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Fixed points and stability of Icosic functional equation in quasi-β-normed spaces

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Abstract

In this paper, we introduce the following icosic functional equation. This pioneering icosic functional equation

$$f(x+10y)-20f(x+9y)+190f(x+8y)-1140f(x+7y)+4845f(x+6y)-15504f(x+5y)$$

$$+38760f(x+4y)-77520f(x+3y)+125970f(x+2y)-167960f(x+y)+184756f(x)$$

$$-167960f(x-y)+125970f(x-2y)-77520f(x-3y)+38760f(x-4y)-15504f(x-5y)$$

$$+4845f(x-6y)-1140f(x-7y)+190f(x-8y)-20\ f(x-9y)+f(x-10y)=20!\ f(y),$$

where $20!\ =\ 2.432902008\times 10^{8}$, is said to be icosic functional equation. Since the functional equation $f(x)=x^{20}$ is the solution. In this paper, we present the general solutions of the said icosic functional equation. We also prove the stability of the icosic functional equation in a quasi-β-normed space.

Keywords

Icosic functional equation, Quasi β-normed space, Fixed point.

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Contents

1. Introduction

The stability problem of functional equations originated from a famous stability question of Ulam [22] in 1940 concerning the stability of group homomorphisms. In fact, let $(G_1,\ .)$ be a group and let $(G_2,*)$ be a metric group with a metric $d(.,.)$. Given $\varepsilon>0$, does there exists $\delta>0$, such that if a mapping $h:G_1\to G_2$ satisfies the inequality $d(h(xy),h(x)h(y))<\varepsilon$, for all $x,y\in G_1$, then there exists a homomorphism $H:G_1\to G_2$ exists with $d(h(x),H(x))\ <\ \varepsilon$ for all $x,y\in G_1$?

The case of approximtely additive functions was solved by Hyers [7] in 1941 under the assumption that both G_1 and G_2 are Banach spaces. Gruber claimed that this kind of stability problems is of particular interest in Probability theory and in the case of functional equations of different types.

In 1978, Rassias [16] proved a generalization of the Hyers Theorem for approximate linear mappings by allowing the Cauchy difference operator

$$CDf(x,y)=f(x+y)-f(x)-f(y)$$

to be controlled by the sum $\varepsilon\left(\|x\|^p+\|y\|^p\right)$. This new concept is known as the Hyers-Ulam-Rassias stability of functional equations, In 1994, a generalization of Rassias theorem was obtained by Gavruta [5] who replaced $\varepsilon\left(\|x\|^p+\|y\|^p\right)$ by a general control function $\phi(x,y)$ in the spirit of the Rassias approach. However, Rassias [] (in 1989) replaced the said sum by the weaker product $\varepsilon\ \left(\|x\|^p\,\|y\|^p\right)$. In recent years, many authors have investigated the stability of various functional equations in various spaces (see, for instance [1, 3, 4, 9, 10, 13–15, 17, 20, 21, 23]).

In 1996, Isac and Rassias [8] were first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. In 2003, Radu [19] proposed the fixed point alternative method for obtaining the existence of exact solutions and error estimations. The fixed point alternative method can be considered as an advantage of this method over the standard direct method. The stability problems of several various functional equations have been extensively investigated by a number of authors using fixed point methods (see, [2, 11, 12, 18, 24]).

In this paper, we introduce the following icosic functional equation. This pioneering icosic functional equation

$$\begin{aligned} &f(x+10y)-20f(x+9y)+190f(x+8y)\\ &\quad-1140f(x+7y)+4845f(x+6y)\\ &\quad-15504f(x+5y)+38760f(x+4y)\\ &\quad-77520f(x+3y)+125970f(x+2y)\\ &\quad-167960f(x+y)+184756f(x)\\ &\quad-167960f(x-y)+125970f(x-2y)\\ &\quad-77520f(x-3y)+38760f(x-4y)\\ &\quad-15504f(x-5y)+4845f(x-6y)\\ &\quad-1140f(x-7y)+190f(x-8y)\\ &\quad-20f(x-9y)+f(x-10y)=20!\ f(y) \qquad (1.1)\end{aligned}$$

where $20! = 2.432902008\times10^{8}$, is said to be icosic functional equation. Since the functional equation $f(x)=x^{20}$ is the solution. In this paper, we present the general solutions of the said icosic functional equation. We also prove the stability of the icosic functional equation in a quasi-β-normed space.

2. Preliminaries

We recall some basic concepts concerning quasi-β-normed spaces introduced by Rassias and Kim [13] in 2009.

Definition 2.1. *Let β be a fix real number with $0<\beta\le1$, and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let X be a linear space over $\mathbb{K}$. A quasi-β-norm $\|.\|$ is a real valued function on X satisfying the following three conditions:*

(i). $\|x\|\ge0$, for all $x\in X$; and $\|x\|=0$ if and only if $x=0$,

(ii). $\|\lambda\|=|\lambda|\ \|x\|$ for all $\lambda\in\mathbb{K}$ and for all $x\in X$,

(iii). there is a constant $K\ge1$ such that

$$\|x+y\|\le K(\|x\|+\|y\|)\ \textit{for all} x,y\in X.$$

A quasi-β-normed space is a pair of $(X,\|.\|)$, where $\|.\|$ is a quasi-β-norm on X. The smallest possible K is called the modules of Concavity of $\|.\|$. A quasi-β-Banach space is a complete quasi-β-normed space.

Definition 2.2. *A quasi-β-norm $\|.\|$ is called a (β,p)-norm $(0<p\le1)$ if*

$$\|x+y\|^p=\|x\|^p+\|y\|^p$$

for all $x,y\in X$.

In this case, a quasi-β-Banach space is called a (β,p)-Banach space. We can refer to [13] for the concept of quasi-normed spaces and p-Banach spaces. Given a p-norm, the formula

$$d(x,y)=\|x-y\|^p$$

gives us a translation invariant metric n X. By the Aoki-Rolewicz theorem, each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, we restrict our attension mainly to p-norms. Using fixed point theorem, Xu, et.al., [24] proved the following important lemma.

Lemma 2.3. *Let $i\in\{-1,1\}$ be a fixed $a,s\in N$ with $a\ge2$, and $\phi: X\to[0,\infty)$ be a function such that there exists an $L<1$ with $\phi(a^ix)\le a^{is\beta}\,L\,\phi(x)$ for all $x\in X$. Let $f:X\to Y$ be a function satisfying*

$$\|f(ax)-a^sf(x)\|_Y\le\phi(x) \qquad (2.1)$$

for all $x,y\in X$. Then there exists a uniquely determined mapping $F:X\to Y$ such that

$$\|f(x)-F(x)\|_Y\le\frac{1}{a^{s\beta}\,|1-L^i|}\,\phi(x) \qquad \forall\, x\in X. \quad (2.2)$$

3. General solution of icosic functional equation

In this section, let X and Y be vector spaces. In the following theorem, we investigate the general solution of icosic functional equation (1.1). Some basic facts on n-additive symmetric mappings can be found in [6].

Theorem 3.1. *A function $f:X\to Y$ is a solution of the icosic functional equation (1.1) if and only if f is of the form $f(x)=A^{20}(x)$ for all $x\in X$, where $A^{20}(x)$ is the diagonal of the 20-additive symmetric mapping $A_{20}:X^{20}\to Y$.*

Proof. Assume that f satisfies functional equation (1.1). Replacing $x=0,y=0$ in (1.1), one finds $f(0)=0$. Replacing $x=0,y=x$ and $x=x,y=-x$ in (1.1) and adding the two resulting equations, we get $f(-x)=f(x)$. Replacing (x,y) with $(0,2x)$, one obtains

$$\begin{aligned} &f(20x)-20f(18x)+190f(16x)\\ &\quad-1140f(14x)+4845f(12x)-15504f(10x)\\ &\quad+38760f(8x)-77520f(6x)+125970f(4x)\\ &\quad-1.216451004\times10^{18}f(2x)=0. \qquad (3.1)\end{aligned}$$

Replacing (x,y) with $(10x,x)$, one gets

$$\begin{aligned} &f(20x)-20f(19x)+190f(18x)-1140f(17x)\\ &+4845f(16x)-15504f(15x)+38760f(14x)\\ &-77520f(13x)+125970f(12x)-167960f(11x)\\ &+184756f(10x)-167960f(9x)+125970f(8x)\\ &-77520f(7x)+38760f(6x)-15504f(5x)\\ &+4845f(4x)-1140f(3x)+190f(2x)\\ &-2.432902008\times10^{18}f(x)=0. \qquad (3.2)\end{aligned}$$

Subtracting equations (3.1) and (3.2), we find

$$\begin{aligned}&20f(19x)-210f(18x)+1140f(17x)\\&-4655f(16x)+15504f(15x)-39900f(14x)\\&+77520f(13x)-121125f(12x)+167960f(11x)\\&-200260f(10x)+167960f(9x)-87210f(8x)\\&+77520f(7x)-116280f(6x)+15504f(5x)\\&+121125f(4x)+1140f(3x)\\&-1.216451004\times10^{18}f(2x)\\&+2.432902008\times10^{18}f(x)=0.\end{aligned}\tag{3.3}$$

Replacing (x,y) with $(9x,x)$, and multiplying the resulting equation by 20, one obtains

$$\begin{aligned}&20f(19x)-400f(18x)+3800f(17x)\\&-22800f(16x)+96900f(15x)-310080f(14x)\\&+775200f(13x)-1550400f(12x)\\&+2519400f(11x)-3359200f(10x)\\&+3695120f(9x)-3359200f(8x)\\&+2519400f(7x)-1550400f(6x)\\&+775200f(5x)-310080f(4x)\\&+96900f(3x)-22800f(2x)\\&-4.865804016\times10^{19}f(x)=0.\end{aligned}\tag{3.4}$$

Subtracting equations (3.3) and (3.4) , we get

$$\begin{aligned}&190f(18x)-2660f(17x)+18145f(16x)\\&-81396f(15x)+270180f(14x)-697680f(13x)\\&+1429275f(12x)-2351440f(11x)\\&+3158940f(10x)-3527160f(9x)\\&+3271990f(8x)-2441880f(7x)\\&+1434120f(6x)-759696f(5x)\\&+431205f(4x)-95760f(3x)\\&-1.216451004\times10^{18}f(2x)\\&+5.109094217\times10^{19}f(x)=0.\end{aligned}\tag{3.5}$$

Replacing (x,y) with $(8x,x)$, and multiplying the resulting equation by 190, one obtains

$$\begin{aligned}&190f(18x)-3800f(17x)+36100f(16x)\\&-216600f(15x)+920550f(14x)-2945760f(13x)\\&+7364400f(12x)-14728800f(11x)\\&+23934300f(10x)-31912400f(9x)\\&+35103640f(8x)-31912400f(7x)\\&+23934300f(6x)-14728800f(5x)\\&+7364400f(4x)-2945760f(3x)\\&+920740f(2x)+4.622513816\times10^{20}f(x)=0.\end{aligned}\tag{3.6}$$

Subtracting equations (3.5) and (3.6), we find

$$\begin{aligned}&1140f(17x)-17955f(16x)+135204f(15x)\\&-650370f(14x)+2248080f(13x)\\&-20775360f(12x)+12377360f(11x)\\&-20761406f(10x)+28385240f(9x)\\&-31831650f(8x)+29470520f(7x)\\&-22500180f(6x)+13969104f(5x)\\&-6933195f(4x)+2850000f(3x)\\&-1.216451004\times10^{18}f(2x)\\&+5.133423238\times10^{20}f(x)=0.\end{aligned}\tag{3.7}$$

Replacing (x,y) with $(7x,x)$, and multiplying the resulting equation by 1140, we get

$$\begin{aligned}&1140f(17x)-22800f(16x)+216600f(15x)\\&-1299600f(14x)+5523300f(13x)\\&-17674560f(12x)+44186400f(11x)\\&-88372800f(10x)+143605800f(9x)\\&-191474400f(8x)+210621840f(7x)\\&-191474400f(6x)+143605800f(5x)\\&-88372800f(4x)+44187540f(3x)\\&-17697360f(2x)-2.773508289\times10^{21}f(x)=0.\end{aligned}\tag{3.8}$$

Subtracting equations (3.7) and (3.8), we get

$$\begin{aligned}&4845f(16x)-81396f(15x)+649230f(14x)\\&-3275220f(13x)+11739435f(12x)\\&-31809040f(11x)+67597440f(10x)\\&-115220560f(9x)+159642750f(8x)\\&-181151320f(7x)+168974220f(6x)\\&-129636696f(5x)+81439605f(4x)\\&-41337540f(3x)-1.216451004\times10^{18}f(2x)\\&+3.286850613\times10^{21}f(x)=0.\end{aligned}\tag{3.9}$$

Replacing (x,y) with $(6x,x)$, and multiplying the resulting equation by 4845, we get

$$\begin{aligned}&4845f(16x)-96900f(15x)+920550f(14x)\\&-5523300f(13x)+23474025f(12x)\\&-75116880f(11x)+187792200f(10x)\\&-375584400f(9x)+610324650f(8x)\\&-813766200f(7x)+895142820f(6x)\\&-813766200f(5x)+610329495f(4x)\\&-375681300f(3x)+188712750f(2x)\\&-1.178741023\times10^{22}f(x)=0.\end{aligned}\tag{3.10}$$

Subtracting equations (3.9) and (3.10), we find

$$\begin{aligned}&7752f(15x)-135660f(14x)+1124040f(13x)\\&-5867295f(12x)+21653920f(11x)\\&-60097380f(10x)+130181920f(9x)\\&-225340950f(8x)+316307440f(7x)\\&-363084300f(6x)+342064752f(5x)\\&-264444945f(4x)+167171880f(3x)\\&-6.08225502\times10^{17}f(2x)\\&+7.53713042\times10^{21}f(x)=0.\end{aligned}\tag{3.11}$$

Replacing (x,y) with $(5x,x)$, and multiplying the resulting equation by 7752, we get

$$\begin{aligned}&7752f(15x)-155040f(14x)+1472880f(13x)\\&-8837280f(12x)+37558440f(11x)\\&-120187008f(10x)+300467520f(9x)\\&-600935040f(8x)+976519440f(7x)\\&-1302025920f(6x)+1432236264f(5x)\\&-1302180960f(4x)+977992320f(3x)\\&-609772320f(2x)-1.885985637\times10^{22}f(x)=0.\end{aligned}\tag{3.12}$$

Subtracting equations (3.11) and (3.12) , we get

$$\begin{aligned}&19380f(14x)-348840f(13x)+2969985f(12x)\\&-15904520f(11x)+60089628f(10x)\\&-170285600f(9x)+375594090f(8x)\\&-660212000f(7x)+938941620f(6x)\\&-1090171512f(5x)+1037736015f(4x)\\&-810820440f(3x)-6.082255014\times10^{17}f(2x)\\&+2.639698679\times10^{22}f(x)=0.\end{aligned}\tag{3.13}$$

Replacing (x,y) with $(4x,x)$, and multiplying the resulting equation by 19380, we get

$$\begin{aligned}&19380f(14x)-387600f(13x)+3682200f(12x)\\&-22093200f(11x)+93896100f(10x)\\&-300467520f(9x)+751168800f(8x)\\&-1502337600f(7x)+2441317980f(6x)\\&-3255452400f(5x)+3584253480f(4x)\\&-327715800f(3x)+2535194700f(2x)\\&-4.714964092\times10^{22}f(x)=0.\end{aligned}\tag{3.14}$$

Subtracting equations (3.13) and (3.14) , we get

$$\begin{aligned}&38760f(13x)-712215f(12x)+6188680f(11x)\\&-33806472f(10x)+130181920f(9x)\\&-375574710f(8x)+842125600f(7x)\\&-1502376360f(6x)+2165280888f(5x)\\&-2546517465f(4x)+2466337560f(3x)\\&-6.082255039\times10^{17}f(2x)\\&+7.354662771\times10^{22}f(x)=0.\end{aligned}\tag{3.15}$$

Replacing (x,y) with $(3x,x)$, and multiplying the resulting equation by 38760, we get

$$\begin{aligned}&38760f(13x)-775200f(12x)+7364400f(11x)\\&-44186400f(10x)+187792200f(9x)\\&-600935040f(8x)+1502376360f(7x)\\&-3005450400f(6x)+4889961600f(5x)\\&-6554316000f(4x)+7348934760f(3x)\\&-7111064640f(2x)-9.429928184\times10^{22}f(x)=0.\end{aligned}\tag{3.16}$$

Subtracting equations (3.15) and (3.16), we get

$$\begin{aligned}&62985f(12x)-1175720f(11x)+10379928f(10x)\\&-57610280f(9x)+225360330f(8x)\\&-660250760f(7x)+1503074040f(6x)\\&-2724680712f(5x)+4007798535f(4x)\\&-4882597200f(3x)-6.082254968\times10^{17}f(2x)\\&+1.678459096\times10^{23}f(x)=0.\end{aligned}\tag{3.17}$$

Replacing (x,y) with $(2x,x)$, and multiplying the resulting equation by 62985, we get

$$\begin{aligned}&62985f(12x)-1259700f(11x)+11967150f(10x)\\&-71802900f(9x)+305225310f(8x)\\&-977779140f(7x)+2453265750f(6x)\\&-4954400100f(5x)+8239382775f(4x)\\&-1.55548004\times10^{10}f(3x)\\&+1.407815526\times10^{10}f(2x)\\&-1.53235333\times10^{23}f(x)=0.\end{aligned}\tag{3.18}$$

Subtracting equations (3.17) and (3.18), we get

$$\begin{aligned}&83980f(11x)-1587222f(10x)+14192620f(9x)\\&-79864980f(8x)+317528380f(7x)\\&-950191710f(6x)+2229719388f(5x)\\&-4231584240f(4x)+6672882840f(3x)\\&-6.082255109\times10^{17}f(2x)\\&+3.210822426\times10^{23}f(x)=0.\end{aligned}\tag{3.19}$$

Replacing (x,y) with (x,x), and multiplying the resulting equation by 83980, we get

$$\begin{aligned}&83980f(11x)-1679600f(10x)+16040180f(9x)\\&-97416800f(8x)+422839300f(7x)\\&-1397763120f(6x)+3661947900f(5x)\\&-7812155520f(4x)+1.38340254\times10^{10}f(3x)\\&-2.06154104\times10^{10}f(2x)\\&-2.043151106\times10^{23}f(x)=0.\end{aligned}\tag{3.20}$$

Subtracting equations (3.19) and (3.20), we get

$$\begin{aligned}&92378f(10x)-1847560f(9x)+17551820f(8x)\\&-105310920f(7x)+447571410f(6x)\\&-1432228512f(5x)+3580571280f(4x)\\&-7161142560f(3x)-6.082254903\times10^{17}f(2x)\\&+5.253973532\times10^{23}f(x)=0.\end{aligned}\tag{3.21}$$

Replacing (x,y) with $(0,x)$, and multiplying the resulting equation by 92378, we get

$$\begin{aligned}&92378f(10x)-1847560f(9x)+17551820f(8x)\\&-105310920f(7x)+447571410f(6x)\\&-1432228512f(5x)+3580571280f(4x)\\&-7161142560f(3x)+1.163685666\times10^{10}f(2x)\\&+1.123733108\times10^{23}f(x)=0.\end{aligned}\tag{3.22}$$

Subtracting equations (3.21) and (3.22), we get

$$\begin{aligned}&-6.082255019\times10^{17}f(2x)\\&+6.37770664\times10^{23}f(x)=0.\\&f(2x)=2^{20}f(x).\end{aligned}\tag{3.23}$$

for all $x\in X$. By [17, Theorems 3.5 and 3.6], f is a generalized polynomial function of degree atmost 20, that is of the form

$$\begin{aligned}f(x)=&A^{20}x+A^{19}x+A^{18}x+A^{17}x+A^{16}x+A^{15}x+A^{14}x\\&+A^{13}x+A^{12}x+A^{11}x+A^{10}x\\&+A^{9}x+A^{8}x+A^{7}x+A^{6}x+A^{5}x\\&+A^{4}x+A^{3}x+A^{2}x+A^{1}x+A^{0}x.\end{aligned}\tag{3.24}$$

Here, $A^0x=A^0$ is an arbitrary element of Y and A^ix is the diagonal of i-additive symmetric map $A_i:X^i\to Y,(i=1,2,...,20)$. By $f(0)=0$ and $f(-x)=-f(x)$ for all $x\in X$, we get $A^0x=A^0=0$ and the function f is even. Thus, we have

$$\begin{aligned}&A^{19}x=A^{17}x=A^{15}x=A^{13}x=A^{11}x=A^{9}x\\&=A^{7}x=A^{5}x=A^{3}x=A^{1}x=0.\end{aligned}$$

It follows that

$$\begin{aligned}f(x)=&A^{20}x+A^{18}x+A^{16}x+A^{14}x\\&+A^{12}x+A^{10}x+A^{8}x+A^{6}x+A^{4}x+A^{2}x.\end{aligned}$$

Utilizing (3.23) and $A^n(rx)=r^nA^n(x)$ whenever $x\in X$ and $r\in Q$, we obtain

$$\begin{aligned}&2^{20}(A^{20}x+A^{18}x+A^{16}x+A^{14}x\\&+A^{12}x+A^{10}x+A^{8}x+A^{6}x+A^{4}x+A^{2}x)\\&=2^{20}A^{20}x+2^{18}A^{18}x+2^{16}A^{16}x\\&+2^{14}A^{14}x+2^{12}A^{12}x+2^{10}A^{10}x\\&+2^{8}A^{8}x+2^{6}A^{6}x+2^{4}A^{4}x+2^{2}A^{2}x\end{aligned}$$

for all $x\in X$ and $r\in Q$. It follows that

$$\begin{aligned}&A^{18}x+A^{16}x+A^{14}x+A^{12}x+A^{10}x\\&+A^{8}x+A^{6}x+A^{4}x+A^{2}x=0.\end{aligned}$$

for all $x\in X$. Hence $f(x)=A^{20}x$. Conversely, assume that $f(x)=A^{20}x$ for all $x\in X$. where $A^{20}x$ is the diagonal of the 20-additive symmetric map$A^{20}:X^{20}\to Y$ form

$$\begin{aligned}A^{20}(x+y)=&A^{20}x+A^{20}y+20A^{19.1}(x,y)+190A^{18.2}(x,y)\\&+1140A^{17.3}(x,y)+4845A^{16.4}(x,y)\\&+15504A^{15.5}(x,y)+38760A^{14.6}(x,y)\\&+77520A^{13.7}(x,y)+125970A^{12.8}(x,y)\\&+167960A^{11.9}(x,y)+184756A^{10.10}(x,y)\\&+167960A^{9.11}(x,y)+125970A^{8.12}(x,y)\\&+77520A^{7.13}(x,y)+38760A^{6.14}(x,y)\\&+15504A^{5.15}(x,y)+4845A^{4.16}(x,y)\\&+1140A^{3.17}(x,y)+190A^{2.18}(x,y)20A^{1.19}(x,y),\end{aligned}$$

and

$$\begin{aligned}&A^{20}(rx)=r^{20}A^{20}x,A^{19.1}(x,ry)=rA^{19.1}(x,y),\\&A^{18.2}(x,ry)=r^{2}A^{18.2}(x,y),\\&A^{17.3}(x,ry)=r^{3}A^{17.3}(x,y),A^{16.4}(x,ry)=r^{4}A^{16.4}(x,y),\\&A^{15.5}(x,ry)=r^{5}A^{15.5}(x,y),\\&A^{14.6}(x,ry)=r^{6}A^{14.6}(x,y),A^{13.7}(x,ry)=r^{7}A^{13.7}(x,y),\\&A^{12.8}(x,ry)=r^{8}A^{12.8}(x,y),\\&A^{11.9}(x,ry)=r^{9}A^{11.9}(x,y),A^{10.10}(x,ry)=r^{10}A^{10.10}(x,y),\\&A^{9.11}(x,ry)=r^{11}A^{9.11}(x,y),\\&A^{8.12}(x,ry)=r^{12}A^{8.12}(x,y),A^{7.13}(x,ry)=r^{13}A^{7.13}(x,y),\\&A^{6.14}(x,ry)=r^{14}A^{6.14}(x,y),\\&A^{5.15}(x,ry)=r^{15}A^{5.15}(x,y),A^{4.16}(x,ry)=r^{16}A^{4.16}(x,y),\\&A^{3.17}(x,ry)=r^{17}A^{3.17}(x,y),\\&A^{2.18}(x,ry)=r^{18}A^{2.18}(x,y),\\&A^{1.19}(x,ry)=r^{19}A^{1.19}(x,y).\end{aligned}$$

for all $x,y\in X$ and $r\in Q$. We see that f satisfies the equation (1.1). This completes the proof of the theorem. □

4. Stability of icosic functional equation

Throughout this section, we will assume that X is a linear space and Y is a (β,p)- Banach space with (β,p)- norm $\|.\|_Y$. Let K be the modulus of concavity of $\|.\|_Y$.

We will establish the following stability for the icosic functional equation in quasi-β-normed spaces. For a given mapping $f : X \to Y$, we define the difference operator

$$\begin{aligned} Df(x,y) &= f(x+10y) - 20f(x+9y) + 190f(x+8y) \\ &- 1140f(x+7y) + 4845f(x+6y) - 15504f(x+5y) \\ &+ 38760f(x+4y) - 77520f(x+3y) + 125970\, f(x+2y) \\ &- 167960\, f(x+y) + 184756\, f(x) - 167960f(x-y) \\ &+ 125970f(x-2y) - 77520\, f(x-3y) + 38760\, f(x-4y) \\ &- 15504\, f(x-5y) + 4845\, f(x-6y) - 1140\, f(x-7y) \\ &+ 190f(x-8y) - 20f(x-9y) + f(x-10y) - 20f(y). \end{aligned} \tag{4.1}$$

Theorem 4.1. *Let $i \in -1,1$ be fixed and $\phi : X\times X \to [0,\infty)$ be a function such that there exists an $L<1$ with $\Phi(2^i x, 2^i y) \le 1048576^{i\beta} L\Phi(x,y)$ for all $x,y \in X$. Let $f : X \to Y$ be the mapping satisfying*

$$\|Df(x,y)\|_Y \le \phi(x,y) \qquad \forall x,y \in X \tag{4.2}$$

Then there exists a unique icosic mapping $N : X \to Y$ such that

$$\|f(x) - N(x)\|_Y \le \frac{1}{1048576^\beta\ |1-L^i|}\ \phi_N(x,y) \quad \forall\, x,y \in X, \tag{4.3}$$

where

$$\begin{aligned} \phi_N(x) &= 92378[K^2\ \phi(0,x) + K^3\ 6.90368948\times 10^{-14} \\ &\phi(0,0) + K^4\ 5.17776711\times 10^{-14}(\phi(x,x) \\ &+ \phi(x,-x)) + K^5\ 3.186318222\times 10^{-14}(\phi(2x,2x) \\ &+ \phi(2x,-2x)) + K^6\ 1.593159111\times 10^{-14}(\phi(3x,3x) \\ &+ \phi(3x,-3x)) + K^7\ 6.372636443\times 10^{-15}(\phi(4x,4x) \\ &+ \phi(4x,-4x)) + K^8\ 1.991448888\times 10^{-15}(\phi(5x,5x) \\ &+ \phi(5x,-5x)) + K^9\ 4.68576109\times 10^{-16}(\phi(6x,6x) \\ &+ \phi(6x,-6x)) + K^{10}\ 7.809603484\times 10^{-17}(\phi(7x,7x) \\ &+ \phi(7x,-7x)) + K^{11}\ 8.22063524\times 10^{-18}(\phi(8x,8x) \\ &+ \phi(8x,-8x)) + K^{12}\ 4.110317623\times 10^{-19}(\phi(9x,9x) \\ &+ \phi(9x,-9x))] + 83980\ [K^3\ \phi(x,x) + K^4\ 5.17776711 \\ &\times 10^{-14}\phi(0,0) + K^5\ 3.186318222\times 10^{-14}(\phi(x,x)+\phi(x,-x)) \\ &+ K^6\ 1.593159111\times 10^{-14}(\phi(2x,2x)+\phi(2x,-2x)) \\ &+ K^7\ 6.372636443\times 10^{-15}(\phi(3x,3x) \\ &+ \phi(3x,-3x)) + K^8\ 1.991448888\times 10^{-15}(\phi(4x,4x) \\ &+ \phi(4x,-4x)) + K^9\ 4.68576109\times 10^{-16}(\phi(5x,5x) \\ &+ \phi(5x,-5x)) + K^{10}\ 7.809603484\times 10^{-17}(\phi(6x,6x) \\ &+ \phi(6x,-6x)) + K^{11}\ 8.22063524\times 10^{-18}(\phi(7x,7x) \\ &+ \phi(7x,-7x)) + K^{12}\ 4.110317623\times 10^{-19}(\phi(8x,8x) \\ &+ \phi(8x,-8x))] + 62985\ [K^4\ \phi(2x,x) \\ &+ K^5\ 3.186318222\times 10^{-14}\phi(0,0) \\ &+ K^6\ 1.593159111\times 10^{-14}(\phi(x,x)+\phi(x,-x)) \\ &+ K^7\ 6.372636443\times 10^{-15}(\phi(2x,2x)+\phi(2x,-2x)) \\ &+ K^8\ 1.991448888\times 10^{-15}(\phi(3x,3x)+\phi(3x,-3x)) \\ &+ K^9\ 4.68576109\times 10^{-16}(\phi(4x,4x)+\phi(4x,-4x)) \\ &+ K^{10}\ 7.809603484\times 10^{-17}(\phi(5x,5x)+\phi(5x,-5x)) \\ &+ K^{11}\ 8.22063524\times 10^{-18}(\phi(6x,6x)+\phi(6x,-6x)) \\ &+ K^{12}\ 4.110317623\times 10^{-19}(\phi(7x,7x)+\phi(7x,-7x))] \\ &+ 38760\ [K^5\ \phi(3x,x) + K^6\ 1.593159111\times 10^{-14}\phi(0,0) \\ &+ K^7\ 6.372636443\times 10^{-15}(\phi(x,x)+\phi(x,-x)) \\ &+ K^8\ 1.991448888\times 10^{-15}(\phi(2x,2x)+\phi(2x,-2x)) \\ &+ K^9\ 4.68576109\times 10^{-16}(\phi(3x,3x)+\phi(3x,-3x)) \\ &+ K^{10}\ 7.809603484\times 10^{-17}(\phi(4x,4x)+\phi(4x,-4x)) \\ &+ K^{11}\ 8.22063524\times 10^{-18}(\phi(5x,5x)+\phi(5x,-5x)) \\ &+ K^{12}\ 4.110317623\times 10^{-19}(\phi(6x,6x)+\phi(6x,-6x))] \\ &+ 19380[K^6\ \phi(4x,x) + K^7\ 6.372636443\times 10^{-15}\phi(0,0) \\ &+ K^8\ 1.991448888\times 10^{-15}(\phi(x,x)+\phi(x,-x)) \\ &+ K^9\ 4.68576109\times 10^{-16}(\phi(2x,2x)+\phi(2x,-2x)) \\ &+ K^{10}\ 7.809603484\times 10^{-17}(\phi(3x,3x)+\phi(3x,-3x)) \\ &+ K^{11}\ 8.22063524\times 10^{-18}(\phi(4x,4x)+\phi(4x,-4x)) \\ &+ K^{12}\ 4.110317623\times 10^{-19}(\phi(5x,5x)+\phi(5x,-5x))] \\ &+ 7752\ [K^7\ \phi(5x,x) + K^8\ 1.991448888\times 10^{-15}\phi(0,0) \\ &+ K^9\ 4.68576109\times 10^{-16}(\phi(x,x)+\phi(x,-x)) \\ &+ K^{10}\ 7.809603484\times 10^{-17}(\phi(2x,2x)+\phi(2x,-2x)) \\ &+ K^{11}\ 8.22063524\times 10^{-18}(\phi(3x,3x)+\phi(3x,-3x)) \\ &+ K^{12}\ 4.110317623\times 10^{-19}(\phi(4x,4x)+\phi(4x,-4x))] \\ &\frac{1}{2}\ [4845\ [K^8\ \phi(6x,x) + K^9\ 4.68576109\times 10^{-16}\phi(0,0) \end{aligned}$$

$$+K^{10}\,7.809603484\times 10^{-17}(\phi(x,x)+\phi(x,-x))$$
$$+K^{11}\,8.22063524\times 10^{-18}(\phi(2x,2x)+\phi(2x,-2x))$$
$$+K^{12}\,4.110317623\times 10^{-19}(\phi(3x,3x)+\phi(3x,-3x))]$$
$$+1140\,[K^{9}\,\phi(7x,x)+K^{10}\,7.809603484\times 10^{-17}\phi(0,0)$$
$$+K^{11}\,8.22063524\times 10^{-18}(\phi(x,x)+\phi(x,-x))$$
$$+K^{12}\,4.110317623\times 10^{-19}(\phi(2x,2x)+\phi(2x,-2x))]$$
$$+190\,[K^{13}\,\phi(8x,x)+K^{14}\,8.22063524\times 10^{-18}\phi(0,0)$$
$$+K^{15}\,4.110317623\times 10^{-19}(\phi(x,x)+\phi(x,-x))]$$
$$+20\,[K^{16}\,\phi(9x,x)+K^{17}\,4.110317623\times 10^{-19}\phi(0,0)]]$$
$$+K^{11}\,\phi(10x,x)+\frac{K^{12}}{2}\,\phi(0,2x)$$
$$+K^{13}\,7.594058428\times 10^{-14}\phi(0,0)$$
$$+K^{14}\,6.90368948\times 10^{-14}(\phi(2x,2x)+\phi(2x,-2x))$$
$$+K^{15}\,5.17776711\times 10^{-14}(\phi(4x,4x)+\phi(4x,-4x))$$
$$+K^{16}\,3.186318222\times 10^{-14}(\phi(6x,6x)+\phi(6x,-6x))$$
$$+K^{17}\,1.593159111\times 10^{-14}(\phi(8x,8x)+\phi(8x,-8x))$$
$$+K^{18}\,6.372636443\times 10^{-15}(\phi(10x,10x)$$
$$+\phi(10x,-10x))+K^{19}\,1.991448888$$
$$\times 10^{-15}(\phi(12x,12x)+\phi(12x,-12x))$$
$$+K^{20}\,4.68576109\times 10^{-16}(\phi(14x,14x)$$
$$+\phi(14x,-14x))+K^{21}\,7.809603484$$
$$\times 10^{-17}(\phi(16x,16x)+\phi(16x,-16x))$$
$$+K^{22}\,8.22063524\times 10^{-18}(\phi(18x,18x)$$
$$+\phi(18x,-18x))+K^{23}\,4.110317623$$
$$\times 10^{-19}(\phi(20x,20x)+\phi(20x,-20x)) \tag{4.4}$$

Proof. Replacing $x=0,\ y=0$ in (4.2), we get

$$\|f(0)\|_Y \le \left(4.110317623\ \times\ 10^{-19}\right)^{\beta}\ \phi(0,0). \tag{4.5}$$

Replacing (x,y) with (x,x) in (4.2), we get

$$\big\|f(11x)-20f(10x)+190f(9x)-1140f(8x)$$
$$+4845f(7x)-15504f(6x)$$
$$+38760f(5x)-77520f(4x)+125970f(3x)$$
$$-167960f(2x)+184756f(x)$$
$$-167960f(0)+125970f(-x)$$
$$-77520f(-2x)+38760f(-3x)$$
$$-15504f(-4x)+4845f(-5x)$$
$$-1140f(-6x)+190f(-7x)$$
$$-20f(-8x)+f(-9x)-20!\ f(x)\big\|\le\phi(x,x) \tag{4.6}$$

Replacing (x,y) in $(x,-x)$ in (4.2), we obtain that

$$\big\|f(-9x)-20f(-8x)+190f(-7x)-1140f(-6x)$$
$$+4845f(-5x)-15504f(-4x)+38760f(-3x)$$
$$-77520f(-2x)+125970f(-x)$$
$$-167960f(0)+184756f(x)$$
$$-167960f(2x)+125970f(3x)$$
$$-77520f(4x)+38760f(5x)$$
$$-15504f(6x)+4845f(7x)$$
$$-1140f(8x)+190f(9x)$$
$$-20f(10x)+f(11x)-20!\ f(x)\big\|\le\phi(x,\ -x) \tag{4.7}$$

From equation (4.6) and (4.7), we get

$$\|f(x)+f(-x)\|_Y \le \left(4.110317623\ \times\ 10^{-19}\right)^{\beta}\ (\phi(x,x)+\phi(x,-x)) \tag{4.8}$$

Replacing (x,y) with $(0,2x)$ in (4.2), one obtains

$$\big\|f(20x)-20f(18x)+190f(16x)-1140f(14x)$$
$$+4845f(12x)-15504f(10x)$$
$$+38760f(8x)-77520f(6x)+125970f(4x)$$
$$-167960f(2x)-167960f(-2x)$$
$$+125970f(-4x)-77520f(-6x)+38760f(-8x)$$
$$-15504f(-10x)+4845f(-12x)$$
$$-1140f(-14x)+190f(-16x)-20f(-18x)$$
$$+f(-20x)-20!\ f(2x)\big\|\le\ \frac{1}{2}\ \phi(0,2x) \tag{4.9}$$

Therefore, from (4.7), (4.8) and (4.9), we have

$$\|f(20x)-20f(18x)+190f(16x)-1140f(14x)$$
$$+4845f(12x)-15504f(10x)+38760f(8x)$$
$$-77520f(6x)+125970f(4x)-1.216451004\times 10^{18}\ f(2x)\|$$
$$\le\ \frac{1}{2}\ \phi(0,2x)+K^{2}\,7.594058428\times 10^{-14}\phi(0,0)$$
$$+K^{3}\,6.90368948\times 10^{-14}\ (\phi(2x,2x)$$
$$+\phi(2x,-2x))+K^{4}\,5.17776711\times 10^{-14}(\phi(4x,4x)$$
$$+\phi(4x,-4x))+K^{5}\,3.186318222\times 10^{-14}(\phi(6x,6x)$$
$$+\phi(6x,-6x))+K^{6}\,1.593159111\times 10^{-14}(\phi(8x,8x)$$
$$+\phi(8x,-8x))+K^{7}\,6.372636443\times 10^{-15}(\phi(10x,10x)$$
$$+\phi(10x,-10x))+K^{8}\,1.991448888\times 10^{-15}(\phi(12x,12x)$$
$$+\phi(12x,-12x))+K^{9}\,4.68576109\times 10^{-16}(\phi(14x,14x)$$
$$+\phi(14x,-14x))+K^{10}\,7.809603484\times 10^{-17}(\phi(16x,16x)$$
$$+\phi(16x,-16x))+K^{11}\,8.22063524\times 10^{-18}(\phi(18x,18x)$$
$$+\phi(18x,-18x))+K^{12}\,4.110317623\times 10^{-19}$$
$$(\phi(20x,20x)+\phi(20x,-20x)) \tag{4.10}$$

Replacing (x,y) with $(10x,x)$ in (4.2), we obtains

$$\|f(20x)-20f(19x)+190f(18x)-1140f(17x) +4845f(16x)-15504f(15x)+38760f(14x) -77520f(13x)+125970f(12x)-167960f(11x) +184756f(10x)-167960f(9x)+125970f(8x) -77520f(7x)+38760f(6x)-15504f(5x) +4845f(4x)-1140f(3x)+190f(2x) -2.432902008\times 10^{18}\ f(x)\| \leq \phi(10x,x) \quad (4.11)$$

From (4.10) and (4.11), we arrive at

$$\|20f(19x)-210f(18x)+1140f(17x)-4655f(16x) +15504f(15x)-39900f(14x)+77520f(13x) -121125f(12x)+167960f(11x)-200260f(10x) +167960f(9x)-87210f(8x)+77520f(7x) -116280f(6x)+15504f(5x)-121125f(4x) +1140f(3x)-1.216451004\times 10^{18}f(2x) +2.432902008\times 10^{18}\ f(x)\| \leq K\ \phi(10x,x) +\frac{K^2}{2}\ \phi(0,2x)+K^3\ 7.594058428\times 10^{-14}\phi(0,0) +K^4\ 6.90368948\times 10^{-14}\ (\phi(2x,2x)+\phi(2x,-2x)) +K^5\ 5.17776711\times 10^{-14}(\phi(4x,4x)+\phi(4x,-4x)) +K^6\ 3.186318222\times 10^{-14}(\phi(6x,6x)+\phi(6x,-6x)) +K^7\ 1.593159111\times 10^{-14}(\phi(8x,8x)+\phi(8x,-8x)) +K^8\ 6.372636443\times 10^{-15}(\phi(10x,10x) +\phi(10x,-10x))+K^9\ 1.991448888 \times 10^{-15}(\phi(12x,12x)+\phi(12x,-12x)) +K^{10}\ 4.68576109\times 10^{-16}(\phi(14x,14x) +\phi(14x,-14x))+K^{11}\ 7.809603484 \times 10^{-17}(\phi(16x,16x)+\phi(16x,-16x)) +K^{12}\ 8.22063524\times 10^{-18}(\phi(18x,18x) +\phi(18x,-18x))+K^{13}\ 4.110317623 \times 10^{-19}(\phi(20x,20x)+\phi(20x,-20x)) \quad (4.12)$$

Replacing (x,y) with $(9x,x)$ in (4.2), we obtains

$$\|20f(19x)-400f(18x)+3800f(17x)-22800f(16x) +96900f(15x)-310080f(14x)+775200f(13x) -1550400f(12x)+2519400f(11x)-3359200f(10x) +3695120f(9x)-3359200f(8x)+2519400f(7x) -1550400f(6x)+775200f(5x)-310080f(4x) +96900f(3x)-22800f(2x) +4.865804016\times 10^{19}f(x)\| \leq 20\ [K\ \phi(9x,x)+K^2\ 4.110317623 \times 10^{-19}\phi(0,0)] \quad (4.13)$$

From (4.12) and (4.13), we assumed that

$$\|190f(18x)-2660f(17x)+18145f(16x) -81396f(15x)+270180f(14x)-697680f(13x) +1429275f(12x)-2351440f(11x)+3158940f(10x) -3527160f(9x)+32711990f(8x)-2441880f(7x) +1434120f(6x)-759696f(5x)+431205f(4x) -95160f(3x)-1.216451004\times 10^{18}\ f(2x) +5.109094217\times 10^{19}f(x)\| \leq 20\ [K^2\ \phi(9x,x)+K^3\ 4.110317623 \times 10^{-19}\phi(0,0)]+K^2\ \phi(10x,x) +\frac{K^3}{2}\ \phi(0,2x)+K^4\ 7.594058428 \times 10^{-14}\phi(0,0)+K^5\ 6.90368948\times 10^{-14}\ (\phi(2x,2x) +\phi(2x,-2x))+K^6\ 5.17776711\times 10^{-14}(\phi(4x,4x) +\phi(4x,-4x))+K^7\ 3.186318222\times 10^{-14}(\phi(6x,6x) +\phi(6x,-6x))+K^8\ 1.593159111\times 10^{-14}(\phi(8x,8x) +\phi(8x,-8x))+K^9\ 6.372636443\times 10^{-15}(\phi(10x,10x) +\phi(10x,-10x))+K^{10}\ 1.991448888\times 10^{-15}(\phi(12x,12x) +\phi(12x,-12x))+K^{11}\ 4.68576109\times 10^{-16}(\phi(14x,14x) +\phi(14x,-14x))+K^{12}\ 7.809603484\times 10^{-17}(\phi(16x,16x) +\phi(16x,-16x))+K^{13}\ 8.22063524 \times 10^{-18}(\phi(18x,18x)+\phi(18x,-18x)) +K^{14}\ 4.110317623\times 10^{-19}(\phi(20x,20x) +\phi(20x,-20x)) \quad (4.14)$$

Replacing (x,y) with $(8x,x)$ in (4.2), one obtains

$$\|190f(18x)-3800f(17x)+36100f(16x) -216600f(15x)+920550f(14x)-2945760f(13x) +7364400f(12x)-1472880f(11x)+23934300f(10x) -31912400f(9x)+35103640f(8x)-31912400f(7x) +23934300f(6x)-1472880f(5x)+7364400f(4x) -2945760f(3x)+920740f(2x)-4.622513816\times 10^{20}\ f(x)\| \leq 190\ [K\ \phi(8x,x)+K^2\ 8.22063524\times 10^{-18}\phi(0,0) +K^3\ 4.110317623\times 10^{-19}(\phi(x,x)+\phi(x,-x)) \quad (4.15)$$

From (4.14) and (4.15), we get

$$\|1140f(17x)-17955f(16x)+135204f(15x) -650370f(14x)+2248080f(13x)-207753360f(12x) +12377360f(11x)-20761406f(10x)+28385240f(9x) -31831650f(8x)+2947520f(7x)-22500180f(6x) +13969104f(5x)-6933195f(4x)+2850000f(3x)$$

$$
\begin{aligned}
&-1.216451004\times 10^{18}\ f(2x)+5.133423238\times 10^{20}f(x)\|\\
&\leq\ 190\ [K^2\ \phi(8x,x)+K^3\ 8.22063524\times 10^{-18}\phi(0,0)\\
&+K^4\ 4.110317623\times 10^{-19}(\phi(x,x)+\phi(x,-x))]\\
&+20\ [K^3\ \phi(9x,x)+K^4\ 4.110317623\times 10^{-19}\phi(0,0)]\\
&+K^3\ \phi(10x,x)+\frac{K^4}{2}\ \phi(0,2x)+K^5\ 7.594058428\\
&\times 10^{-14}\phi(0,0)+K^6\ 6.90368948\times 10^{-14}\ (\phi(2x,2x)\\
&+\phi(2x,-2x))+K^7\ 5.17776711\times 10^{-14}(\phi(4x,4x)\\
&+\phi(4x,-4x))+K^8\ 3.186318222\times 10^{-14}(\phi(6x,6x)\\
&+\phi(6x,-6x))+K^9\ 1.593159111\times 10^{-14}(\phi(8x,8x)\\
&+\phi(8x,-8x))+K^{10}\ 6.372636443\times 10^{-15}(\phi(10x,10x)\\
&+\phi(10x,-10x))+K^{11}\ 1.991448888\\
&\times 10^{-15}(\phi(12x,12x)+\phi(12x,-12x))+K^{12}\ 4.68576109\\
&\times 10^{-16}(\phi(14x,14x)+\phi(14x,-14x))\\
&+K^{13}\ 7.809603484\times 10^{-17}(\phi(16x,16x)\\
&+\phi(16x,-16x))+K^{14}\ 8.22063524\times 10^{-18}\\
&(\phi(18x,18x)+\phi(18x,-18x))+K^{15}\ 4.110317623\\
&\times 10^{-19}(\phi(20x,20x)+\phi(20x,-20x)) \qquad (4.16)
\end{aligned}
$$

Replcing (x,y) with $(7x,x)$ in (4.2), we get

$$
\begin{aligned}
&\|1140f(17x)-22800f(16x)+216600f(15x)\\
&-1299600f(14x)+5523300f(13x)\\
&-17674560f(12x)+44186400f(11x)\\
&-88372800f(10x)+143605800f(9x)\\
&-191474400f(8x)+2201621840f(7x)\\
&-191474400f(6x)+143605800f(5x)\\
&-88372800f(4x)+44187540f(3x)\\
&-17697360f(2x)-2.773508289\times 10^{21}f(x)\|\\
&\leq\ 1140\ [K\ \phi(7x,x)+K^2\ 7.809603484\\
&\times 10^{-17}\phi(0,0)+K^3\ 8.22063524\\
&\times 10^{-18}(\phi(x,x)+\phi(x,-x))\\
&+K^4\ 4.110317623\times 10^{-19}(\phi(2x,2x)+\phi(2x,-2x))]\\
&\qquad (4.17)
\end{aligned}
$$

From (4.16) and (4.17), we get

$$
\begin{aligned}
&\|4845f(16x)-81396f(15x)+649230f(14x)\\
&-3275220f(13x)+11739435f(12x)\\
&-31809040f(11x)-67597440f(10x)\\
&-115220560f(9x)+159642750f(8x)\\
&-181151320f(7x)+168974220f(6x)\\
&-129636696f(5x)+81439605f(4x)\\
&-41337540f(3x)-1.216451004\times 10^{18}f(2x)\\
&+3.286850613\times 10^{21}f(x)\|\\
&\leq\ 1140\ [K^2\ \phi(7x,x)+K^3\ 7.809603484\\
&\times 10^{-17}\phi(0,0)+K^4\ 8.22063524\times 10^{-18}(\phi(x,x)\\
&+\phi(x,-x))+K^5\ 4.110317623\times 10^{-19}(\phi(2x,2x)\\
&+\phi(2x,-2x))]+190\ [K^3\ \phi(8x,x)\\
&+K^4\ 8.22063524\times 10^{-18}\phi(0,0)\\
&+K^5\ 4.110317623\times 10^{-19}(\phi(x,x)\\
&+\phi(x,-x))]+20\ [K^4\ \phi(9x,x)\\
&+K^5\ 4.110317623\times 10^{-19}\phi(0,0)]+\Gamma(x).
\end{aligned}
$$

Where,

$$
\begin{aligned}
\Gamma(x)=&\ K^4\ \phi(10x,x)+\frac{K^5}{2}\ \phi(0,2x)\\
&+K^6\ 7.594058428\times 10^{-14}\phi(0,0)\\
&+K^7\ 6.90368948\times 10^{-14}\ (\phi(2x,2x)\\
&+\phi(2x,-2x))+K^8\ 5.17776711\times 10^{-14}(\phi(4x,4x)\\
&+\phi(4x,-4x))+K^9\ 3.186318222\times 10^{-14}(\phi(6x,6x)\\
&+\phi(6x,-6x))+K^{10}\ 1.593159111\times 10^{-14}(\phi(8x,8x)\\
&+\phi(8x,-8x))+K^{11}\ 6.372636443\times 10^{-15}(\phi(10x,10x)\\
&+\phi(10x,-10x))+K^{12}\ 1.991448888\times 10^{-15}(\phi(12x,12x)\\
&+\phi(12x,-12x))+K^{13}\ 4.68576109\times 10^{-16}(\phi(14x,14x)\\
&+\phi(14x,-14x))+K^{14}\ 7.809603484\times 10^{-17}(\phi(16x,16x)\\
&+\phi(16x,-16x))+K^{15}\ 8.22063524\times 10^{-18}(\phi(18x,18x)\\
&+\phi(18x,-18x))+K^{16}\ 4.110317623\times 10^{-19}\\
&(\phi(20x,20x)+\phi(20x,-20x)) \qquad (4.18)
\end{aligned}
$$

Replcing (x,y) with $(6x,x)$ in (4.2), we get

$$
\begin{aligned}
&\|4845f(16x)-96900f(15x)+920550f(14x)\\
&-5523300f(13x)+23474025f(12x)\\
&-75116880f(11x)+187792200f(10x)\\
&-375584400f(9x)+610324650f(8x)\\
&-813766200f(7x)+895142820f(6x)\\
&-813766200f(5x)+610329495f(4x)\\
&-375681300f(3x)+188712750f(2x)\\
&-1.178741023\times 10^{22}\ f(x)\|\\
&\leq\ 4845\ [K\ \phi(6x,x)+K^2\ 4.68576109\\
&\times 10^{-16}\phi(0,0)+K^3\ 7.809603484\\
&\times 10^{-17}(\phi(x,x)+\phi(x,-x))\\
&+K^4\ 8.22063524\times 10^{-18}(\phi(2x,2x)\\
&+\phi(2x,-2x))+K^5\ 4.110317623\times 10^{-19}(\phi(3x,3x)\\
&+\phi(3x,-3x)) \qquad (4.19)
\end{aligned}
$$

From (4.18) and (4.19), we get

$$\begin{aligned}&\|7752f(15x)-135660f(14x)+1124040f(13x)\\&-5867295f(12x)+21653920f(11x)\\&-60097380f(10x)+130181920f(9x)\\&-225340950f(8x)+316307440f(7x)\\&-363084300f(6x)+342064752f(5x)\\&-264444945f(4x)+167171880f(3x)\\&-6.08225502\times10^{17}\ f(2x)\\&-7.53713042\times10^{21}f(x)\|\\&\leq\ \frac{1}{2}\ \{4845[K^2\ \phi(6x,x)\\&+K^3\ 4.68576109\times10^{-16}\phi(0,0)\\&+K^4\ 7.809603484\times10^{-17}(\phi(x,x)\\&+\phi(x,-x))+K^5\ 8.22063524\times10^{-18}(\phi(2x,2x)\\&+\phi(2x,-2x))+K^6\ 4.110317623\times10^{-19}(\phi(3x,3x)\\&+\phi(3x,-3x))]+1140\ [K^3\ \phi(7x,x)\\&+K^4\ 7.809603484\times10^{-17}\phi(0,0)\\&+K^5\ 8.22063524\times10^{-18}(\phi(x,x)\\&+\phi(x,-x))+K^6\ 4.110317623\times10^{-19}(\phi(2x,2x)\\&+\phi(2x,-2x))]+\ 190\ [K^7\ \phi(8x,x)\\&+K^8\ 8.22063524\times10^{-18}\phi(0,0)\\&+K^9\ 4.110317623\times10^{-19}(\phi(x,x)\\&+\phi(x,-x))]+20\ [K^{10}\ \phi(9x,x)\\&+K^{11}\ 4.110317623\times10^{-19}\phi(0,0)]+K\Gamma(x)\}\end{aligned}$$

Replcing (x,y) with $(5x,x)$ in (4.2), we get

$$\begin{aligned}&\|7752f(15x)-155040f(14x)+1472880f(13x)\\&-8837280f(12x)+37558440f(11x)\\&-120187008f(10x)+300467520f(9x)\\&-600935040f(8x)+976519440f(7x)\\&-1302025920f(6x)+1432236264f(5x)\\&-1302180960f(4x)+977992320f(3x)\\&-609772320\ f(2x)+1.885985637\times10^{22}f(x)\|\\&\leq\ 7752\ [K\ \phi(5x,x)+K^2\ 1.99148888\\&\times10^{-15}\phi(0,0)+K^3\ 4.68576109\times10^{-16}(\phi(x,x)\\&+\phi(x,-x))+K^4\ 7.809603484\times10^{-17}(\phi(2x,2x)\\&+\phi(2x,-2x))+K^5\ 8.22063524\times10^{-18}(\phi(3x,3x)\\&+\phi(3x,-3x))+K^6\ 4.110317623\times10^{-19}\\&(\phi(4x,4x)+\phi(4x,-4x))]\end{aligned}\tag{4.20}$$

From (4.19) and (4.20), one obtains

$$\begin{aligned}&\|19380f(14x)-348840f(13x)+2969985f(12x)\\&-15904520f(11x)+60089628f(10x)\\&-170285600f(9x)+375594090f(8x)\\&-660212000f(7x)+938941620\ f(6x)\\&-1090171512\ f(5x)+10377360\ f(4x)\\&-810820440\ f(3x)\ +6.08225014\times10^{17}f(2x)\\&-2.639698679\times10^{22}\ f(x)\|\ \leq\ \delta(x)\end{aligned}\tag{4.21}$$

where,

$$\begin{aligned}\delta(x)=&\ 7752\ [K^2\ \phi(5x,x)\\&+K^3\ 1.99148888\times10^{-15}\phi(0,0)\\&+K^4\ 4.68576109\times10^{-16}(\phi(x,x)\\&+\phi(x,-x))+K^5\ 7.809603484\\&\times10^{-17}(\phi(2x,2x)+\phi(2x,-2x))\\&+K^6\ 8.22063524\times10^{-18}(\phi(3x,3x)\\&+\phi(3x,-3x))+K^7\ 4.110317623\\&\times10^{-19}(\phi(4x,4x)+\phi(4x,-4x))]\\&+\frac{1}{2}\ \{4845[K^3\ \phi(6x,x)\\&+K^4\ 4.68576109\times10^{-16}\phi(0,0)\\&+K^5\ 7.809603484\times10^{-17}(\phi(x,x)\\&+\phi(x,-x))+K^6\ 8.22063524\times10^{-18}(\phi(2x,2x)\\&+\phi(2x,-2x))+K^7\ 4.110317623\\&\times10^{-19}(\phi(3x,3x)+\phi(3x,-3x))]\\&+1140\ [K^4\ \phi(7x,x)+K^5\ 7.809603484\\&\times10^{-17}\phi(0,0)+K^6\ 8.22063524\times10^{-18}(\phi(x,x)\\&+\phi(x,-x))+K^7\ 4.110317623\\&\times10^{-19}(\phi(2x,2x)+\phi(2x,-2x))]\\&+\ 190\ [K^8\ \phi(8x,x)+K^9\ 8.22063524\\&\times10^{-18}\phi(0,0)+K^{10}\ 4.110317623\\&\times10^{-19}(\phi(x,x)+\phi(x,-x))]\\&+20\ [K^{11}\ \phi(9x,x)+K^{12}\ 4.110317623\\&\times10^{-19}\phi(0,0)]+K^2\Gamma(x)\}\end{aligned}$$

Replacing (x,y) by $(4x,x)$ in (4.2), we get

$$\begin{aligned}&\|19380f(14x)-387600f(13x)+3682200f(12x)\\&-22093200f(11x)+93896100f(10x)\\&-300467520f(9x)+75116880f(8x)\\&-1502337600f(7x)+2441317980f(6x)\\&-3255452400f(5x)+35842534\ f(4x)\\&-3277158000\ f(3x)+2535194700\ f(2x)\\&-4.714964092\times10^{22}\ f(x)\|\\&\leq\ \ 19380\ [K\ \phi(4x,x)+K^2\ 6.376236443\\&\times10^{-15}\phi(0,0)+K^3\ 1.99148888\\&\times10^{-15}(\phi(x,x)+\phi(x,-x))\end{aligned}$$

$$+K^4\,4.68576109\times 10^{-16}(\phi(2x,2x)+\phi(2x,-2x))+K^5\,7.809603484\times 10^{-17}(\phi(3x,3x)+\phi(3x,-3x))+K^6\,8.22063524\times 10^{-18}(\phi(4x,4x)+\phi(4x,-4x))+K^7\,4.110317623\times 10^{-19}(\phi(5x,5x)+\phi(5x,-5x))] \tag{4.22}$$

From (4.21) and (4.22), we obtains that

$$\|38760f(13x)-712215f(12x)+6188680f(11x)-338064f(10x)+130181920f(9x)-375574710f(8x)+842125600f(7x)-1502376360f(6x)+2165280888f(5x)-2546517465f(4x)+2466337560f(3x)-6.08225502\times 10^{17}\,f(2x)-7.53713042\times 10^{21}f(x)\|\leq\ \rho(x).$$

where

$$\rho(x)=19380[K^2\,\phi(4x,x)+K^3\,6.376236443\times 10^{-15}\phi(0,0)+K^4\,1.99148888\times 10^{-15}(\phi(x,x)+\phi(x,-x))+K^5\,4.68576109\times 10^{-16}(\phi(2x,2x)+\phi(2x,-2x))+K^6\,7.809603484\times 10^{-17}(\phi(3x,3x)+\phi(3x,-3x))+K^7\,8.22063524\times 10^{-18}(\phi(4x,4x)+\phi(4x,-4x))+K^8\,4.110317623\times 10^{-19}(\phi(5x,5x)+\phi(5x,-5x))] \tag{4.23}$$

Replcing (x,y) with $(3x,x)$ in (4.2), we gets

$$\|38760f(13x)-775200f(12x)+7364400f(11x)-44186400f(10x)+187792200f(9x)-600935040f(8x)+1502376360f(7x)-3005450400f(6x)+4889961600f(5x)-655316000f(4x)+7348934760f(3x)-7111064640f(2x)-9.429928184\times 10^{22}f(x)\|\leq\ 38760\,[K\,\phi(3x,x)+K^2\,1.593159111\times 10^{-14}\phi(0,0)+K^3\,6.376236443\times 10^{-15}(\phi(x,x)+\phi(x,-x))+K^4\,1.99148888\times 10^{-15}(\phi(2x,2x)+\phi(2x,-2x))+K^5\,4.68576109\times 10^{-16}(\phi(3x,3x)+\phi(3x,-3x))+K^6\,7.809603484\times 10^{-17}(\phi(4x,4x)+\phi(4x,-4x))+K^7\,8.22063524\times 10^{-18}(\phi(5x,5x)+\phi(5x,-5x))+K^8\,4.110317623\times 10^{-19}(\phi(6x,6x)+\phi(6x,-6x))] \tag{4.24}$$

From (4.23) and (4.24), one can get

$$\|62985f(12x)-1175720f(11x)+10379928f(10x)-57610280f(9x)+225360330f(8x)-660250760f(7x)+1503074040f(6x)-2724680f(5x)+4007798535f(4x)-882597200\,f(3x)-6.08225502\times 10^{17}f(2x)-1.678459096\times 10^{23}\,f(x)\|\leq\ 38760\,[K^2\,\phi(3x,x)+K^3\,1.593159111\times 10^{-14}\phi(0,0)+K^4\,6.376236443\times 10^{-15}(\phi(x,x)+\phi(x,-x))+K^5\,1.99148888\times 10^{-15}(\phi(2x,2x)+\phi(2x,-2x))+K^6\,4.68576109\times 10^{-16}(\phi(3x,3x)+\phi(3x,-3x))+K^7\,7.809603484\times 10^{-17}(\phi(4x,4x)+\phi(4x,-4x))+K^8\,8.22063524\times 10^{-18}(\phi(5x,5x)+\phi(5x,-5x))+K^9\,4.110317623\times 10^{-19}(\phi(6x,6x)+\phi(6x,-6x))]+K^3\ \rho(x). \tag{4.25}$$

Replcing (x,y) with $(2x,x)$ in (4.2), we gets

$$\|62985f(12x)-1259700f(11x)+11967150f(10x)-71802900f(9x)+305225310f(8x)-9777779140f(7x)+2453265750f(6x)-4954400100f(5x)+8239382775f(4x)-1.155548004\times 10^{10}\,f(3x)+1.407815526\times 10^{10}f(2x)-1.53236333\times 10^{23}\,f(x)\|\leq\ 62985\,[K\,\phi(2x,x)+K^2\,3.186318222\times 10^{-14}\phi(0,0)+K^3\,1.593159111\times 10^{-14}(\phi(x,x)+\phi(x,-x))+K^4\,6.376236443\times 10^{-15}(\phi(2x,2x)+\phi(2x,-2x))+K^5\,1.99148888$$

$$\times 10^{-15}(\phi(3x,3x)+\phi(3x,-3x)) + K^6\, 4.68576109\times 10^{-16}(\phi(4x,4x) + \phi(4x,-4x)) + K^7\, 7.809603484\times 10^{-17}(\phi(5x,5x) + \phi(5x,-5x)) + K^8\, 8.22063524 \times 10^{-18}(\phi(6x,6x)+\phi(6x,-6x)) + K^9\, 4.110317623\times 10^{-19} (\phi(7x,7x)+\phi(7x,-7x))] \tag{4.26}$$

From (4.25) and (4.26), we get

$$\|83980f(11x)-1587222f(10x)+14192620f(9x) - 79864980f(8x)+317528380f(7x) - 950191710f(6x)+2229719388f(5x) - 4231584240f(4x)+6672882840f(3x) - 6.082255109\times 10^{17}f(2x) + 3.210822426\times 10^{23}f(x)\| \leq \sigma(x)$$

where,

$$\begin{aligned}\sigma(x) = 62985\, [&K^2\,\phi(2x,x) + K^3\, 3.18631822\times 10^{-14}\phi(0,0) + K^4\, 1.593159111(\phi(x,x)+\phi(x,-x)) \\ &+ K^5\, 6.376236443\times 10^{-15}(\phi(2x,2x) + \phi(2x,-2x)) + K^6\, 1.99148888 \times 10^{-15}(\phi(3x,3x)+\phi(3x,-3x)) \\ &+ K^7\, 4.68576109\times 10^{-16}(\phi(4x,4x) + \phi(4x,-4x)) + K^8\, 7.809603484 \times 10^{-17}(\phi(5x,5x)+\phi(5x,-5x)) \\ &+ K^9\, 8.22063524\times 10^{-18}(\phi(6x,6x) + \phi(6x,-6x)) + K^{10}\, 4.110317623 \times 10^{-19}(\phi(7x,7x)+\phi(7x,-7x))] \\ &+ 38760\, [K^3\,\phi(3x,x)+K^4\, 1.593159111 \times 10^{-14}\phi(0,0)+K^5\, 6.376236443 \times 10^{-15}(\phi(x,x)+\phi(x,-x)) \\ &+ K^6\, 1.99148888\times 10^{-15}(\phi(2x,2x) + \phi(2x,-2x)) + K^7\, 4.68576109 \times 10^{-16}(\phi(3x,3x)+\phi(3x,-3x)) \\ &+ K^8\, 7.809603484\times 10^{-17}(\phi(4x,4x) + \phi(4x,-4x)) + K^9\, 8.22063524 \times 10^{-18}(\phi(5x,5x)+\phi(5x,-5x)) \\ &+ K^{10}\, 4.110317623\times 10^{-19}(\phi(6x,6x) + \phi(6x,-6x))] + K^4\,\rho(x)\end{aligned} \tag{4.27}$$

Replacing (x,y) with (x,x) in (4.2), we get

$$\begin{aligned}&\|83980f(11x)-1679600f(10x)+16040180f(9x) - 97416800f(8x)+422839300f(7x) - 1397763120f(6x)+3661947900f(5x) \\ &- 781215520f(4x)+1.38340254\times 10^{10}f(3x) - 2.06154104\times 10^{10}f(2x)-2.043151106\times 10^{23}f(x)\| \\ &\leq\ 83980[K\phi(x,x)+K^2\, 5.17776711 \times 10^{-14}\phi(0,0)+K^3\, 3.186318222 \times 10^{-14}(\phi(x,x)+\phi(x,-x)) \\ &+ K^4\, 1.593159111\times 10^{-14}(\phi(2x,2x) + \phi(2x,-2x))+K^5\, 6.376236443 \times 10^{-15}(\phi(3x,3x)+\phi(3x,-3x)) \\ &+ K^6\, 1.99148888\times 10^{-15}(\phi(4x,4x) + \phi(4x,-4x))+K^7\, 4.68576109 \times 10^{-16}(\phi(5x,5x)+\phi(5x,-5x)) \\ &+ K^8\, 7.809603484\times 10^{-17}(\phi(6x,6x) + \phi(6x,-6x))+K^9\, 8.22063524 (\phi(7x,7x)+\phi(7x,-7x))+K^{10}\, 4.110317623 \\ &\times 10^{-19}(\phi(8x,8x)+\phi(8x,-8x))]\end{aligned} \tag{4.28}$$

From (4.27) and (4.28), we get

$$\begin{aligned}&\|92378f(10x)-1847560f(9x)+17551820f(8x) - 105310920f(7x)+44751410f(6x) - 1432228512f(5x)+3580571280f(4x) \\ &- 716114256f(3x)-6.082254903\times 10^{17}f(2x) + 5.253973532\times 10^{23}\ f(x)\| \\ &\leq\ 83980\ [K^2\ \phi(x,x)+K^3\ 5.17776711 \times 10^{-14}\phi(0,0)+K^4\ 3.186318222 \times 10^{-14}(\phi(x,x)+\phi(x,-x)) \\ &+ K^5\ 1.593159111\times 10^{-14}(\phi(2x,2x) + \phi(2x,-2x))+K^6\ 6.376236443 \times 10^{-15}(\phi(3x,3x)+\phi(3x,-3x)) \\ &+ K^7\ 1.99148888\times 10^{-15}(\phi(4x,4x) + \phi(4x,-4x))+K^8\ 4.68576109 \times 10^{-16}(\phi(5x,5x)+\phi(5x,-5x)) \\ &+ K^9\ 7.809603484\times 10^{-17}(\phi(6x,6x) + \phi(6x,-6x))+K^{10}\ 8.22063524 \times 10^{-18}(\phi(7x,7x)+\phi(7x,-7x)) \\ &+ K^{11}\ 4.110317623\times 10^{-19}(\phi(8x,8x) + \phi(8x,-8x))]+K\ \sigma(x)\end{aligned} \tag{4.29}$$

Replacing (x,y) with $(0,x)$ in (4.2), we get

$$\begin{aligned}
&\|92378f(10x)-1847560f(9x)+17551820f(8x)\\
&-1053109f(7x)+447571410f(6x)\\
&-1432228512f(5x)+3580571280f(4x)\\
&-71161142560f(3x)-1.163685666\times 10^{10}f(2x)\\
&+1.123733108\times 10^{23}\ f(x)\|\\
&\le\ 92378\ [K\ \phi(0,x)+K^2\ 6.90368948\\
&\times 10^{-14}\phi(0,0)+K^3\ 5.17776711\\
&\times 10^{-14}(\phi(x,x)+\phi(x,-x))\\
&+K^4\ 3.186318222\times 10^{-14}(\phi(2x,2x)\\
&+\phi(2x,-2x))+K^5\ 1.593159111\\
&\times 10^{-14}(\phi(3x,3x)+\phi(3x,-3x))\\
&+K^6\ 6.376236443\times 10^{-15}(\phi(4x,4x)\\
&+\phi(4x,-4x))+K^7\ 1.99148888\\
&\times 10^{-15}(\phi(5x,5x)+\phi(5x,-5x))\\
&+K^8\ 4.68576109\times 10^{-16}(\phi(6x,6x)\\
&+\phi(6x,-6x))+K^9\ 7.809603484\\
&\times 10^{-17}(\phi(7x,7x)+\phi(7x,-7x))\\
&+K^{10}\ 8.22063524\times 10^{-18}(\phi(8x,8x)\\
&+\phi(8x,-8x))+K^{11}\ 4.110317623\\
&\times 10^{-19}(\phi(9x,9x)+\phi(9x,-9x))]
\end{aligned} \tag{4.30}$$

From (4.29) and (4.30), we get

$$\begin{aligned}
&\|-6.082255019\times 10^{17}f(2x)\\
&+6.37770664\times 10^{23}f(x)\|\\
&\le\ 92378\ [K^2\ \phi(0,x)+K^3 6.90368948\\
&\times 10^{-14}\phi(0,0)+K^4\ 5.17776711\\
&\times 10^{-14}(\phi(x,x)+\phi(x,-x))\\
&+K^5\ 3.186318222\times 10^{-14}(\phi(2x,2x)\\
&+\phi(2x,-2x))+K^6\ 1.593159111\\
&\times 10^{-14}(\phi(3x,3x)+\phi(3x,-3x))\\
&+K^7\ 6.372636443\times 10^{-15}(\phi(4x,4x)\\
&+\phi(4x,-4x))+K^8\ 1.991448888\\
&\times 10^{-15}(\phi(5x,5x)+\phi(5x,-5x))\\
&+K^9\ 4.68576109\times 10^{-16}(phi(6x,6x)\\
&+\phi(6x,-6x))+K^{10}\ 7.809603484\\
&\times 10^{-17}(\phi(7x,7x)+\phi(7x,-7x))\\
&+K^{11}\ 8.22063524\times 10^{-18}(\phi(8x,8x)\\
&+\phi(8x,-8x))+K^{12}\ 4.110317623\\
&\times 10^{-19}(\phi(9x,9x)+\phi(9x,-9x))]\\
&+83980[K^3\ \phi(x,x)+K^4 5.17776711\\
&\times 10^{-14}\phi(0,0)+K^5\ 3.186318222\\
&\times 10^{-14}(\phi(x,x)+\phi(x,-x))\\
&=K^6\ 1.593159111\times 10^{-14}(\phi(2x,2x)\\
&+\phi(2x,-2x))+K^7\ 6.372636443\\
&\times 10^{-15}(\phi(3x,3x)+\phi(3x,-3x))\\
&+K^8\ 1.991448888\times 10^{-15}(\phi(4x,4x)\\
&+\phi(4x,-4x))+K^9\ 4.68576109\\
&\times 10^{-16}(\phi(5x,5x)+\phi(5x,-5x))\\
&=K^{10}\ 7.809603484\times 10^{-17}(\phi(6x,6x)\\
&+\phi(6x,-6x))+K^{11}\ 8.22063524\\
&\times 10^{-18}(\phi(7x,7x)+\phi(7x,-7x))\\
&+K^{12}\ 4.110317623\times 10^{-19}(\phi(8x,8x)\\
&+\phi(8x,-8x))]+K^2\ \sigma(x)
\end{aligned} \tag{4.31}$$

Therefore, $||f(2x)-2^{20}f(x)||_Y \le \phi_N(x)$ for all $x\ inX$. By Lemma. 2.1, there exists a unique mapping $N : X \to Y$ such that $N(2x) = 2^{20}N(X)$ and

$$||f(x)-N(x)||_Y \le \frac{1}{1048576^{\beta}|1-L^i|}\phi_N(x) \qquad \forall x\in X. \tag{4.32}$$

If remains to show that N is a icosic mapping. Form (4.2), we have

$$\begin{aligned}
&\left|\frac{Df(2^{im}x,2^{im}y)}{1048576^{im}}\right|_Y\\
&\le 1048576^{im\beta}\phi\left(2^{im}x,2^{im}y\right)\\
&\le 1048576^{im\beta}\left(1047576^{i\beta}L\right)^m\phi(x,y)\\
&=L^m\phi(x,y)
\end{aligned} \tag{4.33}$$

for all $x,y\in X$ and $m\in N$. Hence $||DN(x,y)||_Y = 0$ for all $x,y\in X$. Therefore, the mapping $N : X\to Y$ is a icosic mapping. □

The following corollary is an immediate consequence of Theorem 4.1 concerning the stability of icosic functional (1.1).

Corollary 4.2. *Let X be a quasi-α-normal space with quasi-α-normal $||\cdot||_X$, and let Y be a (β,p) Banach sapce with (β,p)-norm$||.||_Y$. Let δ,λ be positive numbers with $\lambda\neq\frac{20^{\beta}}{\alpha}$ and let $f : X\to Y$ be a mapping satisfying*

$$||Df(x,y)||_Y \le \delta\left(||x||_X^{\lambda}+||y||_X^{\lambda}\right) \qquad \forall x,y\in X. \tag{4.34}$$

Then there exist a unique icosic mapping $N : X \to Y$ such that

$$||f(x) - N(x)||_Y \leq \begin{cases} \frac{\delta_{\varepsilon\ \lambda}}{1048576^{\beta} - 2^{\alpha\lambda}}, & \lambda \in \left(0, \frac{20\beta}{\alpha}\right) \\ \frac{2^{\lambda\alpha}\ \delta_{\varepsilon\ \lambda}}{1048576^{\beta}\left(2^{\alpha\lambda} - 1048576^{\beta}\right)} ||x||_X^{\lambda}, & \lambda \in \left(\frac{20\beta}{\alpha}, \infty\right). \end{cases} \tag{4.35}$$

The following example shows that the assumption $\lambda \neq \frac{20\beta}{\alpha}$ cannot be omitted in corollary 4.2. The example is a modification of the well-known example of Gajda [6] for the additive functional inequality (see also [6]).

Example 4.3. *Let $\phi : \mathbb{R} \to \mathbb{R}$ defined by*

$$\phi(x) = \begin{cases} x^{20}, & for\ |x| < 1 \\ 1, & for\ |x| \geq 1 \end{cases} \tag{4.36}$$

consider the function $f : \mathbb{R} \to \mathbb{R}$ to be defined by

$$f(x) = \sum_{n=0}^{\infty} 4^{-20n}\phi(4^n x), \qquad \forall x \in \mathbb{R}.$$

Then f satisfies the following functional inequality

$$|Df(x,y)| \leq \frac{(2.432902008 \times 10^{18})(616666)^3}{616665} (\times)\left(|x|^{20} + |y|^{20}\right) \qquad \forall x, y \in \mathbb{R}. \tag{4.37}$$

Proof. If is easy to see that f is bounded by $\frac{61666}{61665}$ *on* $\mathbb{R}$.
If $|x|^{20} + |y|^{20} = 0$ or $|x|^{20} + |y|^{20} \geq \frac{1}{61666}$.

$$|Df(x,y)| \leq \frac{(2.432902008 \times 10^{18})(616666)}{616665} \leq \frac{(2.432902008 \times 10^{18})(616666)^3}{616665} (\times)\left(|x|^{20} + |y|^{20}\right) \qquad \forall x, y \in \mathbb{R}.$$

Now, Suppose that $0 < |x|^{20} + |y|^{20} < \frac{1}{616666}$. Then there exists a non-negative k. Such that

$$\frac{1}{616666^{k+2}} \leq |x|^{20} + |y|^{20} < \frac{1}{616666^{k+1}}. \tag{4.38}$$

Hence, $61666^k|x|^{20} < \frac{1}{616666}$ and $61666^k|y|^{20} < \frac{1}{61666}$ and

$$4^n(x+10y),\ 4^n(x+9y),\ 4^n(x+8y),\ 4^n(x+7y),\ 4^n(x+6y),$$
$$4^n(x+5y),\ 4^n(x+4y),\ 4^n(x+3y),\ 4^n(x+2y),\ 4^n(x+y),\ 4^n(x),$$
$$4^n(x-y),\ 4^n(x-2y),\ 4^n(x-3y),\ 4^n(x-4y),\ 4^n(x-5y),$$
$$4^n(x-6y),\ 4^n(x-7y),\ 4^n(x-8y),\ 4^n(x-9y),$$
$$4^n(x-10y),\ 4^n(y) \in (-1,1)$$

for all $n = 0, 1, 2, ..., k-1$. Hence $D\phi(4^n(x), 4^n(y)) = 0$ for all $n = 0, 1, 2, ..., k-1$. From the definition of f and inequality (1.67), we obtain that

$$|Df(x,y)| \leq \sum_{n=k}^{\infty} 4^{-20n}(2.432902008 \times 10^{18}) = \frac{2.432902008 \times 10^{18}.4^{20}(1-k)}{616665} \leq \frac{2.432902008 \times 10^{18}(616666^3)}{61665} (\times)\left(|x|^{20} + |y|^{20}\right).$$

Therefore, f satisfies (4.37) for all $x, y \in \mathbb{R}$. Now, we claim that functional equation (2.1) is not stable for $\lambda = 20$ in corollary ($\alpha = \beta = p = 1$). Suppose on the contrary that there exists a icosic mapping $N : \mathbb{R} \to \mathbb{R}$ and constant $c \in \mathbb{R}$ such that $N(x) = cx^{20}$ for all rational numbers x (see [6]). So we obtain the following inequality.

$$|f(x)| \leq (d + |c|)|x|^{20} \qquad \forall x \in Q \tag{4.39}$$

let $m \in N$ with $m+1 > d + |c|$. If x is a rational numbers in $(0, 4^{-m})$, then $4^n x \in (0,1)$ for all $n = 0, 1, 2, ..., m$ and for this case we get

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(4^n x)}{4^{20n}} \geq \sum_{n=0}^{\infty} \frac{(4^n x)}{4^{20n}} = (m+1)x^{20} > (d + |c|)|x|^{20}$$

which contradicts inequality (4.39).

□

References

[1] M. Arunkumar, S. Murthy, G. Ganapathy, Solution and Stability of n - dimensional Quadratic functional equation, ICCMSC, 2012 Computer and Information science (CIS), springer verlag - Germany, vol. 238 (2012), 384 - 394.

[2] M. Arunkumar, S. Murthy, V. Govindan, General solution and generalized Ulam - Hyers stability of a generalized n - type additive quadratic functional equation in Banach spaces and Banach Algebra: Using direct and fixed point method, International journal of Advanced Mathematical sciences, 3(1) (2015) 25 - 64.

[3] Mohan Arunkumar, Abasalt Bodaghi, John Michael Rassias, and Elumalai Sathya, The General Solution and Approximations of a Decic Functional equation in Various normed spaces, Journal of the Chungcheong Mathematical Society, Volume 29, No. 2, May 2016, pp. 287 - 328.

[4] A. Bodaghi, Stability of a mixed type additive and quartic functional equation, Filomat. 28 (2014), no. 8, 1629-1640.

[5] P.Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings , J. Math. Anal. Appl. 184 (1994), 431-436.

[6] Z. Gajda, On the stability of additive mappings, Inter. J. Math. Math. Sci. 14 (1991), 431-434.

[7] D. H. Hyers, On the stability of linear functional equations, Proc. Natl. Acad. Sci. USA, 27 (1941), 222 - 224.

[8] G. Isac and Th. M. Rassias, Stability of ' -Additive Mappings: Applications to Nonlinear Analysis, J. Funct. Anal., 19 (1996), 219-228.

[9] S. S. Jin and Y. H. Lee, Fuzzy stability of the Cauchy additive and quadratic type functional equation, Common korean math. Soc. 27 (2012), 523 - 535.

[10] S. H. Lee, S. M. Im and I. S. Hwang, quadratic functional equations, J. Math. Anal. Appl., 307 (2005) 387 - 394.

[11] R. Murali and V. Vithya, Hyers-Ulam-Rassias Stability of Functional Equations in Matrix Normed Spaces: A Fixed point approach, Assian Journal of Mathematics and Computer Research, 4(3) (2015), 155-163.

[12] J. M. Rassias and M. Eslamian, Fixed points and stability of nonic functional equation in quasi-β-normed spaces, Cont. Anal. Appl. Math. 3 (2015), no. 2, 293-309.

[13] J. M. Rassias and H. M. Kim, Generalized Hyers-Ulam stability for general additive functional equations in quasi-β-normed spaces, J. Math. Anal. Appl. 356 (2009), no. 1, 302-309.

[14] J. M. Rassias, E. Son, and H. M. Kim, On the Hyers-Ulam stability of 3D and 4D mixed type mappings, Far East J. Math. Sci. 48 (2011), no. 1, 83-102.

[15] J. M. Rassias, M. Arunkumar, and T. Namachivayam, Stability Of The Leibniz Additive-Quadratic Functional Equation In Felbin's And Random Normed Spaces: A Direct Method, J. Acad. Res. J. Inter. (2015), 102-110.

[16] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72, (1978), No. 2, 297 - 300.

[17] K. Ravi, J. M. Rassias, S. Pinelas and S. Suresh, General Solution and Stability of Quattuordecic Functional Equation in Quasi β-Normed Spaces, Advances in Pure Mathematics, 2016, 6, 921-941.

[18] K. Ravi, J.M. Rassias and B.V. Senthil Kumar, Ulam-Hyers Stability of Undecic Functional Equation in Quasi-β-Normed Spaces: Fixed Point Method, Tbilisi Mathematical Science, 9(2) (2016), 83-103.

[19] V. Radu, The fixed point alternative and the stability of functional equation in: Seminer on Fixed point theory Cluj Napoca, Vol.IV, 2003, in press.

[20] P. K. Sahoo and J. K. Chung, On the general solution of a quartic functional equation, Bull. Korean. Math. Soc. 40 (4) (2003), 565 - 576.

[21] Y. Shen, W. Chen, On the Stability of Septic and Octic Functional Equations, J. Computational Analysis and Applications, 18(2), (2015), 277-290.

[22] S. M. Ulam, A Collection of Mathematical Problems, Interscience Publ., New York, 1960.

[23] Z.Wang and P. K. Sahoo, Stability of an ACQ- Functional Equation in Various Matrix Normed Spaces, J. Nonlinear Sci. Appl., 8 (2015), 64-85.

[24] T. Z. Xu , J.M. Rassias, M. J. Rassias, and W. X. Xu, A Fixed Point Approach to the Stability of Quintic and Sextic Functional Equations in Quasi-β-Normed Spaces. Journal of Inequalities and Applications, (2010), 1-23.

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