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Ulam-Hyers stability of Euler-Lagrange additive functional equation in intuitionistic fuzzy Banach spaces: Direct and fixed point methods

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Abstract

In this paper, authors verify the generalized Ulam - Hyers stability of the following Euler - Lagrange additive functional equation

$$rf(s(x-y))+sf(r(y-x))+(r+s)f(rx+sy)=(r+s)(rf(x)+sf(y))$$

in Intuitionistic Fuzzy Banach Spaces using direct and fixed point methods.

Keywords

Additive functional equations, Euler - Lagrange functional equations, generalized Ulam - Hyers stability, intuitionistic fuzzy Banach Space, fixed point.

AMS Subject Classification

39B52, 32B72, 32B82.

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Contents

1. Introduction

Fuzzy theory was initiated by Zadeh [37] in 1965. Nowadays, this theory is a powerful tool for modeling uncertainty and vagueness in miscellaneous problems arising in the field of science and engineering. The concept of intuitionistic fuzzy normed spaces, initially has been introduced by Saadati and Park in [32]. Then, Saadati et al. have obtained a modified case of intuitionistic fuzzy normed spaces by improving the separation condition and strengthening some conditions in the definition of [34]. Intuitionistic fuzzy sets and Intuitionistic fuzzy metric spaces are studied in [8] and [28], respectively.

The lessons of stability problems for functional equations is connected to a question of Ulam [36] regarding the stability of group homomorphisms and positively answered for a additive functional equation on Banach spaces by Hyers [19] and Aoki [2]. It was advance generalized and admirable outcome obtained by number of authors; for instance, see [17, 29, 30]. On the other hand, Cădariu and Radu noticed that a fixed point alternative method is very important for the solution of the Ulam problem. In other words, they employed this fixed point method to the investigation of the Cauchy functional equation [13] and for the quadratic functional equation [12] (for more applications of this method, see [7] and [10]). The generalized Hyers-Ulam stability of different functional equations in intuitionistic fuzzy normed spaces has been studied by a number of the authors (see [3–6, 9, 11, 25–27, 35]). Over the last seven decades, the above problem was tackled by numerous authors and its solutions via various forms of functional equations were discussed. We refer the attracted readers for more information on such problems to the monographs [1, 14, 20, 21, 31].

In this paper, authors verify the generalized Ulam - Hyers stability of the following Euler - Lagrange additive functional equation

$$rf(s(x-y))+sf(r(y-x))+(r+s)f(rx+sy) = (r+s)(rf(x)+sf(y)) \quad (1.1)$$

where $r,s \in \mathbb{R}$ with $r \neq 0$ in Intuitionistic Fuzzy Banach Spaces using direct and fixed point methods.

2. Definitions on Intuitionistic Fuzzy Banach Space

Now, we recall the basic definitions and notations in the setting of intuitionistic fuzzy normed space.

Definition 2.1. *A binary operation $*: [0,1]\times[0,1] \longrightarrow [0,1]$ is said to be continuous t-norm if $*$ satisfies the following conditions:*

(1) $*$ is commutative and associative;

(2) $*$ is continuous;

(3) $a*1 = a$ for all $a \in [0,1]$;

(4) $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.

Definition 2.2. *A binary operation $\diamond: [0,1]\times[0,1] \longrightarrow [0,1]$ is said to be continuous t-conorm if $\diamond$ satisfies the following conditions:*

(1') $\diamond$ is commutative and associative;

(2') $\diamond$ is continuous;

(3') $a\diamond 0 = a$ for all $a \in [0,1]$;

(4') $a\diamond b \leq c\diamond d$ whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.

Using the notions of continuous t-norm and t-conorm, Saadati and Park [32] introduced the concept of intuitionistic fuzzy normed space as follows:

Definition 2.3. *The five-tuple $(X,\mu,\nu,*,\diamond)$ is said to be an intuitionistic fuzzy normed space (for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous $t-$conorm, and μ,ν are fuzzy sets on $X\times(0,\infty)$ satisfying the following conditions. For every $x,y \in X$ and $s,t > 0$,*
(IFN1) $\mu(x,t)+\nu(x,t) \leq 1$,
(IFN2) $\mu(x,t) > 0$,
(IFN3) $\mu(x,t) = 1$, if and only if $x = 0$.
(IFN4) $\mu(\alpha x,t) = \mu\left(x,\frac{t}{|\alpha|}\right)$ for each $\alpha \neq 0$,
(IFN5) $\mu(x,t)*\mu(y,s) \leq \mu(x+y,t+s)$,
(IFN6) $\mu(x,\cdot): (0,\infty) \to [0,1]$ is continuous,
(IFN7) $\lim_{t\to\infty}\mu(x,t) = 1$ and $\lim_{t\to 0}\mu(x,t) = 0$,
(IFN8) $\nu(x,t) < 1$,
(IFN9) $\nu(x,t) = 0$, if and only if $x = 0$.
(IFN10) $\nu(\alpha x,t) = \nu\left(x,\frac{t}{|\alpha|}\right)$ for each $\alpha \neq 0$,
(IFN11)$\nu(x,t)\diamond\nu(y,s) \geq \nu(x+y,t+s)$,
(IFN12) $\nu(x,\cdot): (0,\infty) \to [0,1]$ is continuous,
(IFN13) $\lim_{t\to\infty}\nu(x,t) = 0$ and $\lim_{t\to 0}\nu(x,t) = 1$.

In this case, (μ,ν) is called an intuitionistic fuzzy norm.

Example 2.4. *Let $(X,\|\cdot\|)$ be a normed space. Let $a*b = ab$ and $a\diamond b = \min\{a+b,1\}$ for all $a,b \in [0,1]$. For all $x \in X$ and every $t > 0$, consider*

$$\mu(x,t) = \begin{cases} \frac{t}{t+\|x\|} & if \quad t > 0; \\ 0 & if \quad t \leq 0; \end{cases} \quad and$$

$$\nu(x,t) = \begin{cases} \frac{\|x\|}{t+\|x\|} & if \quad t > 0; \\ 0 & if \quad t \leq 0. \end{cases}$$

Then $(X,\mu,\nu,*,\diamond)$ is an IFN-space.

The concepts of convergence and Cauchy sequences in an intuitionistic fuzzy normed space are investigated in [32].

Definition 2.5. *Let $(X,\mu,\nu,*,\diamond)$ be an IFNS. Then, a sequence $x = \{x_k\}$ is said to be intuitionistic fuzzy convergent to a point $L \in X$ if*

$$\lim_{k\to\infty}\mu(x_k-L,t) = 1 \quad and \quad \lim_{k\to\infty}\nu(x_k-L,t) = 0$$

for all $t > 0$. In this case, we write

$$x_k \xrightarrow{IF} L \quad as \quad k \to \infty$$

Definition 2.6. *Let $(X,\mu,\nu,*,\diamond)$ be an IFN-space. Then, $x = \{x_k\}$ is said to be intuitionistic fuzzy Cauchy sequence if*

$$\mu\left(x_{k+p}-x_k,t\right) = 1 \quad and \quad \nu\left(x_{k+p}-x_k,t\right) = 0$$

for all $t > 0$, and $p = 1,2\cdots$.

Definition 2.7. *Let $(X,\mu,\nu,*,\diamond)$ be an IFN-space. Then $(X,\mu,\nu,*,\diamond)$ is said to be complete if every intuitionistic fuzzy Cauchy sequence in $(X,\mu,\nu,*,\diamond)$ is intuitionistic fuzzy convergent $(X,\mu,\nu,*,\diamond)$.*

Hereafter and subsequently, assume that X is a linear space, (Z,μ',ν') is an intuitionistic fuzzy normed space and (Y,μ,ν) an intuitionistic fuzzy Banach space. Now, we use the following notation for a given mapping $f: X \longrightarrow Y$ such that

$$Df_{rx}^{sy} = rf(s(x-y))+sf(r(y-x)) + (r+s)f(rx+sy)-(r+s)(rf(x)+sf(y))$$

where $r,s \in \mathbb{R}$ with $r \neq \pm s$ for all $x,y \in X$.

3. IFNS Stability Results: Direct Method

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.1) in INFS using direct method.

Theorem 3.1. *Let $\eta \in \{1,-1\}$. Let $\varphi, \varphi : X \times X \longrightarrow Z$ be a function such that for some $0 < \left(\frac{p}{r+s}\right)^{\eta} < 1$,*

$$\left.\begin{aligned} &\mu'\left(\varphi\left((r+s)^{n\eta}x,(r+s)^{n\eta}y\right),t\right)\\ &\qquad\geq \mu'\left(p^{n\eta}\varphi(x,y),t\right)\\ &\nu'\left(\varphi\left((r+s)^{n\eta}x,(r+s)^{n\eta}y\right),t\right)\\ &\qquad\leq \nu'\left(p^{n\eta}\varphi(x,y),t\right)\end{aligned}\right\} \tag{3.1}$$

for all $x \in X$ and all $t > 0$ and

$$\left.\begin{aligned} &\lim_{n\to\infty}\mu'\left(\varphi\left((r+s)^{\eta n}x,(r+s)^{\eta n}y\right),a^{\eta n}t\right)=1\\ &\lim_{n\to\infty}\nu'\left(\varphi\left((r+s)^{\eta n}x,(r+s)^{\eta n}y\right),a^{\eta n}t\right)=0\end{aligned}\right\} \tag{3.2}$$

for all $x,y \in X$ and all $t > 0$. Let $f : X \to Y$ be a function satisfying the inequality

$$\left.\begin{aligned} &\mu\left(Df_{rx}^{sy}(x,y),t\right)\geq \mu'\left(\varphi(x,y),t\right)\\ &\nu\left(Df_{rx}^{sy}(x,y),t\right)\leq \nu'\left(\varphi(x,y),t\right)\end{aligned}\right\} \tag{3.3}$$

for all $x,y \in X$ and all $t > 0$. Then there exists a unique additive mapping $\mathscr{A} : X \longrightarrow Y$ satisfying (1.1) and

$$\left.\begin{aligned} &\mu\left(f(x)-\mathscr{A}(x),t\right)\\ &\qquad\geq \mu'\left(\varphi(x,x),(r+s)\,|(r+s)-p|t\right)\\ &\nu\left(f(x)-\mathscr{A}(x),t\right)\\ &\qquad\leq \nu'\left(\varphi(x,x),(r+s)\,|(r+s)-p|t\right)\end{aligned}\right\} \tag{3.4}$$

for all $x \in X$ and all $t > 0$.

Proof. **Case (i):** Let $\eta = 1$.

Setting (x,y) by (x,x) in (3.3), we have

$$\left.\begin{aligned} &\mu\left((r+s)f((r+s)x)-(r+s)^2f(x),t\right)\\ &\qquad\geq \mu'\left(\varphi(x,x),t\right)\\ &\nu\left((r+s)f((r+s)x)-(r+s)^2f(x),t\right)\\ &\qquad\leq \nu'\left(\varphi(x,x),t\right)\end{aligned}\right\} \tag{3.5}$$

for all $x,y \in X$ and all $t > 0$. It follows from (3.5) and (IFN4), (IFN10), we arrive

$$\left.\begin{aligned} &\mu\Big(\frac{f((r+s)x)}{(r+s)}-f(x),\frac{t}{(r+s)^2}\Big)\\ &\qquad\geq \mu'(\varphi(x,x),t)\\ &\nu\Big(\frac{f((r+s)x)}{(r+s)}-f(x),\frac{t}{(r+s)^2}\Big)\\ &\qquad\leq \nu'(\varphi(x,x),t)\end{aligned}\right\} \tag{3.6}$$

for all $x \in X$ and all $t > 0$. Substituting x by $(r+s)^n x$ in (3.6), we have

$$\left.\begin{aligned} &\mu\Big(\frac{f((r+s)^{n+1}x)}{(r+s)}-f((r+s)^nx),\frac{t}{(r+s)^2}\Big)\\ &\qquad\geq \mu'(\varphi((r+s)^nx,(r+s)^nx),t)\\ &\nu\Big(\frac{f((r+s)^{n+1}x)}{(r+s)}-f((r+s)^nx),\frac{t}{(r+s)^2}\Big)\\ &\qquad\leq \nu'(\varphi((r+s)^nx,(r+s)^nx),t)\end{aligned}\right\} \tag{3.7}$$

for all $x \in X$ and all $t > 0$. It is easy to verify from (3.7) and using (3.1), (IFN4), (IFN10) that

$$\left.\begin{aligned} &\mu\Big(\frac{f((r+s)^{n+1}x)}{(r+s)^{(n+1)}}-\frac{f((r+s)^nx)}{(r+s)^n},\frac{t}{(r+s)^{n+2}}\Big)\\ &\qquad\geq \mu'\left(\varphi(x,x),\tfrac{t}{p^n}\right)\\ &\nu\Big(\frac{f((r+s)^{n+1}x)}{(r+s)^{(n+1)}}-\frac{f((r+s)^nx)}{(r+s)^n},\frac{t}{(r+s)^{n+2}}\Big)\\ &\qquad\leq \nu'\left(\varphi(x,x),\tfrac{t}{p^n}\right)\end{aligned}\right\} \tag{3.8}$$

for all $x \in X$ and all $t > 0$. Interchanging t into $p^n t$ in (3.8), we have

$$\left.\begin{aligned} &\mu\Big(\frac{f((r+s)^{n+1}x)}{(r+s)^{(n+1)}}-\frac{f((r+s)^nx)}{(r+s)^n},\frac{t\cdot p^n}{(r+s)^{n+2}}\Big)\\ &\qquad\geq \mu'(\varphi(x,x),t)\\ &\nu\Big(\frac{f((r+s)^{n+1}x)}{(r+s)^{(n+1)}}-\frac{f((r+s)^nx)}{(r+s)^n},\frac{t\cdot p^n}{(r+s)^{n+2}}\Big)\\ &\qquad\leq \nu'(\varphi(x,x),t)\end{aligned}\right\} \tag{3.9}$$

for all $x \in X$ and all $t > 0$. It is easy to see that

$$\frac{f((r+s)^nx)}{(r+s)^n}-f(x)=\sum_{i=0}^{n-1}\frac{f((r+s)^{i+1}x)}{(r+s)^{(i+1)}}-\frac{f((r+s)^ix)}{(r+s)^i} \tag{3.10}$$

for all $x \in X$. From equations (3.9) and (3.10), we get

$$\left.\begin{aligned} &\mu\left(\tfrac{f((r+s)^nx)}{(r+s)^n}-f(x),\textstyle\sum_{i=0}^{n-1}\tfrac{p^i\,t}{(r+s)^{i+2}}\right)\\ &=\mu\left(\textstyle\sum_{i=0}^{n-1}\tfrac{f((r+s)^{i+1}x)}{(r+s)^{(i+1)}}-\tfrac{f((r+s)^ix)}{(r+s)^i},\textstyle\sum_{i=0}^{n-1}\tfrac{p^i\,t}{(r+s)^{i+2}}\right)\\ &\nu\left(\tfrac{f((r+s)^nx)}{(r+s)^n}-f(x),\textstyle\sum_{i=0}^{n-1}\tfrac{p^i\,t}{(r+s)^{i+2}}\right)\\ &=\nu\left(\textstyle\sum_{i=0}^{n-1}\tfrac{f((r+s)^{i+1}x)}{(r+s)^{(i+1)}}-\tfrac{f((r+s)^ix)}{(r+s)^i},\textstyle\sum_{i=0}^{n-1}\tfrac{p^i\,t}{(r+s)^{i+2}}\right)\end{aligned}\right\} \tag{3.11}$$

for all $x \in X$ and all $t > 0$. From equations (3.10) and (3.11), we have

$$\left.\begin{aligned} &\mu\left(\tfrac{f((r+s)^nx)}{(r+s)^n}-f(x),\textstyle\sum_{i=0}^{n-1}\tfrac{p^it}{(r+s)^{i+2}}\right)\\ &\qquad\geq\textstyle\prod_{i=0}^{n-1}\mu\left(\tfrac{f((r+s)^{i+1}x)}{(r+s)^{(i+1)}}-\tfrac{f((r+s)^ix)}{(r+s)^i},\tfrac{p^i\,tr}{(r+s)^{i+2}}\right)\\ &\nu\left(\tfrac{f((r+s)^nx)}{(r+s)^n}-f(x),\textstyle\sum_{i=0}^{n-1}\tfrac{p^it}{(r+s)^{i+2}}\right)\\ &\qquad\leq_{i=0}^{n-1}\nu\left(\tfrac{f((r+s)^{i+1}x)}{(r+s)^{(i+1)}}-\tfrac{f((r+s)^ix)}{(r+s)^i},\tfrac{p^i\,t}{(r+s)^{i+2}}\right)\end{aligned}\right\} \tag{3.12}$$

where

$$\prod_{i=0}^{n-1}c_j=c_1*c_2*\cdots*c_n \quad \textit{and} \quad {}_{i=0}^{n-1}d_j=d_1\diamond d_2\diamond\cdots\diamond d_n$$

for all $x \in X$ and all $t > 0$. Hence

$$\left.\begin{array}{l} \mu\left(\frac{f((r+s)^n x)}{(r+s)^n} - f(x), \sum_{i=0}^{n-1} \frac{p^i\, t}{(r+s)^{i+2}}\right) \\ \quad \geq \prod_{i=0}^{n-1} \mu'(\varphi(x,x),t) = \mu'(\varphi(x,x),t) \\ \nu\left(\frac{f((r+s)^n x)}{(r+s)^n} - f(x), \sum_{i=0}^{n-1} \frac{p^i\, t}{(r+s)^{i+2}}\right) \\ \quad \leq_{i=0}^{n-1} \nu'(\varphi(x,x),t) = \nu'(\varphi(x,x),t) \end{array}\right\} \tag{3.13}$$

for all $x \in X$ and all $t > 0$. Replacing x by $(r+s)^m x$ in (3.13) and using (3.2), (IFN4), (IFN10), we obtain

$$\left.\begin{array}{l} \mu\left(\frac{f((r+s)^{n+m} x)}{(r+s)^{(n+m)}} - \frac{f((r+s)^m x)}{(r+s)^m}, \sum_{i=0}^{n-1} \frac{p^i\, t}{(r+s)^{(i+m+2)}}\right) \\ \quad \geq \mu'(\varphi((r+s)^m x, (r+s)^m x), t) \\ \quad = \mu'\left(\varphi(x,x), \frac{t}{p^m}\right) \\ \nu\left(\frac{f((r+s)^{n+m} x)}{(r+s)^{(n+m)}} - \frac{f((r+s)^m x)}{(r+s)^m}, \sum_{i=0}^{n-1} \frac{p^i\, t}{(r+s)^{(i+m+2)}}\right) \\ \quad \leq \nu'(\varphi((r+s)^m x, (r+s)^m x), t) \\ \quad = \nu'\left(\varphi(x,x), \frac{t}{p^m}\right) \end{array}\right\} \tag{3.14}$$

for all $x \in X$ and all $t > 0$ and all $m, n \geq 0$. Replacing t by $p^m t$ in (3.14), we get

$$\left.\begin{array}{l} \mu\left(\frac{f((r+s)^{n+m} x)}{(r+s)^{(n+m)}} - \frac{f((r+s)^m x)}{(r+s)^m}, \sum_{i=0}^{n-1} \frac{p^{i+m}\, t}{(r+s)^{(i+m+2)}}\right) \\ \quad \geq \mu'(\varphi(x.x),t) \\ \nu\left(\frac{f((r+s)^{n+m} x)}{(r+s)^{(n+m)}} - \frac{f((r+s)^m x)}{(r+s)^m}, \sum_{i=0}^{n-1} \frac{p^{i+m}\, t}{(r+s)^{(i+m+2)}}\right) \\ \quad \leq \nu'(\varphi(x,x),t) \end{array}\right\} \tag{3.15}$$

$$\left.\begin{array}{l} \mu\left(\frac{f((r+s)^{n+m} x)}{(r+s)^{(n+m)}} - \frac{f((r+s)^m x)}{(r+s)^m}, t\right) \\ \quad \geq \mu'\left(\varphi(x,x), \frac{t}{\sum_{i=m}^{n-1} \frac{p^i}{(r+s)^{i+2}}}\right) \\ \nu\left(\frac{f((r+s)^{n+m} x)}{(r+s)^{(n+m)}} - \frac{f((r+s)^m x)}{(r+s)^m}, t\right) \\ \quad \leq \nu'\left(\varphi(x,x), \frac{t}{\sum_{i=m}^{n-1} \frac{p^i}{(r+s)^{i+2}}}\right) \end{array}\right\} \tag{3.16}$$

holds for all $x \in X$ and all $t > 0$ and all $m, n \geq 0$. Since $0 < p < 1$ and $\sum_{i=0}^{n} \left(\frac{p}{1}\right)^i < \infty$. The Cauchy criterion for convergence in IFNS shows that the sequence $\left\{\frac{f((r+s)^n x)}{(r+s)^n}\right\}$ is Cauchy in (Y, μ, ν). Since (Y, μ, ν) is a complete IFN-space this sequence converges to some point $\mathscr{A}(x) \in Y$. So, one can define the mapping $\mathscr{A} : X \longrightarrow Y$ by

$$\lim_{n\to\infty} \mu\left(\frac{f((r+s)^n x)}{(r+s)^n} - \mathscr{A}(x), t\right) = 1,$$

$$\lim_{n\to\infty} \nu\left(\frac{f((r+s)^n x)}{(r+s)^n} - \mathscr{A}(x), t\right) = 0$$

for all $x \in X$ and all $t > 0$. Hence

$$\frac{f((r+s)^n x)}{(r+s)^n} \xrightarrow{IF} \mathscr{A}(x), \quad as \quad n \to \infty.$$

Letting $m = 0$ in (3.16), we arrive

$$\left.\begin{array}{l} \mu\left(\frac{f((r+s)^n x)}{(r+s)^n} - f(x), t\right) \geq \mu'\left(\varphi(x,x), \frac{t}{\sum_{i=0}^{n-1} \frac{p^i}{(r+s)^{i+2}}}\right) \\ \nu\left(\frac{f((r+s)^n x)}{(r+s)^n} - f(x), t\right) \leq \nu'\left(\varphi(x,x), \frac{t}{\sum_{i=0}^{n-1} \frac{p^i}{(r+s)^{i+2}}}\right) \end{array}\right\} \tag{3.17}$$

for all $x \in X$ and all $t > 0$. Letting n tend to infinity in (3.17), we have

$$\left.\begin{array}{l} \mu\left(\mathscr{A}(x) - f(x), t\right) \\ \qquad \geq \mu'(\varphi(x,x), (r+s)\ t((r+s)-p)) \\ \nu\left(\mathscr{A}(x) - f(x), t\right) \\ \qquad \leq \nu'(\varphi(x,x), (r+s)\ t((r+s)-p)) \end{array}\right\} \tag{3.18}$$

for all $x \in X$ and all $t > 0$. To prove $\mathscr{A}$ satisfies (1.1), replacing (x,y) by $((r+s)^n x, (r+s)^n y)$ in (3.3) respectively, we obtain

$$\left.\begin{array}{l} \mu\left(\frac{1}{(r+s)^n} Df_{rx}^{sy}((r+s)^n x, (r+s)^n y), t\right) \\ \qquad \geq \mu'(\varphi((r+s)^n x, (r+s)^n y), (r+s)^n t) \\ \nu\left(\frac{1}{(r+s)^n} Df_{rx}^{sy}((r+s)^n x, (r+s)^n y), t\right) \\ \qquad \leq \nu'(\varphi((r+s)^n x, (r+s)^n y), (r+s)^n t) \end{array}\right\} \tag{3.19}$$

for all $x \in X$ and all $t > 0$. Now,

$$\begin{aligned} &\mu\Big(r\mathscr{A}(s(x-y)) + s\mathscr{A}(r(y-x)) + (r+s)\mathscr{A}(rx+sy) \\ &\qquad - (r+s)(r\mathscr{A}(x) + s\mathscr{A}(y))\Big) \\ &\geq \mu\left(r\mathscr{A}(s(x-y)) - \frac{r}{(r+s)^n} f(s(x-y)), \frac{t}{5}\right) \\ &* \mu\left(s\mathscr{A}(r(y-x)) - \frac{s}{(r+s)^n} f(r(y-x)), \frac{t}{5}\right) \\ &* \mu\left((r+s)\mathscr{A}(rx+sy) + \frac{(r+s)}{(r+s)^n} f(rx+sy), \frac{t}{5}\right) \\ &* \mu\Big(-(r+s)(r\mathscr{A}(x) + s\mathscr{A}(y)) + \frac{(r+s)}{(r+s)^n} \\ &(rf(x) + sf(y)), \frac{t}{5}\Big) * \mu\Big(\frac{r}{(r+s)^n} f(s(x-y)) \\ &+ \frac{s}{(r+s)^n} f(r(y-x)) + \frac{(r+s)}{(r+s)^n} f(rx+sy) \\ &- \frac{(r+s)}{(r+s)^n}(rf(x) + sf(y)), \frac{t}{5}\Big) \end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
&\nu\Big(r\mathscr{A}\left(s\left(x-y\right)\right)+s\mathscr{A}\left(r\left(y-x\right)\right)\\
&+\left(r+s\right)\mathscr{A}\left(rx+sy\right)-\left(r+s\right)\left(r\mathscr{A}\left(x\right)+s\mathscr{A}\left(y\right)\right)\Big)\\
&\geq \nu\Big(r\mathscr{A}\left(s\left(x-y\right)\right)-\frac{r}{(r+s)^n}f\left(s\left(x-y\right)\right),\frac{t}{5}\Big)\\
&*\nu\Big(s\mathscr{A}\left(r\left(y-x\right)\right)-\frac{s}{(r+s)^n}f\left(r\left(y-x\right)\right),\frac{t}{5}\Big)\\
&*\nu\Big(\left(r+s\right)\mathscr{A}\left(rx+sy\right)+\frac{(r+s)}{(r+s)^n}f\left(rx+sy\right),\frac{t}{5}\Big)\\
&*\nu\Big(-\left(r+s\right)\left(r\mathscr{A}\left(x\right)+s\mathscr{A}\left(y\right)\right)\\
&+\frac{(r+s)}{(r+s)^n}\left(rf\left(x\right)+sf\left(y\right)\right),\frac{t}{5}\Big)\\
&*\nu\Big(\frac{r}{(r+s)^n}f\left(s\left(x-y\right)\right)+\frac{s}{(r+s)^n}f\left(r\left(y-x\right)\right)\\
&+\frac{(r+s)}{(r+s)^n}f\left(rx+sy\right)-\frac{(r+s)}{(r+s)^n}\left(rf\left(x\right)+sf\left(y\right)\right),\frac{t}{5}\Big)
\end{aligned} \tag{3.21}$$

for all $x\in X$ and all $t>0$. Also

$$\left.\begin{aligned}
\lim_{n\to\infty}\mu\Big(\tfrac{1}{(r+s)^n}Df_{rx}^{sy}((r+s)^n x,(r+s)^n y),\tfrac{t}{5}\Big)=1\\
\lim_{n\to\infty}\nu\Big(\tfrac{1}{(r+s)^n}Df_{rx}^{sy}((r+s)^n x,(r+s)^n y),\tfrac{t}{5}\Big)=0
\end{aligned}\right\} \tag{3.22}$$

for all $x\in X$ and all $t>0$. Letting $n\to\infty$ in (3.20), (3.21) and using (3.22), we observe that $\mathscr{A}$ fulfills (1.1). Therefore, $\mathscr{A}$ is a additive mapping. In order to prove $\mathscr{A}(x)$ is unique, let $\mathscr{A}'(x)$ be another additive functional equation satisfying (1.1) and (3.4). Hence,

$$\begin{aligned}
&\mu(\mathscr{A}(x)-\mathscr{A}'(x),t)\\
&\geq \mu\left(\mathscr{A}((r+s)^n x)-f((r+s)^n x),\frac{t.(r+s)^n}{2}\right)\\
&*\mu\left(f((r+s)^n x)-\mathscr{A}'((r+s)^n x),\frac{t.(r+s)^n}{2}\right)\\
&\geq \mu'(\varphi((r+s)^n x,(r+s)^n x),\\
&\qquad\qquad \frac{t\,(r+s)^{n+1}}{2}\,|(r+s)-p|\Big)\\
&\geq \mu'\left(\varphi(x,x),\frac{t\ (r+s)^{n+1}\,|(r+s)-p|}{2\cdot p^n}\right)
\end{aligned}$$

$$\begin{aligned}
&\nu(\mathscr{A}(x)-\mathscr{A}'(x),t)\\
&\leq \nu\left(\mathscr{A}((r+s)^n x)-f((r+s)^n x),\frac{t.(r+s)^n}{2}\right)\\
&\diamond\,\nu\left(f((r+s)^n x)-\mathscr{A}'((r+s)^n x),\frac{t.(r+s)^n}{2}\right)\\
&\leq \nu'(\varphi((r+s)^n x,(r+s)^n x),\\
&\qquad\qquad \frac{t\,(r+s)^{n+1}}{2}\,|(r+s)-p|\Big)\\
&\leq \nu'\left(\varphi(x,x),\frac{t\ (r+s)^{n+1}\,|(r+s)-p|}{2\cdot p^n}\right)
\end{aligned}$$

for all $x\in X$ and all $t>0$.
Since $\lim_{n\to\infty}\dfrac{t\ (r+s)^{n+1}\,|(r+s)-p|}{2\ p^n}=\infty$, we obtain

$$\left.\begin{aligned}
\lim_{n\to\infty}\mu'\left(\varphi(x),\tfrac{t\ (r+s)^{n+1}|(r+s)-p|}{2\cdot p^n}\right)=1\\
\lim_{n\to\infty}\nu'\left(\varphi(x),\tfrac{t\ (r+s)^{n+1}|(r+s)-p|}{2\cdot p^n}\right)=0
\end{aligned}\right\}$$

for all $x\in X$ and all $t>0$. Thus

$$\left.\begin{aligned}
\mu(\mathscr{A}(x)-\mathscr{A}'(x),t)=1\\
\nu(\mathscr{A}(x)-\mathscr{A}'(x),t)=0
\end{aligned}\right\}$$

for all $x\in X$ and all $t>0$. Hence, $\mathscr{A}(x)=\mathscr{A}'(x)$. Therefore, $\mathscr{A}(x)$ is unique.

Case 2: For $\eta=-1$. Putting x by $\frac{x}{(r+s)}$ in (3.5), we get

$$\left.\begin{aligned}
\mu\left((r+s)f(x)-(r+s)^2 f\left(\tfrac{x}{(r+s)}\right),t\right)\\
\geq\mu'\left(\varphi\left(\tfrac{x}{2},\tfrac{x}{2}\right),t\right)\\
\nu\left((r+s)f(x)-(r+s)^2 f\left(\tfrac{x}{(r+s)}\right),t\right)\\
\geq\leq\nu'\left(\varphi\left(\tfrac{x}{2},\tfrac{x}{2}\right),t\right)
\end{aligned}\right\} \tag{3.23}$$

for all $x,y\in X$ and all $t>0$. The rest of the proof is similar to that of Case 1. This completes the proof. □

The following corollary is an immediate consequence of Theorem 3.1, regarding the stability of (1.1).

Corollary 3.2. *Suppose that a function $f:X\longrightarrow Y$ satisfies the double inequality*

$$\begin{aligned}
&\mu\left(Df_{rx}^{sy}(x,y),t\right)\\
&\geq\begin{cases}
\mu'(\lambda,t),\\
\mu'\left(\lambda\left(||x||^a+||y||^b\right),t\right),a,b\neq 1\\
\mu'\left(\lambda||x||^a||y||^b,t\right),a+b\neq 1\\
\mu'\left(\lambda\left\{||x||^a\ ||y||^b+\left(||x||^{a+b}+||y||^{a+b}\right)\right\},t\right),\\
\qquad a+b\neq 1
\end{cases}\\
&\nu\left(Df_{rx}^{sy}(x,y),t\right)\\
&\leq\begin{cases}
\nu'(\lambda,t),\\
\nu'\left(\lambda\left(||x||^a+||y||^b\right),t\right),a,b\neq 1\\
\nu'\left(\lambda||x||^a||y||^b,t\right),a+b\neq 1\\
\nu'\left(\lambda\left\{||x||^a\ ||y||^b+\left(||x||^{a+b}+||y||^{a+b}\right)\right\},t\right),\\
\qquad a+b\neq 1
\end{cases}
\end{aligned} \tag{3.24}$$

for all $x,y \in X$ and all $t>0$, where λ, a, b are constants with $\lambda>0$. Then there exists a unique additive mapping $\mathscr{A}: X \longrightarrow Y$ such that

$$\mu(f(x)-\mathscr{A}(x),t) \geq \begin{cases} \mu'(\lambda,(r+s)\ t|(r+s)-1|), \\ \mu'\left(\left[\lambda||x||^a|r+s|^a+\lambda||x||^b|r+s|^b\right], (r+s)\ t\left[|(r+s)-(r+s)^a| + |(r+s)-(r+s)^b|\right]\right), \\ \mu'\left(\lambda||x||^{a+b}|r+s|^{a+b}, (r+s)\ t\left|(r+s)-(r+s)^{a+b}\right|\right), \\ \mu'\left(\lambda||x||^{a+b}|r+s|^{a+b} + \left[\lambda||x||^a|r+s|^a+\lambda||x||^b|r+s|^b\right], (r+s)\ t\left|(r+s)-(r+s)^{a+b}\right|\right) \end{cases}$$

$$\nu(f(x)-\mathscr{A}(x),t) \leq \begin{cases} \nu'(\lambda,(r+s)\ t|(r+s)-p|), \\ \nu'\left(\left[\lambda||x||^a|r+s|^a+\lambda||x||^b|r+s|^b\right], (r+s)\ t\left[|(r+s)-(r+s)^a| + |(r+s)-(r+s)^b|\right]\right), \\ \nu'\left(\lambda||x||^{a+b}|r+s|^{a+b}, (r+s)^2\ t\left|(r+s)-(r+s)^{a+b}\right|\right), \\ \nu'\left(\lambda||x||^{a+b}|r+s|^{a+b} + \left[\lambda||x||^a|r+s|^a+\lambda||x||^b|r+s|^b\right], (r+s)\ t\left|(r+s)-(r+s)^{a+b}\right|\right) \end{cases} \tag{3.25}$$

for all $x \in X$ and all $t>0$.

Proof. By setting

$$\varphi(x,y) = \begin{cases} \lambda, \\ \lambda\left(||x||^a+||y||^b\right), \\ \lambda||x||^a||y||^b, \\ \lambda\left\{||x||^a\ ||y||^b+\left(||x||^{a+b}+||y||^{a+b}\right)\right\}, \end{cases}$$

and

$$p = \begin{cases} (r+s)^0, \\ (r+s)^a+(r+s)^b, \\ (r+s)^{a+b} \\ (r+s)^{a+b}, \end{cases}$$

in Theorem 3.1, we arrive our desired result. □

4. IFNS Stability Results : Fixed Point Method

In this section, we apply a fixed point method for achieving stability of the additive functional equation (1.1). Here, we present the upcoming result due to Margolis and Diaz [24] for fixed point theory.

Theorem 4.1. *[24] Suppose that for a complete generalized metric space (Ω,δ) and a strictly contractive mapping $T: \Omega \longrightarrow \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1}x) = \infty \qquad \forall \qquad n \geq 0,$$

or there exists a natural number n_0 such that
(FP1) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;
(FP2) The sequence $(T^n x)$ is convergent to a fixed to a fixed point y^* of T
(FP3) y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0}x,y) < \infty\}$;
(FP4) $d(y^*,y) \leq \frac{1}{1-L} d(y,Ty)$ for all $y \in \Delta$.

Using the above theorem, we now obtain the generalized Ulam - Hyers stability of the functional equation (1.1).

Theorem 4.2. *Let $f: X \longrightarrow Y$ be a mapping for which there exists a function $K: X \times X \longrightarrow Z$ with the double condition*

$$\left.\begin{array}{l} \lim_{n\to\infty} \mu'\left(K\left(\chi_i^n x, \chi_i^n y\right), \chi^n t\right) = 1 \\ \lim_{n\to\infty} \nu'\left(K\left(\chi_i^n x, \chi_i^n y\right), \chi^n t\right) = 0 \end{array}\right\} \tag{4.1}$$

for all $x,y \in X$ and all $t>0$ where

$$\chi_i = \begin{cases} r+s & if \quad i=0 \\ \frac{1}{r+s} & if \quad i=1 \end{cases} \tag{4.2}$$

and satisfying the double functional inequality

$$\left.\begin{array}{l} \mu\left(Df_{rx}^{sy}(x,y),t\right) \geq \mu'\left(K(x,y),t\right) \\ \nu\left(Df_{rx}^{sy}(x,y),t\right) \leq \nu'\left(K(x,y),t\right) \end{array}\right\} \tag{4.3}$$

for all $x,y \in X$ and all $t>0$. If there exists $L = L(i)$ such that the function

$$\tau(x) = \frac{1}{r+s} K\left(\frac{x}{r+s}, \frac{x}{r+s}\right), \tag{4.4}$$

has the property

$$\left.\begin{array}{l} \mu'\left(L\frac{\tau(\chi_i x)}{\chi_i}, t\right) = \mu'(\tau(x),t) \\ \nu'\left(L\frac{\tau(\chi_i x)}{\chi_i}, t\right) = \nu'(\tau(x),t) \end{array}\right\} \tag{4.5}$$

for all $x \in X$ and all $t>0$, then there exists a unique additive function $\mathscr{A}: X \longrightarrow Y$ satisfying the functional equation (1.1) and

$$\left.\begin{array}{l} \mu(f(x)-\mathscr{A}(x),t) \geq \mu'\left(\tau(x), \frac{L^{1-i}}{1-L}t\right) \\ \nu(f(x)-\mathscr{A}(x),t) \leq \nu'\left(\tau(x), \frac{L^{1-i}}{1-L}t\right) \end{array}\right\} \tag{4.6}$$

for all $x \in X$ and all $t>0$.

Proof. Consider the set

$$\Lambda = \{h | h: X \longrightarrow Y,\ h(0)=0\}$$

and introduce the generalized metric on Λ,

$$d(h,f) = \inf\left\{L \in (0,\infty) : \left\{\begin{array}{l} \mu(h(x)-f(x),t) \geq \mu'(\tau(x),Lt), x \in X, t>0 \\ \nu(h(x)-f(x),t) \leq \nu'(\tau(x),Lt), x \in X, t>0 \end{array}\right\}\right\} \tag{4.7}$$

It is easy to see that (4.7) is complete with respect to the defined metric. Define $J : \Lambda \longrightarrow \Lambda$ by

$$Jh(x) = \frac{1}{\chi_i} h(\chi_i x),$$

for all $x \in \mathscr{X}$. Now, from (4.7) and $h, f \in \Lambda$

$$\inf\left\{ L \in (0,\infty) : \begin{Bmatrix} \left\{\begin{array}{l} \mu(h(x)-f(x),t) \\ \quad \geq \mu'(\tau(x),t)\} \\ \mu(\frac{1}{\chi_i}h(\chi_i x)-\frac{1}{\chi_i}f(\chi_i x),t) \\ \quad \geq \mu'(\tau(\chi_i x),\chi_i t)\} \\ \mu(\frac{1}{\chi_i}h(\chi_i x)-\frac{1}{\chi_i}f(\chi_i x),t) \\ \quad \geq \mu'(\tau(x),Lt)\} \\ \mu(Jh(x)-Jf(x),t) \\ \quad \geq \mu'(\tau(x),Lt)\} \end{array}\right. \\ \left\{\begin{array}{l} \nu(h(x)-f(x),t) \\ \quad \leq \nu'(\tau(x),t)\} \\ \nu(\frac{1}{\chi_i}h(\chi_i x)-\frac{1}{\chi_i}f(\chi_i x),t) \\ \quad \leq \nu'(\tau(\chi_i x),\chi_i t)\} \\ \nu(\frac{1}{\chi_i}h(\chi_i x)-\frac{1}{\chi_i}f(\chi_i x),t) \\ \quad \leq \nu'(\tau(x),Lt)\} \\ \nu(Jh(x)-Jf(x),t) \\ \quad \leq \nu'(\tau(x),Lt)\} \end{array}\right. \end{Bmatrix} \right.$$

for all $x \in X$ and all $t > 0$. This implies J is a strictly contractive mapping on Λ with Lipschitz constant L.

It follows from (4.7),(3.5), we reach

$$\inf\{1 \in (0,\infty) : \left.\begin{Bmatrix} \mu\left(f((r+s)x)-(r+s)f(x),t\right) \\ \geq \mu'\left(K(x,x),(r+s)t\right) \\ \nu\left(f((r+s)x)-(r+s)f(x),t\right) \\ \leq \nu'\left(K(x,x),(r+s)t\right) \end{Bmatrix}\right\} \tag{4.8}$$

for all $x \in X$ and all $t > 0$. Now, from (4.8) and (4.5) for the case $i = 0$, we reach

$$\inf\left\{L^{1-0} \in (0,\infty) : \right. \\ \left\{\begin{array}{l} \left\{\begin{array}{l} \mu\left(f((r+s)x)-(r+s)f(x),t\right) \\ \quad \geq \mu'\left(K(x,x),(r+s)t\right) \\ \mu\left(\frac{f((r+s)x)}{(r+s)}-f(x),t\right) \\ \quad \geq \mu'\left(K(x,x),(r+s)^2 t\right) \\ \mu\left(Jf(x)-f(x),t\right) \geq \mu'\left(\tau(x),Lt\right) \\ \mu\left(Jf(x)-f(x),t\right) \geq \mu'\left(\tau(x),Lt\right) \\ \mu\left(Jf(x)-f(x),t\right) \geq \mu'\left(\tau(x),Lt\right) \end{array}\right. \\ \left\{\begin{array}{l} \nu\left(f((r+s)x)-(r+s)f(x),t\right) \\ \quad \leq \nu'\left(K(x,x),(r+s)t\right) \\ \nu\left(\frac{f((r+s)x)}{(r+s)}-f(x),t\right) \\ \quad \leq \nu'\left(K(x,x),(r+s)^2 t\right) \\ \nu\left(Jf(x)-f(x),t\right) \leq \nu'\left(\tau(x),Lt\right) \\ \nu\left(Jf(x)-f(x),t\right) \leq \nu'\left(\tau(x),Lt\right) \\ \nu\left(Jf(x)-f(x),t\right) \leq \nu'\left(\tau(x),Lt\right) \end{array}\right. \end{array}\right\} \tag{4.9}$$

for all $x \in X$ and all $t > 0$. Again by interchanging x into $\dfrac{x}{(r+s)}$ in (4.8) and (4.5) for the case $i = 1$, we get

$$\inf\left\{L^{1-1} \in (0,\infty) : \right. \\ \left\{\begin{array}{l} \left\{\begin{array}{l} \mu\left(f(x)-(r+s)f\left(\frac{x}{(r+s)}\right),t\right) \\ \geq \mu'\left(K\left(\frac{x}{(r+s)},\frac{x}{(r+s)}\right),(r+s)t\right) \\ \mu\left(f(x)-Jf(x),t\right) \geq \mu'\left(\tau(x),t\right) \\ \mu\left(f(x)-Jf(x),t\right) \geq \mu'\left(\tau(x),t\right) \\ \mu\left(f(x)-Jf(x),t\right) \geq \mu'\left(\tau(x),t\right) \end{array}\right. \\ \left\{\begin{array}{l} \nu\left(f(x)-(r+s)f\left(\frac{x}{(r+s)},\frac{x}{(r+s)}\right),t\right) \\ \leq \nu'\left(K\left(\frac{x}{(r+s)}\right),(r+s)t\right) \\ \nu\left(f(x)-Jf(x),t\right) \leq \nu'\left(\tau(x),t\right) \\ \nu\left(f(x)-Jf(x),t\right) \leq \nu'\left(\tau(x),t\right) \\ \nu\left(f(x)-Jf(x),t\right) \leq \nu'\left(\tau(x),t\right) \end{array}\right. \end{array}\right\} \tag{4.10}$$

for all $x \in X$ and all $t > 0$. Thus, from (4.9) and (4.10), we arrive

$$\inf\left\{L^{1-i} \in (0,\infty) : \begin{Bmatrix} \mu(f(x)-Jf(x),t) \\ \quad \geq \mu'(\tau(x),L^{1-i}t), \\ \nu(f(x)-Jf(x),t) \\ \quad \leq \nu'(\tau(x),L^{1-i}t), \end{Bmatrix}\right\} \tag{4.11}$$

Hence property (FP1) holds.

By (FP2), it follows that there exists a fixed point $\mathscr{A}$ of J in Λ such that

$$\lim_{n\to\infty} \mu\left(\frac{f(\chi_i^n x)}{\chi_i^n} - \mathscr{A}(x),t\right) = 1,$$

$$\lim_{n\to\infty} \nu\left(\frac{f(\chi_i^n x)}{\chi_i^n} - \mathscr{A}(x), t\right) = 0$$

for all $x \in X$ and all $t > 0$.

To order to prove $A : X \longrightarrow Y$ is additive, replacing (x,y) by $(\chi_i^n x, \chi_i^n y)$ and dividing by χ_i^t in (4.3) and using the definition of $\mathscr{A}(x)$, and then letting $t \to \infty$, we see that $\mathscr{A}$ satisfies (1.1) for all $x, y \in X$ and all $t > 0$.

By (FP3), $\mathscr{A}$ is the unique fixed point of J in the set $\Delta = \{\mathscr{A} \in \Lambda : d(f,A) < \infty\}, \mathscr{A}$ is the unique function such that

$$\left.\begin{array}{l} \mu(f(x) - \mathscr{A}(x),t) \geq \mu'(\tau(x), L^{1-i}t), x \in X \\ \nu(f(x) - \mathscr{A}(x),t) \leq \nu'(\tau(x), L^{1-i}t), x \in X \end{array}\right\}$$

for all $x \in X$ and and all $t > 0$. Finally by (FP4), we obtain

$$\left.\begin{array}{l} \mu\left(f(x) - \mathscr{A}(x),t\right) \geq \mu'\left(\tau(x), \frac{L^{1-i}}{1-L}t\right) \\ \nu\left(f(x) - \mathscr{A}(x),t\right) \leq \nu'\left(\tau(x), \frac{L^{1-i}}{1-L}t\right) \end{array}\right\}$$

for all $x \in X$ and all $t > 0$. So, the proof is complete. □

The next corollary is a direct consequence of Theorem 4.2 which shows that (1.1) can be stable.

Corollary 4.3. *Suppose that a function $f : X \longrightarrow Y$ satisfies the double inequality*

$$\begin{array}{l} \mu\left(Df_{rx}^{sy}(x,y),t\right) \\ \geq \begin{cases} \mu'(\lambda,t), \\ \mu'(\lambda(||x||^a + ||y||^a),t), a \neq 1 \\ \mu'(\lambda||x||^a||y||^a,t), \qquad 2a \neq 1 \\ \mu'\left(\lambda\left\{||x||^a\ ||y||^a + \left(||x||^{2a} + ||y||^{2a}\right)\right\},t\right), \\ \qquad\qquad\qquad\qquad 2a \neq 1 \end{cases} \\ \nu\left(Df_{rx}^{sy}(x,y),t\right) \\ \leq \begin{cases} \nu'(\lambda,t), \\ \nu'(\lambda(||x||^a + ||y||^a),t), a \neq 1 \\ \nu'(\lambda||x||^a||y||^a,t), 2a \neq 1 \\ \nu'\left(\lambda\left\{||x||^a\ ||y||^a + \left(||x||^{2a} + ||y||^{2a}\right)\right\},t\right), \\ \qquad\qquad\qquad\qquad 2a \neq 1 \end{cases} \end{array} \tag{4.12}$$

for all $x, y \in X$ and all $t > 0$, where λ, a are constants with $\lambda > 0$. Then there exists a unique additive mapping $\mathscr{A} : X \longrightarrow Y$ such that the double inequality

$$\begin{array}{l} \mu\left(f(x) - \mathscr{A}(x),t\right) \\ \geq \begin{cases} \mu'\left(\frac{\lambda}{(r+s)}, \frac{(r+s)}{1-(r+s)}t\right) \\ \mu'\left(\frac{\lambda||x||^a}{(r+s)}\frac{2}{|r+s|^a}, \frac{(r+s)}{(r+s)^a-(r+s)}t\right) \\ \mu'\left(\frac{\lambda||x||^{2a}}{(r+s)}\frac{1}{|r+s|^{2a}}, \frac{(r+s)}{(r+s)^{2a}-(r+s)}t\right) \\ \mu'\left(\frac{\lambda||x||^{2a}}{(r+s)}\left(\frac{2}{|r+s|^a} + \frac{1}{|r+s|^{2a}}, \frac{(r+s)}{(r+s)^{2a}-(r+s)}t\right)\right) \end{cases} \\ \nu\left(f(x) - \mathscr{A}(x),t\right) \\ \leq \begin{cases} \nu'\left(\frac{\lambda}{(r+s)}, \frac{(r+s)}{1-(r+s)}t\right) \\ \nu'\left(\frac{\lambda||x||^a}{(r+s)}\frac{2}{|r+s|^a}, \frac{(r+s)}{(r+s)^a-(r+s)}t\right) \\ \nu'\left(\frac{\lambda||x||^{2a}}{(r+s)}\frac{1}{|r+s|^{2a}}, \frac{(r+s)}{(r+s)^{2a}-(r+s)}t\right) \\ \nu'\left(\frac{\lambda||x||^{2a}}{(r+s)}\left(\frac{2}{|r+s|^a} + \frac{1}{|r+s|^{2a}}, \frac{(r+s)}{(r+s)^{2a}-(r+s)}t\right)\right) \end{cases} \end{array} \tag{4.13}$$

holds for all $x \in X$ and all $t > 0$.

Proof. Set

$$\begin{array}{l} \mu'\left(K(\chi_i^n x, \chi_i^n y), \chi_i^k t\right) \\ = \begin{cases} \mu'\left(\lambda, \chi_i^k t\right), \\ \mu'\left(\lambda\left(||x||^a + ||y||^a\right), \chi_i^{k-a}t\right), \\ \mu'\left(\lambda||x||^a||y||^a, \chi_i^{k-2a}t\right), \\ \mu'(\lambda\left\{||x||^a\ ||y||^a \right. \\ \qquad \left. + \left(||x||^{2a} + ||y||^{2a}\right)\right\}, \chi_i^{k-2a}t), \end{cases} \\ = \begin{cases} \to 1 \text{ as } k \to \infty \\ \to 1 \text{ as } k \to \infty \\ \to 1 \text{ as } k \to \infty \\ \\ \to 1 \text{ as } k \to \infty \end{cases} \end{array}$$

$$\begin{array}{l} \nu'\left(K(\chi_i^n x, \chi_i^n y), \chi_i^k t\right) \\ = \begin{cases} \nu'\left(\lambda, \chi_i^k t\right), \\ \nu'\left(\lambda\left(||x||^a + ||y||^a\right), \chi_i^{k-a}t\right), \\ \nu'\left(\lambda||x||^a||y||^a, \chi_i^{k-2a}t\right), \\ \nu'(\lambda\left\{||x||^a\ ||y||^a \right. \\ \qquad \left. + \left(||x||^{2a} + ||y||^{2a}\right)\right\}, \chi_i^{k-2a}t), \end{cases} \\ = \begin{cases} \to 0 \text{ as } k \to \infty \\ \to 0 \text{ as } k \to \infty \\ \to 0 \text{ as } k \to \infty \\ \\ \to 0 \text{ as } k \to \infty \end{cases} \end{array}$$

for all $x \in X$ and all $t > 0$. Thus, the relation (4.1) holds. It follows from (4.4), (4.5) and (4.12)

$$\left.\begin{array}{l} \mu'\left(\frac{1}{(r+s)}K\left(\frac{x}{(r+s)}\right),t\right) \\ = \begin{cases} \mu'\left(\frac{\lambda}{(r+s)},t\right) \\ \mu'\left(\frac{\lambda||x||^a}{(r+s)}\frac{2}{|r+s|^a},t\right) \\ \mu'\left(\frac{\lambda||x||^{2a}}{(r+s)}\frac{1}{|r+s|^{2a}},t\right) \\ \mu'\left(\frac{\lambda||x||^{2a}}{(r+s)}\left(\frac{2}{|r+s|^a}+\frac{1}{|r+s|^{2a}}\right),t\right) \end{cases} \\ \nu'\left(\frac{1}{(r+s)}K\left(\frac{x}{(r+s)}\right),t\right) \\ = \begin{cases} \nu'\left(\frac{\lambda}{(r+s)},t\right) \\ \nu'\left(\frac{\lambda||x||^a}{(r+s)}\frac{2}{|r+s|^a},t\right) \\ \nu'\left(\frac{\lambda||x||^{2a}}{(r+s)}\frac{1}{|r+s|^{2a}},t\right) \\ \nu'\left(\frac{\lambda||x||^{2a}}{(r+s)}\left(\frac{2}{|r+s|^a}+\frac{1}{|r+s|^{2a}}\right),t\right) \end{cases} \end{array}\right\}$$

for all $x, y \in X$ and all $t > 0$. Also from (4.5), we have

$$\left.\begin{array}{l} \mu'\left(\frac{\tau(\chi_i x)}{\chi_i},t\right) = \\ \begin{cases} \mu'(\lambda,\chi_i t) \\ \mu'\left(\frac{\lambda||x||^a}{(r+s)}\frac{2}{|r+s|^a},\chi_i^{k1-a}t\right) \\ \mu'\left(\frac{\lambda||x||^{2a}}{(r+s)}\frac{1}{|r+s|^{2a}},\chi_i^{1-2a}t\right) \\ \mu'\left(\frac{\lambda||x||^{2a}}{(r+s)}\left(\frac{2}{|r+s|^a}+\frac{1}{|r+s|^{2a}}\right),\chi_i^{1-2a}t\right) \end{cases} \\ \nu'\left(\frac{\tau(\chi_i x)}{\chi_i},t\right) = \\ \begin{cases} \nu'(\lambda,\chi_i t) \\ \nu'\left(\frac{\lambda||x||^a}{(r+s)}\frac{2}{|r+s|^a},\chi_i^{1-a}t\right) \\ \nu'\left(\frac{\lambda||x||^{2a}}{(r+s)}\frac{1}{|r+s|^{2a}},\chi_i^{1-2a}t\right) \\ \nu'\left(\frac{\lambda||x||^{2a}}{(r+s)}\left(\frac{2}{|r+s|^a}+\frac{1}{|r+s|^{2a}}\right),\chi_i^{1-2a}t\right) \end{cases} \end{array}\right\}$$

for all $x \in X$ and all $t > 0$. Hence, the inequality (4.6) is true for

	L	$a, i=0$	L	$a, i=1$
1.	$(r+s)$	0	$(r+s)^{-1}$	0
2.	$(r+s)^{1-a}$	$a<1$	$(r+s)^{a-1}$	$a>1$
3.	$(r+s)^{1-a}$	$2a<1$	$(r+s)^{2a-1}$	$2a>1$
4.	2^{1-2a}	$2a<1$	$(r+s)^{2a-1}$	$2a>1.$

Now, for condition 1. and $i = 0$, we have

$$\left.\begin{array}{r} \mu(f(x)-\mathscr{A}(x),t) \geq \mu'\left(\tau(x),\frac{(r+s)^{1-0}}{1-(r+s)}t\right) \\ = \mu'\left(\frac{\lambda}{(r+s)},\frac{(r+s)}{1-(r+s)}t\right) \\ \nu(f(x)-\mathscr{A}(x),t) \leq \nu'\left(\tau(x),\frac{(r+s)^{1-0}}{1-(r+s)}t\right) \\ = \nu'\left(\frac{\lambda}{(r+s)},\frac{(r+s)}{1-(r+s)}t\right) \end{array}\right\}$$

for all $x \in X$ and all $t > 0$. Also, for condition 1. and $i = 1$, we get

$$\left.\begin{array}{r} \mu(f(x)-\mathscr{A}(x),t) \geq \mu'\left(\tau(x),\frac{(((r+s))^{-1})^{1-1}}{1-((r+s))^{-1}}t\right) \\ = \mu'\left(\frac{\lambda}{(r+s)},\frac{(r+s)}{(r+s)-1}t\right) \\ \nu(f(x)-\mathscr{A}(x),t) \leq \nu'\left(\tau(x),\frac{(((r+s))^{-1})^{1-1}}{1-((r+s))^{-1}}t\right) \\ = \nu'\left(\frac{\lambda}{(r+s)},\frac{(r+s)}{(r+s)-1}t\right) \end{array}\right\}$$

for all $x \in X$ and all $t > 0$. Again, for condition 2. and $i = 0$, we obtain

$$\left.\begin{array}{r} \mu(f(x)-\mathscr{A}(x),t) \geq \mu'\left(\tau(x),\frac{((r+s)^{1-a})^{1-0}}{1-((r+s)^{1-a})}t\right) \\ = \mu'\left(\frac{\lambda||x||^a}{(r+s)}\frac{2}{|r+s|^a},\frac{(r+s)}{(r+s)^a-(r+s)^k}t\right) \\ \nu(f(x)-\mathscr{A}(x),t) \leq \nu'\left(\tau(x),\frac{((r+s)^{1-a})^{1-0}}{1-((r+s)^{1-a})}t\right) \\ = \nu'\left(\frac{\lambda||x||^a}{(r+s)}\frac{2}{|r+s|^a},\frac{(r+s)}{(r+s)^a-(r+s)}t\right) \end{array}\right\}$$

for all $x \in X$ and all $t > 0$. Also, for condition 2. and $i = 1$, we arrive

$$\left.\begin{array}{r} \mu(f(x)-\mathscr{A}(x),t) \geq \mu'\left(\tau(x),\frac{((r+s)^{a-1})^{1-1}}{1-((r+s)^{a-1})}t\right) \\ = \mu'\left(\frac{\lambda||x||^a}{(r+s)}\frac{2}{|r+s|^a},\frac{(r+s)}{(r+s)^k-(r+s)^a}t\right) \\ \nu(f(x)-\mathscr{A}(x),t) \leq \nu'\left(\tau(x),\frac{((r+s)^{a-1})^{1-1}}{1-((r+s)^{a-1})}t\right) \\ = \nu'\left(\frac{\lambda||x||^a}{(r+s)}\frac{2}{|r+s|^a},\frac{(r+s)}{(r+s)-(r+s)^a}t\right) \end{array}\right\}$$

for all $x \in X$ and all $t > 0$. The rest of the proof is similar to that of previous cases. This finishes the proof. □

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