Malaya Journal of Matematik, Vol. 6, No. 1, 286-290, 2018

https://doi.org/10.26637/MJM0601/0032

A geometric modeling of 2-dimensional parabolic problem with periodic boundary condition

F. Kanca[1]*, I. Baglan[2] and G. Öztürk[2]

Abstract

In this work, we tried to find the solution of a linear 2-dimensional parabolic equation with periodic boundary conditions. It showed the existence, uniqueness of solution by theoretical. Also we consider numerical solution for this problem by using finite differences method. Finally, we give a geometric modeling of the solution which corresponds to a surface.

Keywords

Two dimensional parabolic equation, egg box surface, geometric modeling.

AMS Subject Classification

65N22, 53A05.

[1] *Department of Management Information Systems, Kadir Has University, 34083, Istanbul, Turkey.*

[2] *Department of Mathematics, Kocaeli University, 41380, Kocaeli, Turkey.*

***Corresponding author**: 1 fatma.kanca@khas.edu.tr; 1 [2]isakinc@kocaeli.edu.tr, ogunay@kocaeli.edu.tr

Article History: Received **6** June **2017**; Accepted **7** August **2017**

Contents

1. Introduction

Geometric modeling is a speedily widening area of study with significant practices in graphics, manufacturing and designing with computer. Geometric modeling is the guide of projecting geometric models, which are geometric assets such as curves, surfaces. In geometric modeling one can see two methods used , that are parametric modeling and implicit modeling. In implicit modeling, the surface is given with level sets, i.e., with the form $H(x_1,x_2,x_3)=0$. In parametric one, it is simple to produce points on surfaces and curves, and this makes it a very useful way in computer aided design. Total arguments of geometric modeling can be analyzed from [1]. In [5], parametric equation of a surface in $\mathbb{R}^3$ is constructed with

$$X(s,\theta)=(x_1(s,\theta),x_2(s,\theta),x_3(s,\theta)),\ (s,\theta)\in[s_1,s_2]\times[\theta_1,\theta_2]$$

where the coordinates x_1, x_2, x_3 are differentiable maps with the parameters s and θ. In $\mathbb{R}^2$, a curve can be defined with the parameter s by decreasing the third coordinate.

Boundary conditions loaded around the sides of the surface control the shape of it. Then we ask how we can take the boundary conditions to generate a surface. The answer is to choose boundary conditions for riching a hoped form. In partial differential equation method, boundary conditions are generally united with the boundary curves in 3-dimensional space.

Many scientists are interested in two dimensional parabolic equation [2, 4, 8–10, 12]. These problems with non-local boundary conditions are becoming more difficult, especially, periodic boundary conditions [2, 7, 9].

Numerical solutions of such problems have been studied extensively for non-local boundary value problems in 1-dimension . For solving such problems, the following methods have been tried, finite difference method, finite element method etc. Among which the most commonly used is the explicit and implicit schemes of the finite difference method [2, 3].

Aim of this study is to find solution of parabolic partial differential 2-dimensional equation with non-local boundary conditions using Fourier series and finite difference method.

In recent work, we cope with the problem:

$$\begin{aligned}\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial s^2}+\frac{\partial^2 v}{\partial \theta^2}+g(s,\theta,t), \qquad (1.1)\\ (s,\theta,t) &\in \Omega:=\{0<s<\pi,\ 0<\theta<\pi,\ 0<t<T\}\end{aligned}$$

$$\begin{aligned} v(0,\theta,t) &= v(\pi,\theta,t)\ ,t\in[0,T] \\ v(s,0,t) &= v(s,\pi,t)\ ,t\in[0,T] \end{aligned} \tag{1.2}$$

$$\begin{aligned} v_s(s,0,t) &= v_s(s,\pi,t)\ ,t\in[0,T] \\ v_\theta(s,0,t) &= v_\theta(s,\pi,t)\ ,t\in[0,T] \end{aligned} \tag{1.3}$$

$$v(s,\theta,t)=\vartheta(s,\theta)\ ,s\in[0,\pi] \tag{1.4}$$

for a parabolic 2-dimensional equation with the periodic boundary conditions. The function $\vartheta(s,\theta)$ and $g(s,\theta,t)$ are given functions on $[0,\pi]$ and $\overline{\Omega}$ respectively. We denote the solution of problem (1.1)-(1.4) by $v(s,\theta,t)$. The first condition means initial condition, the other condition means periodic Dirichlet and Neumann condition [7]. We prove the existence, uniqueness and build an numerical iteration algorithm for the solution. We use Fourier method for the considered problem (1.1)-(1.4) and give the stability of method for the solution. Also, we investigate an example for the solution of problem (1.1)-(1.4). Finally, we give a geometric modeling of the solution of problem (1.1)-(1.4) with initial conditions which corresponds to a surface known as a translation surface (particular egg box surface).

2. Existence and uniqueness of the solution

Now, we stand the solution of (1.1)-(1.4) by

$$\begin{aligned} v(s,\theta,t) &= \frac{v_0(t)}{4}+\sum_{m,n=1}^{\infty}\left(v_{cmn}\cos ms\ \cos n\theta+v_{csmn}\cos ms\ \sin n\theta\right. \\ &\quad \left.+v_{scmn}\sin ms\cos n\theta+v_{smn}\sin ms\sin n\theta\right) \end{aligned} \tag{2.1}$$

where $\vartheta(s,\theta)$ in (1.1) has to be the type

$$\begin{aligned} \vartheta(s,\theta) &= \frac{\vartheta_0}{4}+\sum_{m,n=1}^{\infty}\left(\vartheta_{cmn}\cos ms\cos n\theta+\vartheta_{csmn}\cos ms\sin n\theta\right. \\ &\quad \left.+\vartheta_{scmn}\sin ms\cos n\theta+\vartheta_{smn}\sin ms\sin n\theta\right). \end{aligned} \tag{2.2}$$

Then we carry out the Fourier series, we get Fourier coefficients:

$$\begin{aligned} v_0(t) &= \vartheta_0(0,0)+\frac{4}{\pi^2}\int_0^t g_{cmn}(\tau)d\tau \\ v_{cmn}(t) &= \vartheta_{cmn}(0,0)+\frac{4}{\pi^2}\int_0^t e^{-\left[(2m)^2+(2n)^2\right](t-\tau)}g_{cmn}(\tau)d\tau \\ v_{csmn}(t) &= \vartheta_{csmn}(0,0)+\frac{4}{\pi^2}\int_0^t e^{-\left[(2m)^2+(2n)^2\right](t-\tau)}g_{csmn}(\tau)d\tau \\ v_{scmn}(t) &= \vartheta_{scmn}(0,0)+\frac{4}{\pi^2}\int_0^t e^{-\left[(2m)^2+(2n)^2\right](t-\tau)}g_{scmn}(\tau)d\tau \\ v_{smn}(t) &= \vartheta_{smn}(0,0)+\frac{4}{\pi^2}\int_0^t e^{-\left[(2m)^2+(2n)^2\right](t-\tau)}g_{smn}(\tau)d\tau \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \vartheta_0(0,0) &= v_0(0,0),\ \vartheta_{cmn}(0,0)=v_{cmn}(0,0)e^{-\left[(2m)^2+(2n)^2\right]t}, \\ \vartheta_{csmn}(0,0) &= v_{csmn}(0,0)e^{-\left[(2m)^2+(2n)^2\right]t}, \\ \vartheta_{scmn}(0,0) &= v_{scmn}(0,0)e^{-\left[(2m)^2+(2n)^2\right]t}, \\ \vartheta_{smn}(0,0) &= v_{smn}(0,0)e^{-\left[(2m)^2+(2n)^2\right]t}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} g_{cmn}(t) &= \frac{4}{\pi^2}\int_0^\pi\int_0^\pi g(\xi,\eta,\tau)\cos m\xi\ \cos n\eta\ d\xi d\eta \\ g_{csmn}(t) &= \frac{4}{\pi^2}\int_0^\pi\int_0^\pi g(\xi,\eta,\tau)\cos m\xi\ \sin n\eta\ d\xi d\eta \\ g_{scmn}(t) &= \frac{4}{\pi^2}\int_0^\pi\int_0^\pi g(\xi,\eta,\tau)\sin m\xi\ \cos n\eta\ d\xi d\eta \\ g_{smn}(t) &= \frac{4}{\pi^2}\int_0^\pi\int_0^\pi g(\xi,\eta,\tau)\sin m\xi\ \sin n\eta\ d\xi d\eta. \end{aligned} \tag{2.5}$$

Let find the solution of system (1.1) by

$$\begin{aligned} v(s,\theta,t) &= \frac{1}{4}\left(\vartheta_0(0,0)+\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi g(\xi,\eta,\tau)d\xi d\eta d\tau\right) \\ &+\sum_{m,n=1}^{\infty}\vartheta_{cmn}(0,0)\cos ms\cos n\theta \\ &+\sum_{m,n=1}^{\infty}\left(\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi e^{-\left[(2m)^2+(2n)^2\right](t-\tau)}g(\xi,\eta,\tau)\cos m\xi\ \cos n\eta\ d\xi d\eta d\tau\right)\cos ms\cos n\theta \\ &+\sum_{m,n=1}^{\infty}\vartheta_{csmn}(0,0)\cos ms\sin n\theta \\ &+\sum_{m,n=1}^{\infty}\left(\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi e^{-\left[(2m)^2+(2n)^2\right](t-\tau)}g(\xi,\eta,\tau)\cos m\xi\ \sin n\eta\ d\xi d\eta d\tau\right)\cos ms\sin n\theta \\ &+\sum_{m,n=1}^{\infty}\vartheta_{scmn}(0,0)\sin ms\cos n\theta \\ &+\sum_{m,n=1}^{\infty}\left(\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi e^{-\left[(2m)^2+(2n)^2\right](t-\tau)}g(\xi,\eta,\tau)\sin m\xi\ \cos n\eta\ d\xi d\eta d\tau\right)\sin ms\cos n\theta \\ &+\sum_{m,n=1}^{\infty}\vartheta_{smn}(0,0)\sin ms\sin n\theta \\ &+\sum_{m,n=1}^{\infty}\left(\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi e^{-\left[(2m)^2+(2n)^2\right](t-\tau)}g(\xi,\eta,\tau)\sin m\xi\ \sin n\eta\ d\xi d\eta d\tau\right)\sin ms\sin n\theta. \end{aligned}$$

Theorem 2.1. *Let $g(s,\theta,t)$, $\vartheta(s,\theta)$ be continuous functions according to all parameters then the solution of system (1.1) has unique solutions.*

Here $\vartheta(s,\theta)\varepsilon C([0,\pi]\times[0,\pi])$ and $g(s,\theta,t)\,\varepsilon C\left(\overline{\Omega}\right)$. The equation (2.10) and $\sum\limits_{k=1}^{\infty}\frac{\partial}{\partial s}$, $\sum\limits_{k=1}^{\infty}\frac{\partial}{\partial\theta}$ are convergent uniformly in $\partial\Omega$ then their majorizing sums are absolutely convergent. Therefore their sums $v(s,\theta)$, $v_s(s,t)$ and $v_\theta(s,t)$ are continuous in $\overline{\Omega}$. In addition, $\sum\limits_{k=1}^{\infty}\frac{\partial}{\partial t}$ and $\sum\limits_{k=1}^{\infty}\frac{\partial^2}{\partial s^2}$, $\sum\limits_{k=1}^{\infty}\frac{\partial^2}{\partial\theta^2}$ are uniformly convergent . Finally, $v_t(s,t)$ is continuous in Ω because the majorizing sum of $\sum\limits_{k=1}^{\infty}\frac{\partial}{\partial t}$ is absolutely convergent.

So $v(s,\theta,t)$ has unique solution.

Definition 2.2. *$v(s,\theta,t)\in C^{2,2,1}(\Omega)\cap C^{1,1,0}\left(\overline{\Omega}\right)$ is called the classical solution of two dimensional parabolic equation.*

3. Finite Differences Method for Discretizing

In this section, we use implicit finite-difference method for discreazing problem (1.1)-(1.3):

$$\frac{1}{\tau}\left(v_{i,j}^{k+1}-v_{i,j}^{k}\right) = \frac{1}{h^2}\left(v_{i-1,j}^{k+1}-2v_{i,j}^{k+1}+v_{i+1,j}^{k+1}\right)+ \frac{1}{h^2}\left(v_{i,j-1}^{k+1}-2v_{i,j}^{k+1}+v_{i,j+1}^{k+1}\right)+g_{i,j}^{k}, \tag{3.1}$$

$$v_{i,j}^{0}=\phi_i, \tag{3.2}$$

$$v_{0,j}^{k}=v_{M+1,j}^{k},\ v_{M+1,j}^{k}=\frac{v_{1,j}^{k}-v_{M,j}^{k}}{2} \tag{3.3}$$

$$v_{i,0}^{k}=v_{i,M+1}^{k}, v_{i,M+1}^{k}=\frac{v_{i,1}^{k}-v_{i,M}^{k}}{2}$$

where, we discretize the computing domain $[0,\pi]\times[0,\pi]\times[0,T]$ by $s_i=ih,\ i=0,1,...,M,\ \theta_j=jh,\ j=0,1,...,M$ and $t_k=k\tau,\ k=0,1,...,N,$ where $h=\pi/M$ and $\tau=T/N$ are the space and time steps, respectively and M,N are positive integers, $v_{i,j}^{k}=v(s_i,\theta_j,t_k), g_{i,j}^{k}=g(s_i,\theta_j,t_k)$.

4. Visualization

In $\mathbb{R}^3$, a surface can be defined with Monge patch. That is, at a point (s,θ), a surface is given by a differentialable function $x_3(s,\theta)$.

The translation surface is one of the surprising surface in $\mathbb{R}^3$ with the following definition as a Monge patch constructed by $x_3(s,\theta)=f_1(s)+f_2(\theta)$, where f_1 and f_2 are differential functions:

Definition 4.1. *A surface in $\mathbb{R}^3$ given with the parametrization*

$$X:U\subset\mathbb{R}^2\to\mathbb{R}^3:(s,\theta)\mapsto(s,\theta,\ f_1(s)+f_2(\theta)),$$

is called a translation surface, where $x_3(s,\theta)=f_1(s)+f_2(\theta)$, f_1 and f_2 are differentialable functions.

It can be seen that translation surface is constructed by two curves. For this reason, researchers use the translation surface to design in architecture, see [6]. A translation surface which is minimal, but not flat was found by H. Scherk in 1835 and it is known as Scherk's surface [11].

Now, in order to illustrate the behavior of our numerical method, we consider the following examples:

Example 4.2. *Let*

$$\vartheta(s,\theta) = \sin 2s+\cos 2\theta$$
$$g(s,\theta,t) = \frac{9}{2}(\sin 2s+\cos 2\theta)\exp(t)$$

be the functions which satisfies the initial condition. Then we have the analytical solution

$$v(s,\theta,t)=(\sin 2s+\cos 2\theta)\exp(t).$$

The step sizes are $h=0.0393$, $\tau=0.005$.

Let us take $T=1$.

The comparisons between the analytical solution and the approximate solution are shown in Figures 1,2.

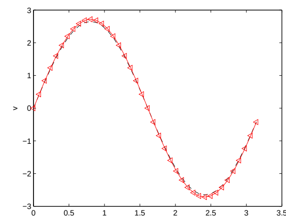

Figure 1: The analytical and approximate solutions of $v(\frac{\pi}{2},\theta,1)$.

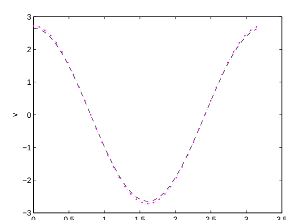

Figure 2: The analytical and approximate solutions of $v(s,\frac{\pi}{2},1)$.

In Figure 3, we give the graph of the solution of problem (1.1)-(1.4) which corresponds to the translation surface. This surface is known as egg box surface.

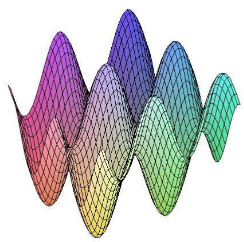

Figure 3: Translation surface corresponds the solution of $v(s,\theta,1)$ (egg box surface).

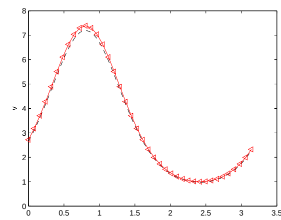

Figure 4: The analytical and approximate solutions of $v(\frac{\pi}{2},\theta,1)$.

Example 4.3. *Let*

$$\begin{aligned}
\vartheta(s,\theta) &= \exp(\sin 2s)+\exp(\cos 2\theta) \\
g(s,\theta,t) &= \exp(t+\sin 2s)(1-4(\cos^2 2s-\sin 2s)) \\
&\quad +\exp(t+\cos 2\theta)(1-4(\sin^2 2\theta-\cos 2\theta))
\end{aligned}$$

be the functions which satisfies the initial condition. Then we have the analytical solution

$$v(s,\theta,t)=(\exp(\sin 2s)+\exp(\cos 2\theta))\exp(t).$$

The step sizes are $h=0.0393$, $\tau=0.005$.

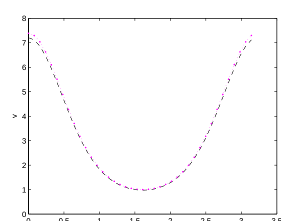

Figure 5: The analytical and approximate solutions of $v(s,\frac{\pi}{2},1)$.

Let us take $T=1$.

The comparisons between the exact solution and the numerical finite difference solution are shown in Figures 4,5.

Finally, in Figure 6, we give the graph of the solution of problem (1.1)-(1.4) which corresponds to the translation surface.

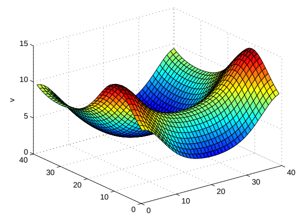

Figure 6: Translation surface corresponds the solution of $v(s,\theta,1)$.

References

[1] J. Barnak and A. Fisher, Parametrization and reconstruction from 3D scattered points based on neural network and PDE techniques, *IEEE Transactions on Visualization and Computer Graphics*, 7(2001), 1-16.

[2] J. R. Cannon, The solution of the heat conduction subject to the specification of energy, *Quart Appl.Math.*, 21(1983), 155-160.

[3] J. R. Cannon, *The one-dimensional heat equation*, Encyclopedia of Mathematics and Its Applications, Addison-Wesley,California, 1984.

[4] C. Chen, F. Liu, I. Turner, V. Anh, Numerical Schemes and multivariate extrapolation of a two dimensional anomalaus sub-diffusion equation, *Numer. Alg.*, 62(2013), 383-409.

[5] G. Farin, *Curves and surfaces for computer aided geometric design*, Second edition, Academic Press, New York, 1990.

[6] J. Glymph, D. Schelden, C. Ceccato, J. Mussel and H. Schober, A parametric strategy for free-from glass structures using quadrilateral planar facets, *Automation in Construction*, 13(2004), 187-202.

[7] G. W. Hill, On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon, *Acta Mathematica*, 8(1886), 1-36.

[8] N. I. Kamynin, A boundary value problem in the theory of the heat conduction with non classical boundary condition, *Ussr Comput. Math. Phys.*, 4(1964), 33-59.

[9] Y. Lin, S. Xu, H. M. Yin, Finite difference approximations for a class of Nonlocal parabolic equation, *BIT*, 31(1991), 245-255.

[10] Y. S. Choi, K. Y. Chan, A parabolic equation with a non local boundary condition arising from electrochemistry, *Nonlinear Analysis*, 18(1992), 317-331.

[11] H. F. Scherk, Bemerkungen ber die kleinste Flche innerhalb gegebener Grenzen, *J. R. Angew. Math.*, 13(1835), 185-208.

[12] P. R. Sharma, G. Methi, Solution of two dimensional parabolic equation subject to non-local conditions using homotopy Perturbation method, *Jour. of App.Com. Sci.*, 1(2012), 12-16.

⋆⋆⋆⋆⋆⋆⋆⋆⋆

ISSN(P):2319 − 3786

Malaya Journal of Matematik

ISSN(O):2321 − 5666

⋆⋆⋆⋆⋆⋆⋆⋆⋆