{4\ e}\right), \end{cases} \tag{3.75}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$.

Theorem 3.5. *Let $p = \pm 1$ and $\lambda : \mathscr{S}_1^2 \longrightarrow \mathscr{S}_3$ be a function satisfying the conditions (3.1) and (3.40) for all $x_1,x_2,x_3,\cdots,x_n \in \mathscr{S}_1$, for some $t > 0$ with (3.2) and (3.41) for all $x \in \mathscr{S}_1$ and all $s > 0$. Let $f : \mathscr{S}_1 \longrightarrow \mathscr{S}_2$ be a mapping fulfilling the inequality*

$$N(F_{AQ}(x_1,x_2,x_3,\cdots,x_n),s) \geq N'(\lambda(x_1,x_2,x_3,\cdots,x_n), s) \tag{3.76}$$

for all $x_1,x_2,x_3,\cdots,x_n \in \mathscr{S}_1$ and all $s > 0$. Then there exists a unique Additive mapping $\mathscr{A} : \mathscr{S}_1 \longrightarrow \mathscr{S}_2$ and a unique quadratic mapping $\mathscr{Q} : \mathscr{S}_1 \longrightarrow \mathscr{S}_2$ which satisfies (1.5) and

$$\begin{aligned} &N(f(x) - \mathscr{A}(x) - \mathscr{Q}(x), s) \\ &\geq \min\left\{N'\left(\Lambda_A(x,x,\cdots,x), \frac{s|2-t|}{2a}\right),\right. \\ &\qquad N'\left(\Lambda_A(-x,-x,\cdots,-x), \frac{s|2-t|}{2a}\right), \\ &\qquad N'\left(\Lambda_Q(x,x,\cdots,x), \frac{s|4-t|}{2e}\right), \\ &\qquad \left. N'\left(\Lambda_Q(-x,-x,\cdots,-x), \frac{s|4-t|}{2e}\right)\right\} \end{aligned} \tag{3.77}$$

where a, e, $\Lambda_A(x,x,\cdots,x), \Lambda_Q(x,x,\cdots,x)$ and $\mathscr{A}(x), \mathscr{Q}(x)$ are respectively defined in (3.5),(3.44), (3.6),(3.45) and (3.7),(3.46) for all $x \in \mathscr{S}_1$ and all $s > 0$.

Proof. Let $f_O(x) = \frac{f(x)-f(-x)}{2}$ for all $x \in \mathscr{S}_1$. It is easy to verify that $f_O(0) = 0$ and $f_O(-x) = -f_O(x)$ for all $x \in \mathscr{S}_1$. By definition of $f_O(x)$, we have

$$\begin{aligned} &N(f_O(x_1,x_2,x_3,\cdots,x_n),s) \\ &= N\left(\frac{1}{2}(f(x_1,x_2,x_3,\cdots,x_n)\right. \\ &\qquad \left. -f(-x_1,-x_2,-x_3,\cdots,-x_n)), s\right) \\ &= N(f(x_1,x_2,x_3,\cdots,x_n) - f(-x_1,-x_2,-x_3,\cdots,-x_n), 2s) \\ &\geq \min\{N(f(x_1,x_2,x_3,\cdots,x_n),s), \\ &\qquad N(f(-x_1,-x_2,-x_3,\cdots,-x_n),s)\} \end{aligned} \tag{3.78}$$

for all $x_1,x_2,x_3,\cdots,x_n \in \mathscr{S}_1$ and all $s > 0$. Hence, by Theorem 3.1,

$$\begin{aligned} &N(f_O(x) - \mathscr{A}(x), s) \\ &\geq \min\left\{N'\left(\Lambda_A(x,x,\cdots,x), \frac{s|2-t|}{a}\right),\right. \\ &\qquad \left. N'\left(\Lambda_A(-x,-x,\cdots,-x), \frac{s|2-t|}{a}\right)\right\} \end{aligned} \tag{3.79}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$.
Also, let $f_E(x) = \frac{f(x)+f(-x)}{2}$ for all $x \in \mathscr{S}_1$. It is easy to verify that $f_E(0) = 0$ and $f_E(-x) = f_E(x)$ for all $x \in \mathscr{S}_1$. By definition of $f_E(x)$, we have

$$\begin{aligned} &N(f_E(x_1,x_2,x_3,\cdots,x_n),s) \\ &= N\left(\frac{1}{2}(f(x_1,x_2,x_3,\cdots,x_n)\right. \\ &\qquad \left. +f(-x_1,-x_2,-x_3,\cdots,-x_n)), s\right) \\ &= N(f(x_1,x_2,x_3,\cdots,x_n) \\ &\qquad +f(-x_1,-x_2,-x_3,\cdots,-x_n), 2s) \\ &\geq \min\{N(f(x_1,x_2,x_3,\cdots,x_n),s), \\ &\qquad N(f(-x_1,-x_2,-x_3,\cdots,-x_n),s)\} \end{aligned} \tag{3.80}$$

for all $x_1, x_2, x_3, \cdots, x_n \in \mathscr{S}_1$ and all $s > 0$. Hence, by Theorem 3.3,

$$N(f_E(x) - \mathscr{Q}(x), s) \geq \min\left\{N'\left(\Lambda_Q(x, x, \cdots, x), \frac{s|4-t|}{e}\right), N'\left(\Lambda_Q(-x, -x, \cdots, -x), \frac{s|4-t|}{e}\right)\right\} \quad (3.81)$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Define

$$f(x) = f_O(x) + f_E(x) \quad (3.82)$$

for all $x \in \mathscr{S}_1$. Using (3.79), (3.81) in (3.82), we arrive

$$\begin{aligned} N(f(x) - \mathscr{A}(x) - \mathscr{Q}(x), 2s) &= N(f_O(x) + f_E(x) - \mathscr{A}(x) - \mathscr{Q}(x), 2s) \\ &\geq \min\{N(f_O(x) - \mathscr{A}(x), s), N(f_E(x) - \mathscr{Q}(x), s)\} \\ &\geq \min\left\{N'\left(\Lambda_A(x, x, \cdots, x), \frac{s|2-t|}{2a}\right), N'\left(\Lambda_A(-x, -x, \cdots, -x), \frac{s|2-t|}{2a}\right),\right. \\ &\qquad N'\left(\Lambda_Q(x, x, \cdots, x), \frac{s|4-t|}{2e}\right), \left. N'\left(\Lambda_Q(-x, -x, \cdots, -x), \frac{s|4-t|}{2e}\right)\right\} \end{aligned}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. □

The following corollary is the immediate consequence of Theorem 3.5 concerning the stabilities of (1.5).

Corollary 3.6. *Let $f : \mathscr{S}_1 \longrightarrow \mathscr{S}_2$ be a mapping. If there exist real numbers d and b such that*

$$N(F_{AQ}(x_1, x_2, x_3, \cdots, x_n), s) \geq \begin{cases} N(d, s) \\ N\left(d\sum_{i=1}^{n}||x_i||^b, s\right), & b \neq 1, 2; \\ N\left(d\prod_{i=1}^{n}||x_i||^b, s\right), & nb \neq 1, 2; \\ N\left(d\sum_{i=1}^{n}||x_i||^{b_i}, s\right), & b_i \neq 1, 2; \\ N\left(d\prod_{i=1}^{n}||x_i||^{b_i}, s\right), & \sum_{i=1}^{n} b_i \neq 1, 2; \end{cases} \quad (3.83)$$

for all $x_1, x_2, x_3, \cdots, x_n \in \mathscr{S}_1$ and all $s > 0$, then there exists a unique Additive mapping $\mathscr{A} : \mathscr{S}_1 \longrightarrow \mathscr{S}_2$ and a unique quadratic mapping $\mathscr{Q} : \mathscr{S}_1 \longrightarrow \mathscr{S}_2$ such that

$$N(f(x) - \mathscr{A}(x) - \mathscr{Q}(x), s) \geq \begin{cases} \min\left\{N'\left(d, \frac{s|2-1|}{2\,a}\right), N'\left(d, \frac{s|4-1|}{4\,e}\right), \right\} \\ \min\left\{N'\left(nd||x||^b, \frac{s|2-2^b|}{2\,a}\right), N'\left(nd||x||^b, \frac{s|4-2^b|}{4\,e}\right)\right\} \\ \min\left\{N'\left(||x||^{nb}, \frac{s|2-2^{nb}|}{2\,a}\right), N'\left(||x||^{nb}, \frac{s|4-2^{nb}|}{4\,e}\right)\right\} \\ \min\left\{N'\left(\sum_{i=1}^{n} d||x||^{b_i}, \sum_{i=1}^{n}\frac{s|2-2^{b_i}|}{2\,a}\right), N'\left(\sum_{i=1}^{n} d||x||^{b_i}, \sum_{i=1}^{n}\frac{s|4-2^{b_i}|}{4\,e}\right)\right\} \\ \min\left\{N'\left(d||x||^{\sum_{i=1}^{n} b_i}, \frac{s|2-2^{\sum_{i=1}^{n} b_i}|}{2\,a}\right), N'\left(d||x||^{\sum_{i=1}^{n} b_i}, \frac{s|4-2^{\sum_{i=1}^{n} b_i}|}{4\,e}\right)\right\} \end{cases} \quad (3.84)$$

for all $x \in \mathscr{S}_1$ and all $s > 0$.

4. Fuzzy Stability Results: Fixed Point Method

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.5) in Fuzzy normed space using fixed point method. Now, we will recall the fundamental results in fixed point theory.

Theorem 4.1. *(Banach's contraction principle) Let (X, d) be a complete metric space and consider a mapping $T : X \longrightarrow X$ which is strictly contractive mapping, that is*
$(A_1)(Tx, Ty) \leq Ld(x, y)$ for some (Lipschitz constant) $L < 1$. Then,
(i) The mapping T has one and only fixed point $x^* = T(x^*)$;
(ii)The fixed point for each given element x^* is globally attractive, that is
$(A_2) lim_{n\to\infty} T^n x = x^*$, for any starting point $x \in X$;
(iii) One has the following estimation inequalities:
$(A_3) d(T^n x, x^*) \leq \frac{1}{1-L}\ d(T^n x, T^{n+1} x), \forall\ n \geq 0, \forall\ x \in X$;
$(A_4) d(x, x^*) \leq \frac{1}{1-L}\ d(x, x^*), \forall\ x \in X$.

Theorem 4.2. [40] *(The alternative of fixed point) Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \longrightarrow X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*
$(F_1) d(T^n x, T^{n+1} x) = \infty \quad \forall\ n \geq 0$,
or
(F_2) there exists a natural number n_0 such that:
$(FPC1)$ $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;
$(FPC2)$The sequence $(T^n x)$ is convergent to a fixed point y^* of T
$(FPC3)$ y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^{n_0} x, y) < \infty\}$;
$(FPC4)$ $d(y^*, y) \leq \frac{1}{1-L}\ d(y, Ty)$ for all $y \in Y$.

Using 4.2, we prove the stability results of functional equation (1.5).

Theorem 4.3. *Let $f:\mathscr{S}_1\to\mathscr{S}_2$ be an odd mapping for which there exist a mapping $\lambda:\mathscr{S}_1^2\longrightarrow\mathscr{S}_3$ with the condition*

$$\lim_{q\to\infty} N'\left(\lambda\left(C_c^q x_1, C_c^q x_2, C_c^q x_n\right), C_c^q s\right)=1 \tag{4.1}$$

for all $x\in\mathscr{S}_1$ and all $s>0$ where

$$C_c=\begin{cases} 2 & if \quad c=0,\\ \frac{1}{2} & if \quad c=1\end{cases} \tag{4.2}$$

and satisfying the functional inequality

$$N\left(F_{AQ}(x_1,x_2,x_3,\cdots,x_n),s\right)\geq N'\left(\lambda\left(x_1,x_2,x_3,\cdots,x_n\right),s\right) \tag{4.3}$$

for all $x_1,x_2,x_3,\cdots,x_n\in\mathscr{S}_1$ and all $s>0$. If there exists $L=L(c)$ such that the function

$$\Lambda_{A_F}(x,x,x,\cdots,x)=\Lambda_A\left(\frac{x}{2},\frac{x}{2},\frac{x}{2},\cdots,\frac{x}{2}\right), \tag{4.4}$$

where $\Lambda_A(x,x,x,\cdots,x)$ is defined in (3.6) with the property

$$N'\left(\frac{1}{C_c}\Lambda_{A_F}(C_cx,C_cx,C_cx,\cdots,C_cx),s\right) = N'\left(\Lambda_{A_F}(x,x,x,\cdots,x),L\,s\right), \tag{4.5}$$

for all $x\in\mathscr{S}_1$ and all $s>0$. Then there exists a unique additive mapping $\mathscr{A}:\mathscr{S}_1\longrightarrow\mathscr{S}_2$ satisfying the functional equation (1.5) and

$$N\left(f(x)-\mathscr{A}(x),s\right)\geq N'\left(\Lambda_{A_F}(x,x,x,\cdots,x),\left[\frac{L^{1-c}}{1-L}\right]\frac{s}{a}\right), \tag{4.6}$$

for all $x\in\mathscr{S}_1$ and all $s>0$.

Proof. Consider the set

$$\mathscr{C}=\{f_1|\,f_1:\mathscr{S}_1\longrightarrow\mathscr{S}_2,\ f_1(0)=0\}$$

and introduce the generalized metric on $\mathscr{C}$ as follows:

$$\begin{aligned} d(f_1,f_2)=\inf\{\rho\in(0,\infty):N\left(f_1(x)-f_2(x),s\right)\\ \geq N'\left(\Lambda_{A_F}(x,x,x,\cdots,x,\rho s)\right),x\in\mathscr{S}_1,s>0\}. \end{aligned} \tag{4.7}$$

It is easy to see that (4.7) is complete with respect to the defined metric. Define $J:\mathscr{C}\longrightarrow\mathscr{C}$ by

$$Jf(x)=\frac{1}{C_c}f(C_cx),$$

for all $x\in\mathscr{S}_1$. Now, from (4.7), $f_1,f_2\in\mathscr{C}$ and $x\in\mathscr{S}_1, s>0$, we arrive

$$\begin{aligned} & d(f_1,f_2)\leq\rho\\ \Rightarrow & N\left(f_1(x)-f_2(x),s\right)\geq N'\left(\Lambda_{A_F}(x,x,x,\cdots,x),\rho s\right)\\ \Rightarrow & N\left(\frac{1}{C_c}f_1(C_cx)-\frac{1}{C_c}f_2(C_ix),s\right)\\ & \qquad\geq N'\left(\Lambda_{A_F}(C_cx,C_cx,C_cx,\cdots,C_cx),C_c\,\rho\,s\right),\\ \Rightarrow & N\left(Jf_1(x)-Jf_2(x),s\right)\geq N'\left(\Lambda_{A_F}(x,x,x,\cdots,x),L\rho s\right),\\ \Rightarrow & d(f_1,f_2)\leq L\,\rho. \end{aligned}$$

This implies J is a strictly contractive mapping on $\mathscr{C}$ with Lipschitz constant L. It follows from (3.23), we reach

$$N\left(\frac{f(2x)}{2}-f(x),\frac{a}{2}s\right)\geq N'\left(\Lambda_A\left(x,x,x,\cdots,x\right),s\right) \tag{4.8}$$

for all $x\in\mathscr{S}_1$ and all $s>0$. It follows from (4.7) and (4.5) for the case $c=0$, we reach

$$N\left(Jf(x)-f(x),as\right)\geq N'\left(\Lambda_A\left(x,x,x,\cdots,x\right),Ls\right) \tag{4.9}$$

for all $x\in\mathscr{S}_1$ and all $s>0$. Again replacing $x=\frac{x}{2}$ in (4.8) , we get

$$N\left(f(x)-2f\left(\frac{x}{2}\right),as\right)\geq N'\left(\Lambda_A\left(\frac{x}{2},\frac{x}{2},\frac{x}{2},\cdots,\frac{x}{2}\right),s\right) \tag{4.10}$$

for all $x\in\mathscr{S}_1$ and all $s>0$. It follows from (4.7) and (4.5) for the case $c=1$, we reach

$$N\left(f(x)-Jf(x),as\right)\geq N'\left(\Lambda_A\left(x,x,x,\cdots,x\right),s\right) \tag{4.11}$$

for all $x\in\mathscr{S}_1$ and all $s>0$. Combining (4.9) and (4.11), we arrive

$$N\left(f(x)-Jf(x),as\right)\geq N'\left(\Lambda_A\left(x,x,x,\cdots,x\right),L^{1-c}\,s\right) \tag{4.12}$$

for all $x\in\mathscr{S}_1$ and all $s>0$. Hence property (FPC1) holds. It follows from property (FPC2) that there exists a fixed point $\mathscr{A}$ of J in $\mathscr{C}$ such that

$$\mathscr{A}(x)=\lim_{q\to\infty}\frac{1}{C_c{}^q}f(C_c^q x) \tag{4.13}$$

for all $x\in\mathscr{S}_1$. In order to show that $\mathscr{A}$ satisfies (1.5) the proof is similar clues to of Theorem 3.1. By property (FPC3), $\mathscr{A}$ is the unique fixed point of J in the set

$$\mathscr{D}=\{\mathscr{A}\in\mathscr{C}:d(f,\mathscr{A})<\infty\},$$

such that

$$N\left(f(x)-\mathscr{A}(x),a\,s\right)\geq N'\left(\Lambda_A\left(x,x,x,\cdots,x\right),L^{1-c}\,s\right)$$

for all $x\in\mathscr{S}_1$ and all $s>0$. Finally, by property (FPC4), we obtain

$$N\left(f(x)-\mathscr{A}(x),s\right)\geq N'\left(\Lambda_A\left(x,x,x,\cdots,x\right),\left[\frac{L^{1-c}}{1-L}\right]\frac{s}{a}\right)$$

for all $x\in\mathscr{S}_1$ and all $s>0$. This finishes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 4.3 concerning the stabilities of (1.5).

Corollary 4.4. *Let $f : \mathscr{S}_1 \longrightarrow \mathscr{S}_2$ be an odd mapping. If there exist real numbers d and b such that*

$$N\left(F_{AQ}(x_1,x_2,x_3,\cdots,x_n),s\right) \geq \begin{cases} N'(d,s) \\ N'\left(d\sum_{i=1}^{n}||x_i||^b,s\right), & b\neq 1; \\ N'\left(d\prod_{i=1}^{n}||x_i||^b,s\right), & nb\neq 1; \\ N'\left(d\sum_{i=1}^{n}||x_i||^{b_i},s\right), & b_i\neq 1; \\ N'\left(d\prod_{i=1}^{n}||x_i||^{b_i},s\right), & \sum_{i=1}^{n} b_i\neq 1; \end{cases} \tag{4.14}$$

for all $x_1,x_2,x_3,\cdots,x_n \in \mathscr{S}_1$ and all $s>0$, then there exists a unique Additive mapping $\mathscr{A} : \mathscr{S}_1 \longrightarrow \mathscr{S}_2$ such that

$$N\left(f(x)-\mathscr{A}(x),s\right) \geq \begin{cases} N'\left(4d,(n+3)\frac{s}{|a|}\right); \\ N'\left((2n+2+2^b)d||x||^b,\frac{(n+3)2^b\ s}{a|2-2^b|}\right); \\ N'\left(2d||x||^{nb}d||x||^b,\frac{(n+3)2^{nb}\ s}{a|2-2^{nb}|}\right); \\ N'\left((3+2^b)d||x||^{b_1}+3||x||^{b_2}+\sum_{i=3}^{n}(n-2)||x||^{b_i},\right. \\ \qquad \left.\left(\sum_{i=1}^{n}\frac{(n+3)2^{b_i}\ s}{a|2-2^{b_i}|}\right)\right); \\ N'\left(2d||x||^{\sum_{i=1}^{n}b_i},\frac{(n+3)2^{\sum_{i=1}^{n}b_i}\ s}{a|2-2^{\sum_{i=1}^{n}b_i}|}\right); \end{cases} \tag{4.15}$$

for all $x\in\mathscr{S}_1$ and all $s>0$.

Proof. If we take

$$N'\left(\lambda(x_1,x_2,x_3,\cdots,x_n),s\right) = \begin{cases} N'(d,s); \\ N'\left(d\sum_{i=1}^{n}||x_i||^b,s\right); \\ N'\left(d\prod_{i=1}^{n}||x_i||^b,s\right); \\ N'\left(d\sum_{i=1}^{n}||x_i||^{b_i},s\right); \\ N'\left(d\prod_{i=1}^{n}||x_i||^{b_i},s\right); \end{cases} \tag{4.16}$$

for all $x_1,x_2,x_3,\cdots,x_n \in \mathscr{S}_1$ and all $s>0$. Now

$$\begin{aligned} &N'\left(\lambda(C_c^q x_1,C_c^q x_2,C_c^q x_3,\cdots,C_c^q x_n),C_c^q\ s\right) \\ &= \begin{cases} N'\left(d,C_c^q\ s\right); \\ N'\left(d\sum_{i=1}^{n}||C_c^q x_i||^b,C_c^q\ s\right); \\ N'\left(d\prod_{i=1}^{n}||C_c^q x_i||^b,C_c^q\ s\right); \\ N'\left(d\sum_{i=1}^{n}||C_c^q x_i||^{b_i},C_c^q\ s\right); \\ N'\left(d\prod_{i=1}^{n}||C_c^q x_i||^{b_i},C_c^q\ s\right); \end{cases} \\ &= \begin{cases} \to 1 \text{ as } q\to\infty, \\ \to 1 \text{ as } q\to\infty, \\ \to 1 \text{ as } q\to\infty, \\ \to 1 \text{ as } q\to\infty, \\ \to 1 \text{ as } q\to\infty. \end{cases} \end{aligned}$$

Thus, (4.1) holds. But from (4.4), (3.6) and (4.16), we have

$$\begin{aligned} &N'\left(\Lambda_{A_F}(x,x,x,\cdots,x),s\right) \\ &= N'\left(\Lambda_A\left(\frac{x}{2},\frac{x}{2},\frac{x}{2},\cdots,\frac{x}{2}\right),s\right), \\ &= \min\left\{N'\left(\lambda\left(-x,\underbrace{x,x,\cdots,x}_{n-1\ \ times}\right),s\right),\right. \\ &\qquad N'\left(\lambda\left(x,\underbrace{-x,-x,\cdots,-x}_{n-1\ \ times}\right),s\right), \\ &\qquad N'\left(\lambda\left(x,x,\underbrace{0,\cdots,0}_{n-2\ \ times}\right),ns\right), \\ &\qquad \left.N'\left(\lambda\left(2x,\underbrace{0,0,\cdots,0}_{n-1\ \ times}\right),s\right)\right\} \\ &= N'\left(\lambda\left(-x,\underbrace{x,x,\cdots,x}_{n-1\ \ times}\right)+\lambda\left(x,\underbrace{-x,-x,\cdots,-x}_{n-1\ \ times}\right)\right. \\ &\qquad +\lambda\left(x,x,\underbrace{0,\cdots,0}_{n-2\ \ times}\right) \\ &\qquad \left.+\lambda\left(2x,\underbrace{0,0,\cdots,0}_{n-1\ \ times}\right),(n+3)s\right) \\ &= \begin{cases} N'\left(4d,(n+3)s\right); \\ N'\left((2n+2+2^b)d||x||^b,(n+3)s\right); \\ N'\left(2d||x||^{nb},(n+3)s\right); \\ N'\left((3+2^b)d||x||^{b_1}+3||x||^{b_2}\right. \\ \qquad \left.+\sum_{i=3}^{n}(n-2)||x||^{b_i},(n+3)s\right); \\ N'\left(2d||x||^{\sum_{i=1}^{n}b_i},(n+3)s\right); \end{cases} \end{aligned} \tag{4.17}$$

for all $x\in\mathscr{S}_1$ and all $s>0$. Now, similarly by (4.5), (3.6) and (4.16), we prove

$$\begin{aligned} &N'\left(\frac{1}{C_c}\Lambda_{A_F}(C_cx,C_cx,C_cx,\cdots,C_cx),s\right) \\ &= \begin{cases} N'\left(\frac{1}{C_c}4d,(n+3)s\right); \\ N'\left(\frac{1}{C_c}(2n+2+2^b)d||C_cx||^b,(n+3)s\right); \\ N'\left(\frac{1}{C_c}2d||C_cx||^{nb},(n+3)s\right); \\ N'\left(\frac{1}{C_c}\left[(3+2^b)d||C_cx||^{b_1}+3||C_cx||^{b_2}\right.\right. \\ \qquad \left.\left.+\sum_{i=3}^{n}(n-2)||C_cx||^{b_i}\right],(n+3)s\right); \\ N'\left(\frac{1}{C_c}2d||C_cx||^{\sum_{i=1}^{n}b_i},(n+3)s\right); \end{cases} \\ &= \begin{cases} N'\left(4d,(n+3)C_c^{-1}s\right); \\ N'\left((2n+2+2^b)d||x||^b,(n+3)C_c^{b-1}s\right); \\ N'\left(2d||x||^{nb},(n+3)C_c^{nb-1}s\right); \\ N'\left((3+2^b)d||x||^{b_1}+3||x||^{b_2}\right. \\ \qquad \left.+\sum_{i=3}^{n}(n-2)||x||^{b_i},(n+3)\left(\sum_{i=1}^{n}C_c^{b_i-1}\right)s\right); \\ N'\left(2d||C_cx||^{\sum_{i=1}^{n}b_i},(n+3)C_c^{\sum_{i=1}^{n}b_i-1}C_cs\right); \end{cases} \\ &= N'\left(\Lambda_{A_F}(x,x,x,\cdots,x),L\ s\right) \end{aligned} \tag{4.18}$$

Hence, the inequality (4.6) holds for the following cases.

$$L = C_c^{-1} = 2^{-1} \quad if \quad c = 0$$

$$\begin{aligned} N(f(x) - \mathscr{A}(x), s) &\geq N'\left(\Lambda_{A_F}(x,x,x,\cdots,x), \left[\frac{L^{1-c}}{1-L}\right]\frac{s}{a}\right) \\ &= N'\left(\Lambda_{A_F}(x,x,x,\cdots,x), \left[\frac{(2^{-1})^{1-0}}{1-2^{-1}}\right]\frac{s}{a}\right) \\ &= N'\left(4d, (n+3)\frac{s}{a}\right). \end{aligned}$$

$$L = \frac{1}{C_c^{-1}} = 2 \quad if \quad c = 1$$

$$\begin{aligned} N(f(x) - \mathscr{A}(x), s) &\geq N'\left(\Lambda_{A_F}(x,x,x,\cdots,x), \left[\frac{L^{1-c}}{1-L}\right]\frac{s}{a}\right) \\ &= N'\left(\Lambda_{A_F}(x,x,x,\cdots,x), \left[\frac{2^{1-1}}{1-2}\right]\frac{s}{a}\right) \\ &= N'\left(4d, (n+3)\frac{s}{-a}\right). \end{aligned}$$

$$L = C_c^{b-1} = 2^{b-1} \quad for \quad b < 1 \quad if \quad c = 0$$

$$\begin{aligned} N(f(x) - \mathscr{A}(x), s) &\geq N'\left(\Lambda_{A_F}(x,x,x,\cdots,x), \left[\frac{L^{1-c}}{1-L}\right]\frac{s}{a}\right) \\ &= N'\left(\Lambda_{A_F}(x,x,x,\cdots,x), \left[\frac{(2^{b-1})^{1-0}}{1-2^{b-1}}\right]\frac{s}{a}\right) \\ &= N'\left((2n+2+2^b)d||x||^b, \frac{(n+3)2^b\, s}{a(2-2^b)}\right). \end{aligned}$$

$$L = \frac{1}{C_c^{b-1}} = 2^{1-b} \quad for \quad b > 1 \quad if \quad c = 1$$

$$\begin{aligned} N(f(x) - \mathscr{A}(x), s) &\geq N'\left(\Lambda_{A_F}(x,x,x,\cdots,x), \left[\frac{L^{1-c}}{1-L}\right]\frac{s}{a}\right) \\ &= N'\left(\Lambda_{A_F}(x,x,x,\cdots,x), \left[\frac{(2^{1-b})^{1-1}}{1-2^{1-b}}\right]\frac{s}{a}\right) \\ &= N'\left((2n+2+2^b)d||x||^b, \frac{(n+3)2^b\, s}{a(2^b-2)}\right). \end{aligned}$$

$$L = C_c^{nb-1} = 2^{nb-1} \quad for \quad nb < 1 \quad if \quad c = 0$$

$$\begin{aligned} N(f(x) - \mathscr{A}(x), s) &\geq N'\left(\Lambda_{A_F}(x,x,x,\cdots,x), \left[\frac{L^{1-c}}{1-L}\right]\frac{s}{a}\right) \\ &= N'\left(\Lambda_{A_F}(x,x,x,\cdots,x), \left[\frac{(2^{nb-1})^{1-0}}{1-2^{nb-1}}\right]\frac{s}{a}\right) \\ &= N'\left(2d||x||^{nb}d||x||^b, \frac{(n+3)2^{nb}\, s}{a(2-2^{nb})}\right). \end{aligned}$$

$$L = \frac{1}{C_c^{nb-1}} = 2^{1-nb} \quad for \quad nb > 1 \quad if \quad c = 1$$

$$\begin{aligned} N(f(x) - \mathscr{A}(x), s) &\geq N'\left(\Lambda_{A_F}(x,x,x,\cdots,x), \left[\frac{L^{1-c}}{1-L}\right]\frac{s}{a}\right) \\ &= N'\left(\Lambda_{A_F}(x,x,x,\cdots,x), \left[\frac{(2^{1-nb})^{1-0}}{1-2^{1-nb}}\right]\frac{s}{a}\right) \\ &= N'\left(2d||x||^{nb}d||x||^b, \frac{(n+3)2^{nb}\, s}{a(2^{nb}-2)}\right). \end{aligned}$$

$$L = C_c^{b_i-1} = 2^{b_i-1} \quad for \quad b_i < 1 \quad if \quad c = 0$$

$$\begin{aligned} N(f(x) - \mathscr{A}(x), s) &\geq N'\left(\Lambda_{A_F}(x,x,x,\cdots,x), \left[\frac{L^{1-c}}{1-L}\right]\frac{s}{a}\right) \\ &= N'\left(\Lambda_{A_F}(x,x,x,\cdots,x), \left[\frac{(2^{b_i-1})^{1-0}}{1-2^{b_i-1}}\right]\frac{s}{a}\right) \\ &= N'\left((3+2^b)d||x||^{b_1} + 3||x||^{b_2} + \sum_{i=3}^{n}(n-2)||x||^{b_i}, \right. \\ &\qquad \left. (n+3)\left(\sum_{i=1}^{n}\frac{2^{b_i}}{2-2^{b_i}}\right)\frac{s}{a}\right). \end{aligned}$$

$$L = \frac{1}{C_c^{b_i-1}} = 2^{1-b_i} \quad for \quad b_i > 1 \quad if \quad c = 1$$

$$\begin{aligned} N(f(x) - \mathscr{A}(x), s) &\geq N'\left(\Lambda_{A_F}(x,x,x,\cdots,x), \left[\frac{L^{1-c}}{1-L}\right]\frac{s}{a}\right) \\ &= N'\left(\Lambda_{A_F}(x,x,x,\cdots,x), \left[\frac{(2^{1-b_i})^{1-1}}{1-2^{1-b_i}}\right]\frac{s}{a}\right) \\ &= N'\left((3+2^b)d||x||^{b_1} + 3||x||^{b_2} + \sum_{i=3}^{n}(n-2)||x||^{b_i}, \right. \\ &\qquad \left. (n+3)\left(\sum_{i=1}^{n}\frac{2^{b_i}}{2^{b_i}-2}\right)\frac{s}{a}\right). \end{aligned}$$

$$L = C_c^{\sum_{i=1}^n b_i - 1} = 2^{\sum_{i=1}^n b_i - 1} \quad for \quad \sum_{i=1}^{n} b_i < 1 \quad if \quad c = 0$$

$$\begin{aligned} N(f(x) - \mathscr{A}(x), s) &\geq N'\left(\Lambda_{A_F}(x,x,x,\cdots,x), \left[\frac{L^{1-c}}{1-L}\right]\frac{s}{a}\right) \\ &= N'\left(\Lambda_{A_F}(x,x,x,\cdots,x), \left[\frac{(2^{\sum_{i=1}^n b_i - 1})^{1-0}}{1-2^{\sum_{i=1}^n b_i - 1}}\right]\frac{s}{a}\right) \\ &= N'\left(2d||x||^{\sum_{i=1}^n b_i}, \frac{(n+3)2^{\sum_{i=1}^n b_i}\, s}{a(2-2^{\sum_{i=1}^n b_i})}\right). \end{aligned}$$

$$L = \frac{1}{C_c^{\sum_{i=1}^n b_i - 1}} = 2^{1-\sum_{i=1}^n b_i} \quad for \quad \sum_{i=1}^n b_i > 1 \quad if \quad c = 1$$

$$\begin{aligned} &N(f(x) - \mathscr{A}(x), s) \\ &\geq N'\left(\Lambda_{A_F}(x,x,x,\cdots,x), \left[\frac{L^{1-c}}{1-L}\right]\frac{s}{a}\right) \\ &= N'\left(\Lambda_{A_F}(x,x,x,\cdots,x), \left[\frac{(2^{1-\sum_{i=1}^n b_i})^{1-0}}{1-2^{1-\sum_{i=1}^n b_i}}\right]\frac{s}{a}\right) \\ &= N'\left(2d||x||^{nb}d||x||^b, \frac{(n+3)2^{\sum_{i=1}^n b_i} s}{a(2^{\sum_{i=1}^n b_i} - 2)}\right). \end{aligned}$$

Hence the proof is complete. □

Theorem 4.5. *Let $f : \mathscr{S}_1 \to \mathscr{S}_2$ be an even mapping for which there exist a mapping $\lambda : \mathscr{S}_1^2 \longrightarrow \mathscr{S}_3$ with the condition*

$$\lim_{q\to\infty} N'\left(\lambda\left(C_c^q x_1, C_c^q x_2, C_c^q x_n\right), C_c^{2q} s\right) = 1 \tag{4.19}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$ where

$$C_c = \begin{cases} 2 & if \quad c = 0, \\ \frac{1}{2} & if \quad c = 1 \end{cases} \tag{4.20}$$

and satisfying the functional inequality

$$N(F_{AQ}(x_1,x_2,x_3,\cdots,x_n), s) \geq N'\left(\lambda\left(x_1,x_2,x_3,\cdots,x_n\right), s\right) \tag{4.21}$$

for all $x_1,x_2,x_3,\cdots,x_n \in \mathscr{S}_1$ and all $s > 0$. If there exists $L = L(c)$ such that the function

$$\Lambda_{Q_F}(x,x,x,\cdots,x) = \Lambda_Q\left(\frac{x}{2},\frac{x}{2},\frac{x}{2},\cdots,\frac{x}{2}\right), \tag{4.22}$$

where $\Lambda_Q(x,x,x,\cdots,x)$ is defined in (3.45) with the property

$$\begin{aligned} &N'\left(\frac{1}{C_c^2}\Lambda_{Q_F}(C_c x, C_c x, C_c x, \cdots, C_c x), s\right) \\ &\qquad = N'\left(\Lambda_{Q_F}(x,x,x,\cdots,x), L\, s\right), \end{aligned} \tag{4.23}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$. Then there exists a unique quadratic mapping $\mathscr{Q} : \mathscr{S}_1 \longrightarrow \mathscr{S}_2$ satisfying the functional equation (1.5) and

$$N(f(x) - \mathscr{Q}(x), s) \geq N'\left(\Lambda_{Q_F}(x,x,x,\cdots,x), \left[\frac{L^{1-c}}{1-L}\right]\frac{s}{e}\right), \tag{4.24}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$.

Proof. Consider the set

$$\mathscr{C} = \{f_2 | f_2 : \mathscr{S}_1 \longrightarrow \mathscr{S}_2, \ f_2(0) = 0\}$$

and introduce the generalized metric on $\mathscr{C}$ as follows:

$$\begin{aligned} d(f_1, f_2) = \inf\{\rho \in (0,\infty) : N(f_1(x) - f_2(x), s) \\ \geq N'\left(\Lambda_{Q_F}(x,x,x,\cdots,x,\rho s)\right), x \in \mathscr{S}_1, s > 0\}. \end{aligned} \tag{4.25}$$

It is easy to see that (4.25) is complete with respect to the defined metric. Define $J : \mathscr{C} \longrightarrow \mathscr{C}$ by

$$Jf(x) = \frac{1}{C_c^2} f(C_c x),$$

for all $x \in \mathscr{S}_1$. The rest of the proof is similar to that of Theorem 4.3. □

The following corollary is an immediate consequence of Theorem 4.5 concerning the stabilities of (1.5).

Corollary 4.6. *Let $f : \mathscr{S}_1 \longrightarrow \mathscr{S}_2$ be an even mapping. If there exist real numbers d and b such that*

$$N(F_{AQ}(x_1,x_2,x_3,\cdots,x_n), s) \geq \begin{cases} N'(d, s) \\ N'\left(d\sum_{i=1}^n ||x_i||^b, s\right), & b \neq 2; \\ N'\left(d\sum_{i=1}^n ||x_i||^{b_i}, s\right), & b_i \neq 2; \end{cases} \tag{4.26}$$

for all $x_1,x_2,x_3,\cdots,x_n \in \mathscr{S}_1$ and all $s > 0$, then there exists a unique quadratic mapping $\mathscr{Q} : \mathscr{S}_1 \longrightarrow \mathscr{S}_2$ such that

$$N(f(x) - \mathscr{Q}(x), s) \geq \begin{cases} N'\left(9d, (2n+7)\frac{s}{3|e|}\right); \\ N'\left([(7+2n+2)n^b + (n-1)^b + 1]d||x||^b, \frac{(2n+7)2^b\, s}{e|4-2^b|}\right); \\ N'\left((6.n^{b_1}+1)||x||^{b_1} + [3.n^{b_2} + (n-1)^{b_2}]||x||^{b_2} \right. \\ \qquad \left. + \sum_{i=3}^n 2(n-2)n^{b_i}||x||^{b_i}, \left(\sum_{i=1}^n \frac{(2n+7)2^{b_i}\, s}{e|4-2^{b_i}|}\right)\right); \end{cases} \tag{4.27}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$.

Theorem 4.7. *Let $f : \mathscr{S}_1 \to \mathscr{S}_2$ be a mapping for which there exist a mapping $\lambda : \mathscr{S}_1^2 \longrightarrow \mathscr{S}_3$ with the conditions (4.1) and (4.19) for all $x \in \mathscr{S}_1$ and all $s > 0$ and satisfying the functional inequality*

$$N(F_{AQ}(x_1,x_2,x_3,\cdots,x_n), s) \geq N'\left(\lambda\left(x_1,x_2,x_3,\cdots,x_n\right), s\right) \tag{4.28}$$

for all $x_1,x_2,x_3,\cdots,x_n \in \mathscr{S}_1$ and all $s > 0$. If there exists $L = L(c)$ such that the functions (4.4) and (4.22) with the properties (4.5) and (4.23) for all $x \in \mathscr{S}_1$ and all $s > 0$. Then there exists a unique additive mapping $\mathscr{A} : \mathscr{S}_1 \longrightarrow \mathscr{S}_2$ and a unique quadratic mapping $\mathscr{Q} : \mathscr{S}_1 \longrightarrow \mathscr{S}_2$ satisfying the functional equation (1.5) and

$$\begin{aligned} &N(f(x) - \mathscr{A}(x) - \mathscr{Q}(x), s) \\ &\geq \min\left\{N'\left(\Lambda_{A_F}(x,x,x,\cdots,x), \left[\frac{L^{1-c}}{1-L}\right]\frac{s}{a}\right),\right. \\ &\qquad N'\left(\Lambda_{A_F}(-x,-x,-x,\cdots,-x), \left[\frac{L^{1-c}}{1-L}\right]\frac{s}{a}\right), \\ &\qquad N'\left(\Lambda_{Q_F}(x,x,x,\cdots,x), \left[\frac{L^{1-c}}{1-L}\right]\frac{s}{e}\right), \\ &\qquad \left. N'\left(\Lambda_{Q_F}(-x,-x,-x,\cdots,-x), \left[\frac{L^{1-c}}{1-L}\right]\frac{s}{e}\right)\right\} \end{aligned} \tag{4.29}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$.

Proof. The proof of the theorem is similar ideas and clues used in Theorem 3.5. Hence the details of the proofs are omitted. □

The following corollary is an immediate consequence of Theorem 4.5 concerning the stabilities of (1.5).

Corollary 4.8. *Let $f : \mathscr{S}_1 \longrightarrow \mathscr{S}_2$ be a mapping. If there exist real numbers d and b such that*

$$N\left(F_{AQ}(x_1,x_2,x_3,\cdots,x_n),s\right) \geq \begin{cases} N'\left(d,s\right) \\ N'\left(d\sum_{i=1}^{n}||x_i||^b,s\right), & b \neq 1,2; \\ N'\left(d\sum_{i=1}^{n}||x_i||^{b_i},s\right), & b_i \neq 1,2; \end{cases} \tag{4.30}$$

for all $x_1,x_2,x_3,\cdots,x_n \in \mathscr{S}_1$ and all $s > 0$, then there exists a unique additive mapping $\mathscr{A} : \mathscr{S}_1 \longrightarrow \mathscr{S}_2$ and a unique quadratic mapping $\mathscr{Q} : \mathscr{S}_1 \longrightarrow \mathscr{S}_2$ such that

$$N\left(f(x)-\mathscr{A}(x)-\mathscr{Q}(x),s\right) \geq \begin{cases} \min\left\{N'\left(4d,(n+3)\frac{s}{|a|}\right),N'\left(9d,(2n+7)\frac{s}{3|e|}\right)\right\}; \\ \min\left\{N'\left((2n+2+2^b)d||x||^b, \frac{(n+3)2^b\ s}{a|2-2^b|}\right),\right. \\ \quad N'\left([(7+2n+2)n^b+(n-1)^b+1]d||x||^b,\right. \\ \quad \left.\left.\frac{(2n+7)2^b\ s}{e|4-2^b|}\right)\right\}; \\ \min\left\{N'\left((3+2^b)d||x||^{b_1}+3||x||^{b_2}\right.\right. \\ \quad \left.+\sum_{i=3}^{n}(n-2)||x||^{b_i}, \left(\sum_{i=1}^{n}\frac{(n+3)2^{b_i}\ s}{a|2-2^{b_i}|}\right)\right), \\ \quad N'\left((6.n^{b_1}+1)||x||^{b_1}+[3.n^{b_2}+(n-1)^{b_2}]\right. \\ \quad ||x||^{b_2}+\sum_{i=3}^{n}2(n-2)n^{b_i}||x||^{b_i}, \\ \quad \left.\left.\left(\sum_{i=1}^{n}\frac{(2n+7)2^{b_i}\ s}{e|4-2^{b_i}|}\right)\right)\right\}; \end{cases} \tag{4.31}$$

for all $x \in \mathscr{S}_1$ and all $s > 0$.

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