Malaya Journal of Matematik, Vol. 6, No. 2, 299-309, 2018
https://doi.org/10.26637/MJM0602/0001

Analysis of a quasistatic contact problem with wear and damage for thermo-viscoelastic materials

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Abstract
We consider a quasistatic contact problem for an thermo visco-elastic body with wear and damage between a thermo-viscoelastic body and a rigid obstacle. The contact is frictional and bilateral which results in the wear and damage of contacting surface. The evolution of the wear function is described with Archard's law.The evolution of the damage is described by an inclusion of parabolic type. We establish a variational formulation for the model and we prove the existence of a unique weak solution to the problem. The proof is based on a classical existence and uniqueness result on parabolic inéqualities, differential equations and fixed point argument

Keywords
thermoviscoelastic, variationel inequality, wear,damage field, fixed point.

AMS Subject Classification
74F05, 74M15, 74H20, 74R20, 74H25 .

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Article History: Received **15** February **2018**; Accepted **13** March **2018**

Contents

1. Introduction

Considerable progress has been achieved recently in modeling, mathematical analysis and numerical simulations of various contact processes and, as a result, a general mathematical theory of contact mechanics is currently maturing. It is concerned with the mathematical structures which underlie general contact problems with different constitutive laws (i.e., different materials), varied geometries and settings, and different contact conditions, see for instance [13, 21, 22] and the references therein. The theory's aim is to provide a sound, clear and rigorous background for the constructions of models for contact between deformable bodies; proving existence, uniqueness and regularity results; assigning precise meaning to solutions; and the necessary setting for finite element approximations of the solutions.

The modelisation of a contact phenomenon is determined by a set of assumptions influencing on the form and structure of partial differential equations system or on boundary conditions of the associated mathematical model.

Among the assumptions influencing the partial differential equations system :

-*Hypothesis about the geometry of the deformation* (small deformation or others).

-*Hypothesis about the mechanical process* (quasi-static or dynamic).

-*Hypothesis about the laws of material behavior* (elastic, viscoelastic,...).

The model equations can be influenced by additional phenomena (thermal, piezoelectric,...).

The boundary conditions on the contact surface are described in both normal direction and in the tangential plane, these are called boundary conditions of friction.

In the direction of normal, we have unilateral and bilateral contact (when there is no separation between the body and the obstacle). The normal compliance (when the obstacle is deformable).

The boundary conditions are also influenced by the consideration of various underlying phenomena accompanying the contact with friction, adhesion, wear, thermal effects, the dependence of the threshold friction versus slip or the slip speed can influence the boundary conditions of the mathematical model.

The constitutive laws with internal variables has been used in various publications in order to model the effect of internal variables in the behavior of real bodies. Some of the internal state variables considered by many authors are the temperature and the damage field. Wear is one of the processes which reduce the lifetime of modern machine elements. It represents the unwanted removal of materials from surfaces of contacting bodies occurring in relative motion.

In this paper we consider a mathematical frictional contact, between thermo-viscoelastic body and a rigid obstacle, with damage and wear. For this, we consider rate-type constitutive equation of the form

$$\sigma = \mathscr{A}\varepsilon(\dot{\mathbf{u}})+\mathscr{G}\big(\varepsilon(\mathbf{u}),\zeta\big)+\mathscr{F}(\theta,\zeta), \quad \text{in } \Omega\times(0,T) \quad (1.1)$$

In which $\mathbf{u}$, σ represent, respectively, the displacement field and the stress field where the dot above denotes the derivative with respect to the time variable, θ represents the temperature, ζ is the damage field, $\mathscr{A}$ and $\mathscr{G}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, and $\mathscr{F}$ is a nonlinear constitutive function which describe the behavior of the material. The differential inclusion used for the evolution of the damage field is

$$\dot{\zeta} - \kappa\Delta\zeta + \partial\psi_K(\zeta) \ni \phi\big(\varepsilon(\mathbf{u}),\zeta\big), \quad \text{in } \Omega\times(0,T), \quad (1.2)$$

Where $\psi_K(\zeta)$ denote the subdifferential of the indicator function of the set K of admissible damage functions defined by

$$K = \{\zeta \in H^1(\Omega); 0 \le \zeta \le 1, \text{ a.e. in } \Omega\}, \quad (1.3)$$

and ϕ is given constitutive function which describe the source of the damage in the system. When $\zeta = 0$ the material is completely damaged, when $\zeta = 1$ the material is undamaged, , and for $0 < \zeta < 1$ there is partial damage.

The thermo-viscoelastic constitutive law (1.1) includes a temperature effects described by the parabolic equation given by

$$\dot{\theta} - \kappa_0\Delta\theta = \psi\big(\sigma,\varepsilon(\mathbf{u}),\theta\big) + q, \quad \text{in } \Omega\times(0,T), \quad (1.4)$$

Analysis of a dynamic thermo-elastic-viscoplastic contact problem.was studies in [3] *A frictional contact problem with wear involving elastic-viscoplastic materials with damage and thermal effects*.can be found in [4].*Dynamic evolution of damage in elastic-thermo-viscoplastic materials*.was studies in [18].

In this paper we study a quasistatic problem of frictional bilateral contact with wear and damage. We model the material behavior with an thermo viscoelastic constitutive law .

This article is organized as follows. In Section 2 we describe the mathematical models for the frictional contact problem between thermo-viscoelastic body and a rigid obstacle with damage. The contact is modelled with normal compliance and wear. In Section 3 we introduce some notation, list the assumptions on the problem's data, and derive the variational formulation of the model. In Section 4 we state and prove our main existence and uniqueness result, the prove is carried out in several steps and is based on arguments of evolutionary variational inequalities, a classical existence and uniqueness result on parabolic inequalities, differential equations and the Banach fixed point theorem.

2. Problem Statement

Let us consider a thermo-viscoelastic body occupying a bounded domain Ω of the space $\mathbb{R}^d (d = 2,3)$. Tthe boundary Γ of Ω is assumed to be Lipschitz continuous, and is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that the measure of Γ_1 is strictly positive. The body and the rigid obstacle are in bilateral frictional contact along the part Γ_3. Let $T > 0$ and let $[0,T]$ be the time interval of interest We admit a possible external heat source applied in $\Omega\times(0,T)$, given by the functions $\mathbf{q}$. The body Ω is clamped on $\Gamma_1\times(0,T)$. The surface tractions $\mathbf{f}_2$ act on $\Gamma_2\times(0,T)$ and a body forces of density $\mathbf{f}_0$ acts on $\Omega\times(0,T)$. We model the materials with thermo-viscoelastic constituve law with damage. We also assume that the normal derivative of ζ represent a homogeneous Neumann boundary conditions where

$$\frac{\partial\zeta}{\partial\nu} = 0$$

The body and the rigid obstacle can enter in contact along the part Γ_3. We introduce the wear function $\omega : \Gamma_3\times(0,T) \to \mathbb{R}^+$ which measures the wear of the surface. The wear is identified as the normal depth of the material that is lost. Let g be the initial gap between the body and the rigid foundation . Let p_ν and p_τ denote the normal and tangential compliance functions. We denote by $\mathbf{v}^*$ and $\alpha^* = \|\mathbf{v}^*\|$ the tangential velocity and the tangential speed at the contact surface between the body and the rigid foundation. We use the modified version of Archard's law:

$$\dot{\omega} = -\kappa_\omega \mathbf{v}^*\sigma_\nu$$

To describe the evolution of wear, where $\kappa_\omega > 0$ is a wear coefficient. We introduce the unitary vector $\delta : \Gamma_3 \to \mathbb{R}^d$ defined by $\delta = \mathbf{v}^*/\|\mathbf{v}^*\|$. When the contact arises, some material of the contact surfaces worn out and immediately removed from the system. This process is measured by the wear function ω.

With the assumption above, the classical formulation of the friction contact problem with wear and damage between a thermo-viscoelastics body and rigid obstacle is following.

Problem P

Find a displacement field $\mathbf{u} : \Omega \times (0,T) \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times (0,T) \to \mathbb{S}^d$, a temperature $\theta : \Omega \times (0,T) \to \mathbb{R}$, a damage $\zeta : \Omega \times (0,T) \to \mathbb{R}$, and a wear $\omega : \Gamma_3 \times (0,T) \to \mathbb{R}^+$ such that

$$\sigma = \mathscr{A}\varepsilon(\dot{\mathbf{u}})+\mathscr{G}\big(\varepsilon(\mathbf{u}),\zeta\big)+\mathscr{F}(\theta,\zeta), \text{in } \Omega \times (0,T) \quad (2.1)$$

$$\dot{\theta} - \kappa_0\Delta\theta = \psi\big(\sigma,\varepsilon(\dot{\mathbf{u}}),\theta\big)+\mathbf{q}, \quad \text{in } \Omega \times (0,T), \quad (2.2)$$

$$\dot{\zeta} - \kappa\Delta\zeta + \partial\psi_K(\zeta) \ni \phi\left(\varepsilon(\mathbf{u}),\zeta\right), \text{in} \Omega \times (0,T) \quad (2.3)$$

$$Div\sigma + \mathbf{f}_0 = 0 \quad \text{in } \Omega \times (0,T), \quad (2.4)$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_1 \times (0,T), \quad (2.5)$$

$$\sigma \cdot \nu = \mathbf{f}_2 \quad \text{on } \Gamma_2 \times (0,T), \quad (2.6)$$

$$\sigma_\nu = p_\nu(\mathbf{u}_\nu - \omega - g) \quad \text{on } \Gamma_3 \times (0,T), \quad (2.7)$$

$$\sigma_\tau = -p_\tau(\mathbf{u}_\nu - \omega - g)\frac{\mathbf{v}^*}{\|\mathbf{v}^*\|}, \quad \text{on } \Gamma_3 \times (0,T), \quad (2.8)$$

$$\dot{\omega} = -\kappa_\omega\alpha^*\sigma_\nu = \kappa_\omega\alpha^* p_\nu(\mathbf{u}_\nu - \omega - g), \text{on } \Gamma_3 \times (0,T) \quad (2.9)$$

$$k_1\frac{\partial\theta}{\partial\nu} + B\theta = 0, \quad \text{on } \Gamma \times (0,T), \quad (2.10)$$

$$\frac{\partial\zeta}{\partial\nu} = 0, \quad \text{on } \Gamma_1 \times (0,T), \quad (2.11)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \theta(0) = \theta_0, \quad \zeta(0) = \zeta_0, \quad \text{in } \Omega, \quad (2.12)$$

$$\omega(0) = \omega_0, \quad \text{on } \Gamma_3 \quad (2.13)$$

Equations (2.1) and(2.2)represent the thermo-viscoelastic constitutive law with damage, the evolution of the damage is governed by the inclusion of parabolic type given by the relation (2.3). Equation (2.4) is the equilibrium equations for the stress. Equations (2.5) and (2.6) represent the displacement and traction boundary condition, respectively. Condition (2.7) and (2.8)represents the frictional bilateral contact with wear described above.

Next, the equation (2.9) represents the ordinary differential equation which describes the evolution of the wear function. Equations (2.10) and (2.11) represent, respectively on Γ, a Fourier boundary condition for the temperature and an homogeneous Neumann boundary condition for the damage field on Γ. the functions $\mathbf{u}_0$, θ_0, $\zeta_0$and ω_0 in (2.12) and (2.13) are the initial data.

3. Variational formulation and preliminaries

In this section, we list the assumptions on the data and derive a variational formulation for the contact problem. To this end, we need to introduce some notation and preliminary material. Here and below, $\mathbb{S}^d$ represent the space of second-order symmetric tensors on $\mathbb{R}^d$. We recall that the inner products and the corresponding norms on $\mathbb{S}^d$ and $\mathbb{R}^d$ are given by

$$\mathbf{u}.\mathbf{v} = u_i.v_i, \quad |\mathbf{v}| = (\mathbf{v}.\mathbf{v})^{\frac{1}{2}}, \quad \forall \mathbf{u},\mathbf{v} \in \mathbb{R}^d,$$
$$\sigma.\tau = \sigma_{ij}.\tau_{ij}, \quad |\tau| = (\tau \cdot \tau)^{\frac{1}{2}}, \quad \forall \sigma,\tau \in \mathbb{S}^d.$$

Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. Now, to proceed with the variational formulation, we need the following function spaces:

$$\begin{aligned} H &= \{\mathbf{v} = (v_i); v_i \in L^2(\Omega)\}, \\ H_1 &= \{\mathbf{v} = (v_i); v_i \in H^1(\Omega)\}, \\ \mathscr{H} &= \{\tau = (\tau_{ij}); \tau_{ij} = \tau_{ji} \in L^2(\Omega)\}, \\ \mathscr{H}_1 &= \{\tau = (\tau_{ij}) \in \mathscr{H}; div\tau \in H\}. \end{aligned}$$

The spaces H, H_1, $\mathscr{H}$ and $\mathscr{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\mathbf{u},\mathbf{v})_H = \int_\Omega \mathbf{u}.\mathbf{v}dx, \qquad (\mathbf{u},\mathbf{v})_{H_1} = \int_\Omega \mathbf{u}.\mathbf{v}dx + \int_\Omega \nabla\mathbf{u}.\nabla\mathbf{v}dx,$$
$$(\sigma,\tau)_{\mathscr{H}} = \int_\Omega \sigma.\tau dx, \qquad (\sigma,\tau)_{\mathscr{H}_1} = \int_\Omega \sigma.\tau dx + \int_\Omega div\sigma.Div\tau dx,$$

and the associated norms $\|\cdot\|_H$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathscr{H}}$, and $\|\cdot\|_{\mathscr{H}_1}$ respectively. Here and below we use the notation

$$\begin{aligned} \nabla\mathbf{u} &= (u_{i,j}), \\ \varepsilon(\mathbf{u}) &= (\varepsilon_{ij}(\mathbf{u})), \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j}+u_{j,i}), \quad \forall u \in H_1 \\ Div\sigma &= (\sigma_{ij,j}), \quad \forall \sigma \in \mathscr{H}_1. \end{aligned}$$

For every element $\mathbf{v} \in H_1$, we also use the notation $\mathbf{v}$ for the trace of $\mathbf{v}$ on Γ and we denote by v_ν and $\mathbf{v}_\tau$ the *normal* and the *tangential* components of $\mathbf{v}$ on the boundary Γ given by

$$v_\nu = \mathbf{v}.\nu, \quad \mathbf{v}_\tau = \mathbf{v} - v_\nu\nu.$$

Let H'_Γ be the dual of $H_\Gamma = H^{\frac{1}{2}}(\Gamma)^d$ and let $(\cdot,\cdot)_{-\frac{1}{2},\frac{1}{2},\Gamma}$ denote the duality pairing between H'_Γ and H_Γ. For every element $\sigma \in \mathscr{H}_1$ let $\sigma\nu$ be the element of H'_Γ given by

$$(\sigma\nu,\mathbf{v})_{-\frac{1}{2},\frac{1}{2},\Gamma} = (\sigma,\varepsilon(\mathbf{v}))_{\mathscr{H}} + Div\sigma,\mathbf{v})_H \quad \forall \mathbf{v} \in H_1.$$

Denote by σ_ν and σ_τ the *normal* and the *tangential* traces of $\sigma \in \mathscr{H}_1$, respectively. If σ is continuously differentiable on $\Omega \cup \Gamma$, then

$$\sigma_\nu = (\sigma\nu)\cdot\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu\nu,$$
$$(\sigma\nu,\mathbf{v})_{-\frac{1}{2},\frac{1}{2},\Gamma} = \int_\Gamma \sigma\nu\cdot\mathbf{v}da$$

for all $\mathbf{v} \in H_1$, where da is the surface measure element.

To obtain the variational formulation of the problem (2.1) (2.13), we introduce for the displacement field we need the closed subspace of H_1 defined by

$$V = \left\{\mathbf{v} \in H_1; \mathbf{v} = 0 \text{ on } \Gamma_1\right\}.$$

Since $meas\Gamma_1 > 0$, the following Korn's inequality holds:

$$\|\varepsilon(\mathbf{v})\|_{\mathscr{H}} \geq c_K\|\mathbf{v}\|_{H_1} \quad \forall \mathbf{v} \in V, \tag{3.1}$$

Where the constant c_K denotes a positive constant which may depends only on Ω, Γ_1 . Over the space V we consider the inner product given by

$$(\mathbf{u},\mathbf{v})_V = (\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v}))_{\mathscr{H}}, \quad \forall \mathbf{u},\mathbf{v} \in V, \tag{3.2}$$

Let $\|\cdot\|_V$ be the associated norm. It follows from Korn's inequality (3.1) that the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent on V. Then $(V,\|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (3.2), there exists a constant $c_0 > 0$, depending only on Ω, Γ_1 and Γ_3 such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \leq c_0\|\mathbf{v}\|_V \quad \forall \mathbf{v} \in V. \tag{3.3}$$

We also introduce the spaces

$$E_0 = L^2(\Omega), \quad E_1 = H^1(\Omega),$$

The spaces V, E_1 are real Hilbert spaces endowed with the canonical inner products denoted by $(\cdot,\cdot)_V$, $(\cdot,\cdot)_{E_1}$. The associate norms will be denoted by $\|\cdot\|_V$, $\|\cdot\|_{E_1}$, respectively. We recall the following standard result for parabolic variational inequalities used in section 4 (see [2, p124]). Let V and H be real Hilbert spaces such that V is dense in H and the injection map is continuous. The space H is identified with its own dual and with a subspace of the dual V' of V . We write

$$V \subset H \subset V'.$$

We say that the inclusions above define a Gelfand triple. We denote by $\|\cdot\|_V$, $\|\cdot\|_H$, and $\|\cdot\|_{V'}$, the norms on the spaces V , H and V' respectively, and we use $(\cdot,\cdot)_{V'\times V}$ for the duality pairing between V' and V. Note that if $f \in H$ then

$$(f,\mathbf{v})_{V'\times V} = (f,\mathbf{v})_H ,\forall \mathbf{v} \in H.$$

Theorem 3.1. *Let $V \subset H \subset V'$ be a Gelfand triple. Let K be a nonempty, closed, and convex set of V . Assume that $\mathfrak{a}(\cdot,\cdot) : V \times V \to \mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants $\zeta > 0$ and c_0,*

$$\mathfrak{a}(\mathbf{v},\mathbf{v}) = c_0\,\|\mathbf{v}\|_H^2 \geqslant \zeta\,\|\mathbf{v}\|_V^2 \ ,$$

$$\forall \mathbf{v} \in H$$

Then, for every $\mathbf{u}_0 \in K$ and $f \in L^2(0,T;H)$, there exists a unique function $\mathbf{u} \in H^1(0,T;H) \cap L^2(0,T;V)$ such that

$$\mathbf{u}(0) = \mathbf{u}_0, \mathbf{u}(t) \in K \quad for\,all\,t \in [0,T],$$

and almost all $t \in (0,T)$,

$$(\dot{\mathbf{u}}(t), \mathbf{v} - \mathbf{u}(t))_{V'\times V} + \mathfrak{a}(\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t)) \geqslant (f(t), \mathbf{v} - \mathbf{u}(t))_H ,$$

$\forall \mathbf{v} \in K$,

Finally, for any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$, $W^{k,p}(0,T;X)$, where $1 \leq p \leq \infty$, $k \geq 1$. We denote by $C(0,T;X)$ and $C^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$\|f\|_{C(0,T;X)} = \max_{t\in[0,T]} \|f(t)\|_X,$$
$$\|f\|_{C^1(0,T;X)} = \max_{t\in[0,T]} \|f(t)\|_X + \max_{t\in[0,T]} \|\dot{f}(t)\|_X,$$

respectively.

In the study of the Problem **P**, we consider the following assumptions:
The *viscosity function* $\mathscr{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

(a) There exists $L_{\mathscr{A}} > 0$ such that
$$|\mathscr{A}(x,\xi_1) - \mathscr{A}(x,\xi_2)| \leq L_{\mathscr{A}}|\xi_1 - \xi_2|$$
for all $\xi_1,\xi_2 \in \mathbb{S}^d$, a.e. $x \in \Omega$.
(b) There exists $m_{\mathscr{A}} > 0$ such that
$$(\mathscr{A}(x,\xi_1) - \mathscr{A}(x,\xi_2))\cdot(\xi_1 - \xi_2) \geq m_{\mathscr{A}}|\xi_1 - \xi_2|^2 \tag{3.4}$$
for all $\xi_1,\xi_2 \in \mathbb{S}^d$, a.e. $x \in \Omega$.
(c) The mapping $x \mapsto \mathscr{A}(x,\xi)$ is Lebesgue measurable on Ω, for any $\xi \in \mathbb{S}^d$.
(d) The mapping $x \mapsto \mathscr{A}(x,0)$ is continuous on $\mathbb{S}^d$, a.e. $x \in \Omega$.

The *elasticity operator* $\mathscr{G} : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies:

(a) There exists $L_{\mathscr{G}} > 0$ such that
$$|\mathscr{G}(x,\xi_1,\zeta_1) - \mathscr{G}(x,\xi_2,\zeta_2)| \leq L_{\mathscr{G}}\big(|\xi_1 - \xi_2| + |\zeta_1 - \zeta_2|\big),$$
for all $\xi_1,\xi_2 \in \mathbb{S}^d$, for all $\zeta_1,\zeta_2 \in \mathbb{R}$, a.e. $x \in \Omega$. (3.5)
(b)The mapping $x \mapsto \mathscr{G}(x,\xi,\zeta)$ is Lebesgue measurable on Ω, for any $\xi \in \mathbb{S}^d$,and for all $\zeta \in \mathbb{R}$.
(c) The mapping $x \mapsto \mathscr{G}(x,0,0)$ belongs to $\mathscr{H}$.

The *thermal expansion operator* $\mathscr{F} : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{S}^d$ satisfies:

(a) There exists $L_{\mathscr{F}} > 0$ such that $|\mathscr{F}(x,\theta_1,\zeta_1) - \mathscr{F}(x,\theta_2,\zeta_2)| \leq L_{\mathscr{F}}\big(|\theta_1 - \theta_2| + |\zeta_1 - \zeta_2|\big)$, for all $\theta_1,\theta_2 \in \mathbb{R}$, for all $\zeta_1,\zeta_2 \in \mathbb{R}$, a.e. $x \in \Omega$.
(b) The mapping $x \mapsto \mathscr{F}(x,\theta,\zeta)$ is Lebesgue measurable on Ω, for any $\theta,\zeta \in \mathbb{R}$. (3.6)
(c) The mapping $x \mapsto \mathscr{F}(x,0,0)$ belongs to $\mathscr{H}$.

The *damage source function* $\phi : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies:

$$\left\{\begin{array}{l} \text{(a) There exists } L_\phi > 0 \text{ such that } |\phi(x,\xi_1,\zeta_1) - \phi(x,\xi_2,\zeta_2)| \le L_\phi\big(|\xi_1-\xi_2|+|\zeta_1-\zeta_2|\big), \text{ for all } \xi_1,\xi_2 \in \mathbb{S}^d \text{ and } \zeta_1,\zeta_2 \in \mathbb{R} \text{ a.e. } x \in \Omega,\\ \text{(b) The mapping } x \mapsto \phi(x,\xi,\zeta) \text{ is Lebesgue measurable on } \Omega, \text{ for any } \xi \in \mathbb{S}^d \text{ and } \zeta \in \mathbb{R},\\ \text{(c) The mapping } x \mapsto \phi(x,0,0) \text{ belongs to } L^2(\Omega),\\ \text{(d) } \phi(x,\xi,\zeta) \text{ is bounded for all } \xi \in \mathbb{S}^d,\ \zeta \in \mathbb{R} \text{ a.e. } x \in \Omega. \end{array}\right. \tag{3.7}$$

The *nonlinear constitutive function* $\psi : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies:

$$\left\{\begin{array}{l} \text{(a) There exists } L_\psi > 0 \text{ such that } |\psi(x,\sigma_1,\xi_1,\theta_1) - \psi(x,\sigma_2,\xi_2,\theta_2)| \le L_\psi\big(|\sigma_1-\sigma_2|+|\xi_1-\xi_2|+|\theta_1-\theta_2|\big), \text{ for all } \sigma_1,\sigma_2,\xi_1,\xi_2 \in \mathbb{S}^d \text{ and } \theta_1,\theta_2 \in \mathbb{R} \text{ a.e. } x \in \Omega,\\ \text{(b) The mapping } x \mapsto \psi(x,\sigma,\xi,\theta) \text{ is Lebesgue measurable on } \Omega, \text{ for any } \sigma,\xi \in \mathbb{S}^d \text{ and } \theta \in \mathbb{R},\\ \text{(c) The mapping } x \mapsto \psi(x,0,0,0) \text{ belongs to } L^2(\Omega),\\ \text{(d) } \psi(x,\sigma,\xi,\theta) \text{ is bounded for all } \sigma,\xi \in \mathbb{S}^d,\ \theta \in \mathbb{R} \text{ a.e. } x \in \Omega. \end{array}\right. \tag{3.8}$$

The *normal compliance function* $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies:

$$\left\{\begin{array}{l} \text{(a) There exists } L_\nu > 0 \text{ such that } |p_\nu(x,u_1) - p_\nu(x,u_2)| \le L_\nu|u_1-u_2| \text{ for all } u_1,u_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3.\\ \text{(b) } (p_\nu(x,u_1)-p_\nu(x,u_2))(u_1-u_2) \ge 0 \text{ for all } u_1,u_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3.\\ \text{(c) The mapping } x \mapsto p_\nu(x,u) \text{ is measurable on } \Gamma_3 \text{ for all } u \in \mathbb{R}.\\ \text{(d) } p_\nu(x,u) = 0 \text{ for all } u \le 0, \text{ a.e. } x \in \Gamma_3. \end{array}\right. \tag{3.9}$$

The *tangential contact function* $p_\tau : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies:

$$\left\{\begin{array}{l} \text{(a) There exists } L_\tau > 0 \text{ such that } |p_\tau(x,u_1) - p_\tau(x,u_2)| \le L_\tau|u_1-u_2| \text{ for all } u_1,u_2 \in \mathbb{R}, \text{ a.e. } x \in \Omega.\\ \text{(b) The mapping } x \mapsto p_\tau(x,u) \text{ is Lebesgue measurable on } \Gamma_3 \text{ for all } u \in \mathbb{R}.\\ \text{(c) The mapping } x \mapsto p_\tau(x,0) \text{ belongs to } L^2(\Gamma_3). \end{array}\right. \tag{3.10}$$

We also suppose the following regularities

$$\mathbf{f}_0 \in L^2(0,T;L^2(\Omega)^d), \quad \mathbf{f}_2 \in L^2(0,T;L^2(\Gamma_2)^d), \quad \mathbf{q} \in L^2(0,T;L^2(\Omega)), \tag{3.11}$$

$$\mathbf{u}_0 \in V, \tag{3.12}$$

$$\zeta_0 \in K, \tag{3.13}$$

$$\omega_0 \in L^2(\Gamma_3), \tag{3.14}$$

$$p_\nu(.,u) \in L^2(\Gamma_3), p_\tau(.,u) \in L^2(\Gamma_3), u \in \mathbb{R}, \tag{3.15}$$

$$g \in L^2(\Gamma_3), \quad g \ge 0, \text{ a.e. on } \Gamma_3 \tag{3.16}$$

where K is the set of admissible damage functions defined in (1.3).

Using the Riesz representation theorem, we define the linear mappings $\mathbf{f} : [0,T] \to V$ as follows:

$$(\mathbf{f}(t),\mathbf{v})_V = \int_\Omega \mathbf{f}_0(t)\cdot\mathbf{v}\,dx + \int_{\Gamma_2} \mathbf{f}_2(t)\cdot\mathbf{v}\,da \quad \forall \mathbf{v} \in V \tag{3.17}$$

Next, we define the mappings $a : E_1 \times E_1 \to \mathbb{R}$, the wear functional $j : V \times V \times L^2(\Gamma_3) \to \mathbb{R}$, respectively by

$$a(\zeta,\xi) = \kappa \int_\Omega \nabla\zeta\cdot\nabla\xi\,dx, \tag{3.18}$$

$$j(\mathbf{u},\mathbf{v},\omega) = \int_{\Gamma_3} (p_\nu(\mathbf{u}_\nu - \omega - g)\mathbf{v}_\nu)\,da + \int_{\Gamma_3} (p_\tau(\mathbf{u}_\nu - \omega - g)\mathbf{v}_\nu)\|\mathbf{v}_\tau - \mathbf{v}^*\|\,da, \quad \text{for all} \mathbf{u},\mathbf{v} \in V, \omega \in L^2(\Gamma_3) \tag{3.19}$$

We note that conditions (3.14) imply

$$\mathbf{f} \in L^2(0,T;V) \tag{3.20}$$

By a standard procedure based on Green's formula, we derive the following variational formulation of the mechanical problem (2.1) (2.13).

Problem PV

Find a displacement field $\mathbf{u} : [0,T] \to V$, a stress field $\sigma : [0,T] \to \mathscr{H}$, a temperature $\theta : [0,T] \to V$, a damage $\zeta : [0,T] \to E_1$, a wear $\omega : [0,T] \to L^2(\Gamma_3)$ such that

$$\sigma = \mathscr{A}\varepsilon(\dot{\mathbf{u}}) + \mathscr{G}\big(\varepsilon(\mathbf{u}),\zeta\big) + \mathscr{F}(\theta,\zeta), \text{in } \Omega\times(0,T) \tag{3.21}$$

$$\begin{aligned}&(\sigma,\varepsilon(\mathbf{v})-\varepsilon(\dot{\mathbf{u}}(t)))_{\mathscr{H}} + j(\mathbf{u}(t),\mathbf{v},\omega(t))\\ &-j(\mathbf{u}(t),\dot{\mathbf{u}}(t),\omega(t)) \ge (\mathbf{f}(t),\mathbf{v}-\dot{\mathbf{u}}(t))_V\\ &\qquad \forall\mathbf{v}\in V, \quad \text{a.e} t \in (0,T),\end{aligned} \tag{3.22}$$

$$\begin{aligned}&(\dot{\theta},\mathbf{v}) + a(\theta,\mathbf{v}) = (\psi(\sigma,\varepsilon(\mathbf{u}),\theta),\mathbf{v}) + (\mathbf{q},\mathbf{v})\\ &\qquad \forall \mathbf{v} \in V,\end{aligned} \tag{3.23}$$

$$\begin{aligned}&(\dot{\zeta}(t),\xi-\zeta(t))_{L^2(\Omega)} + a(\zeta(t),\xi-\zeta(t))\\ &\ge \Big(\phi\big(\varepsilon(\mathbf{u}(t)),\zeta(t)\big),\xi-\zeta(t)\Big)_{L^2(\Omega)},\\ &\qquad \forall\xi \in K, \text{a.e} t \in (0,T),\end{aligned} \tag{3.24}$$

$$\dot{\omega} = \kappa_\omega\alpha^* p_\nu(\mathbf{u}_\nu - \omega - g) \tag{3.25}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \theta(0) = \theta_0, \zeta(0) = \zeta_0, \tag{3.26}$$

$$\omega(0) = \omega_0, \tag{3.27}$$

We notice that the variational Problem **PV** is formulated in terms of a displacement field, a stress field, a temperature, a damage, and a wear. The existence of the unique solution of problem **PV** is stated and proved in the next section.

4. Existence and uniqueness result

Our main existence and uniqueness result is the following.

Theorem 4.1. *Assume that (3.4)–(3.16) hold. Then there exists a unique solution of Problem PV. Moreover, the solution satisfies*

$$\mathbf{u} \in C^1(0,T;V), \tag{4.1}$$

$$\sigma \in C(0,T;\mathscr{H}_1), \tag{4.2}$$

$$\theta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;V), \tag{4.3}$$

$$\zeta \in H^1(0,T;E_0) \cap L^2(0,T;E_1), \tag{4.4}$$

$$\omega \in C^1(0,T;L^2(\Gamma_3)), \tag{4.5}$$

The functions $\mathbf{u}$, σ, θ, ζ and ω which satisfy (3.21)-(3.27) are called a weak solution of the contact Problem **P**. We conclude that, under the assumptions (3.4)– (3.16), the mechanical problem (2.1)–(2.13) has a unique weak solution satisfying (4.1) (4.5).

The proof of Theorem 4.1 will be done in several steps and is based on arguments of nonlinear equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities and Banach fixed point theorem. To this end, we assume in what follows that (3.4) (3.16) hold, and we consider that C is a generic positive constant which depends on Ω, Γ_1, Γ_1, Γ_3, p_ν, p_τ, $\mathscr{A}$, $\mathscr{G}$, $\mathscr{F}$, ψ, ϕ, κ, and T. but does not depend on t nor of the rest of input data, and whose value may change from place to place.

In order to prove the theorem, we consider for $\omega \in \mathscr{C}^1(0,T;\mathscr{L}^2(\Gamma_3))$; $\eta \in \mathscr{C}(0,T;\mathscr{H})$; $h \in \mathscr{C}(0,T;V)$; $\mu \in \mathscr{C}(0,T;V')$ and $\chi \in \mathscr{C}(0,T;\mathscr{L}^2(\Gamma_3))$, the following four auxiliary problems.

First step

Let $\omega \in \mathscr{C}^1(0,T;\mathscr{L}^2(\Gamma_3))$, $\eta \in \mathscr{C}(0,T;\mathscr{H})$and $h \in \mathscr{C}(0,T;V)$ we consider the following variational problem.

Problem $PV_{\omega\eta h}$
Let $\mathbf{v}_{\omega\eta h} = \dot{\mathbf{u}}_{\omega\eta h}$
Find a displacement field $\mathbf{v}_{\omega\eta h} : [0,T] \to V$, a stress field $\sigma_{\omega\eta h} : [0,T] \to \mathscr{H}$ such that

$$\sigma_{\omega\eta h} = \mathscr{A}\varepsilon(\mathbf{v}_{\omega\eta h}(t)) + \eta(t) \tag{4.6}$$

$$\begin{aligned}&(\sigma_{\omega\eta h}(t), \varepsilon(\mathbf{v}) - \varepsilon(\mathbf{v}_{\omega\eta h}(t)))_{\mathscr{H}} +\\ &j(h(t),\mathbf{v},\omega(t)) - j(h(t),\mathbf{v}_{\omega\eta h},\omega(t))\\ &\geq (\mathbf{f}(t),\mathbf{v}-\mathbf{v}_{\omega\eta h}(t))_V \quad \forall \mathbf{v} \in V,\ t \in (0,T),\end{aligned} \tag{4.7}$$

$$\mathbf{u}_{\omega\eta h}(0) = \mathbf{u}_0$$

We have the following result for the problem $\mathbf{PV}_{\omega\eta h}$.

Lemma 4.2. *$PV_{\omega\eta h}$ has a unique weak solution such that $\mathbf{v}_{\omega\eta h} \in C(0,T;V)$, and $\sigma_{\omega\eta h} \in C(0,T;\mathscr{H}_1)$ to the problem (4.6) and (4.7).*

Proof. We define the operators $A : V \to V$ such that:

$$(A\mathbf{u},\mathbf{v})_V = (\mathscr{A}\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v}))_{\mathscr{H}}, \tag{4.8}$$

It follows from (3.4)(a) and (4.8) that:

$$\|A\mathbf{u} - A\mathbf{v}\|_V \leq L_A \|\mathbf{u}-\mathbf{v}\|_V \quad \forall \mathbf{u},\mathbf{v} \in V. \tag{4.9}$$

Wich shows that $A : V \to V$ is Lipschitz continuous. Now by (3.4)(b) and (4.8) we find:

$$(A\mathbf{u} - A\mathbf{v}, \mathbf{u}-\mathbf{v})_V \geq m_A \|\mathbf{u}-\mathbf{v}\|_V^2 \quad \forall \mathbf{u},\mathbf{v} \in V. \tag{4.10}$$

Wich shows that $A : V \to V$ is a strongly monotone operator on V.

Moreover, using Riesz representation theorem we may define the functions $\mathbf{F}_\eta : [0,T] \to V$ by

$$\mathbf{F}_\eta(t) = \mathbf{f}(t) - \eta(t) \quad \forall t \in [0,T],$$

Since A is a strongly monotone operator and Lipschitz continuous operator on V and since $j(h(t),\mathbf{v},\omega(t))$is a proper convex lower semicontinuous functional, it follows from classical result on elliptic inequalities (see[6]) that there exists a unique function $\mathbf{v}_{\omega nh}(t) \in V$ such that

$$\begin{aligned}&(\mathscr{A}\varepsilon(\mathbf{v}_{\omega\eta h}(t)), \varepsilon(\mathbf{v}) - \varepsilon(\mathbf{v}_{\omega\eta h}(t)))_{\mathscr{H}} +\\ &j(h(t),\mathbf{v},\omega(t)) - j(h(t),\mathbf{v}_{\omega\eta h}(t),\omega(t))\\ &\geq (\mathbf{F}_\eta(t),\mathbf{v}-\mathbf{v}_{\omega\eta h}(t))_V \quad \forall \mathbf{v} \in V,\ t \in (0,T),\end{aligned} \tag{4.11}$$

We use the relation (4.6), the assumptions (3.4), we obtain that

$$\sigma_{\omega nh}(t) \in \mathscr{H}$$

Using the definition (3.17) for $\mathbf{f}$, we deduce

$$Div\sigma_{\omega\eta h}(t) + \mathbf{f}_0(t) = 0 \tag{4.12}$$

With the regularity assumption (3.11) we see that

$$Div\sigma_{\omega\eta h}(t) \in H \text{ therefore } \sigma_{\omega\eta h}(t) \in \mathscr{H}_1$$

Let now t_1, $t_2 \in [0,T]$, and denote $\eta(t_i) = \eta_i$, $\mathbf{f}(t_i) = \mathbf{f}_i$, $h(t_i) = h_i$, $\mathbf{v}_{\omega\eta h}(t_i) = \mathbf{v}_i$, $\sigma_{\omega\eta h}(t_i) = \sigma_i$, for $i = 1,2$. Using the relation (4.11) we find that

$$\begin{aligned}&(A\mathbf{v}_1 - A\mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2)_V \leq (\mathbf{f}_1 - \mathbf{f}_2, \mathbf{v}_1 - \mathbf{v}_2)_V +\\ &(\eta_2 - \eta_1, \varepsilon(\mathbf{v}_1 - \mathbf{v}_2))_{\mathscr{H}} + j(h_1,\mathbf{v}_2,\omega) -\\ &j(h_1,\mathbf{v}_1,\omega) + j(h_2,\mathbf{v}_1,\omega) - j(h_2,\mathbf{v}_2,\omega)\end{aligned} \tag{4.13}$$

From the definition of the functional j given by (3.19) we have

$$\begin{aligned}&j(h_1,\mathbf{v}_2,\omega) - j(h_1,\mathbf{v}_1,\omega) + j(h_2,\mathbf{v}_1,\omega) - j(h_2,\mathbf{v}_2,\omega)\\ &= \int_{\Gamma_3} \{p_\nu(h_{1\nu} - \omega - g) - p_\nu(h_{2\nu} - \omega - g)\}(v_{2\nu} - v_{1\nu})\,da\\ &+ \int_{\Gamma_3} \{p_\tau(h_{1\tau} - \omega - g) - p_\tau(h_{2\tau} - \omega - g)\}(\|v_{2\tau} - v^*\| - \|v_{1\tau} - v^*\|)\,da\end{aligned}$$

We use (3.3), (3.9) and (3.10) to deduce that

$$\begin{aligned}&j(h_1,\mathbf{v}_2,\omega) - j(h_1,\mathbf{v}_1,\omega) + j(h_2,\mathbf{v}_1,\omega)\\ &-j(h_2,\mathbf{v}_2,\omega) \leq C\|h_1 - h_2\|_V \|\mathbf{v}_1 - \mathbf{v}_2\|_V\end{aligned} \tag{4.14}$$

The relation (3.2), the estimate (4.10) and the inequality (4.14) combined with (4.13) give us

$$m_A \|\mathbf{v}_1 - \mathbf{v}_2\|_V \leq C(\|\mathbf{f}_1 - \mathbf{f}_2\|_V + \|\eta_1 - \eta_2\|_{\mathscr{H}} + \|h_1 - h_2\|_V) \quad (4.15)$$

The inequality (4.15) and the regularity of the function $\mathbf{f}$, h and η show that

$$\mathbf{v}_{\omega\eta h} \in C(0,T;V)$$

From assumption (3.4) and the relation (4.6) we have

$$\|\sigma_1 - \sigma_2\|_{\mathscr{H}} \leq C(\|\mathbf{v}_1 - \mathbf{v}_2\|_V + \|\eta_1 - \eta_2\|_{\mathscr{H}}) \quad (4.16)$$

and from (4.12) we have.

$$Div\sigma(t_i) + \mathbf{f}_0(t) = 0,\ i = 1,2. \quad (4.17)$$

The regularity of the function η, $\mathbf{v}$, $\mathbf{f}_0$ and the relation (4.16)-(4.17)show that

$$\sigma_{\omega\eta h} \in C(0,T;\mathscr{H}_1)$$

Let $\omega \in C(0,T;L^2(\Gamma_3)), h \in C(0,T;V)$ and let $\eta \in C(0,T;\mathscr{H})$ be given. We consider the following operator

$$\Lambda_{\omega\eta} : C(0,T;V) \to C(0,T;V)$$

Defined by

$$\Lambda_{\omega\eta} h = \mathbf{u}_0 + \int_0^t \mathbf{v}_{\omega\eta h}(s)ds \quad \forall h \in C(0,T;V) \quad (4.18)$$

□

Lemma 4.3. *Let the assumptions (3.4)-(3.16) hold. Then the operator $\Lambda_{\omega\eta}$ has a unique fixed point $h_{\omega\eta} \in C(0,T;V)$,*

Proof. Let $h_1, h_2 \in C(0,T;V)$ and let $\eta \in C(0,T;\mathscr{H})$,we use the relation $\mathbf{v}_{\omega\eta h_i} = \mathbf{v}_i$and $\sigma_{\omega\eta h_i} = \sigma_i$ for $i = 1,2$.

Using similar arguments as those used in (4.15) we find that

$$m_{\mathscr{A}} \|\mathbf{v}_1(t) - \mathbf{v}_2(t)\|_V \leq C\|h_1(t) - h_2(t)\|_V \quad \forall t \in [0,T] \quad (4.19)$$

From (4.18)(a) and (4.19) we have

$$\|\Lambda_{\omega\eta} h_1 - \Lambda_{\omega\eta} h_2\|_V \leq C \int_0^t \|h_1(t) - h_2(t)\|_V\, ds \quad \forall t \in [0,T] \quad (4.20)$$

Reitrating this inequality m times, we obtain

$$\|\Lambda_{\omega\eta} h_1 - \Lambda_{\omega\eta} h_2\|_{C(0,T;V)} \leq \frac{C^m T^m}{m!} \|h_1 - h_2\|_{C(0,T;V)}\, ds \quad \forall t \in [0,T]$$

This show that for m large enough the operator $\Lambda^m_{\omega\eta}$ is a contraction in the Banach space. Thus, from Banach's fixed point theorem the operator $\Lambda_{\omega\eta}$ has a unique fixed point $h^*_{\omega\eta} \in C(0,T;V)$. □

For $\eta \in \mathscr{C}(0,T;\mathscr{H})$, let $h^*_{\omega\eta}$ be the fixed point given by the above lemma, i.e. $h_{\omega\eta^*} = \mathbf{v}_{\omega\eta^* h}$. In the sequel we denote by $(\mathbf{v}_{\omega\eta}, \sigma_{\omega\eta}) \in \mathscr{C}(0,T;V) \times \mathscr{C}(0,T;\mathscr{H}_1)$ the unique solution of Problem $\mathbf{PV}_{\omega\eta h}$, i.e. $\mathbf{v}_{\omega\eta} = \mathbf{v}_{\omega\eta^* h}$, $\sigma_{\omega\eta} = \sigma_{\omega\eta^* h}$. Also, we denote by $\mathbf{u}_{\omega\eta} : [0,T] \to V$ the function defined by

$$\mathbf{u}_{\omega\eta}(t) = \int_0^t \mathbf{v}(s)ds + \mathbf{u}_0, \quad \forall t \in [0,T]. \quad (4.21)$$

From Lemma 4.2 we deduce that

$$\mathbf{u}_{\omega\eta} \in C^1(0,T;V)$$

Now we consider the following problem

Problem PV$\omega\eta$

Find a displacement field $\mathbf{u}_{\omega\eta} : [0,T] \to V$ such that for all $t \in [0,T]$

$$\begin{aligned}&(A\varepsilon(\dot{\mathbf{u}}_{\omega\eta}(t)), \varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}_{\omega\eta}(t)))_{\mathscr{H}} + j(\mathbf{u}_{\omega\eta}(t), \mathbf{v}, \omega(t)) - \\ &j(\mathbf{u}_{\omega\eta}(t), \dot{\mathbf{u}}_{\omega\eta}(t), \omega(t)) + (\eta(t), \varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}_{\omega\eta}(t)))_{\mathscr{H}} \\ &\geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}_{\omega\eta}(t))_V, \quad \forall \mathbf{v} \in V,\ t \in (0,T),\end{aligned} \quad (4.22)$$

$$u_{\omega\eta}(0) = \mathbf{u}_0 \quad (4.23)$$

We have the following result for the problem $\mathbf{PV}_{\omega\eta}$.

Lemma 4.4. *$PV_{\omega\eta}$ has a unique weak solution satisfying the regularity* (4.1).

Proof. For each $\omega \in C(0,T;L^2(\Gamma_3))$ and $\eta \in C(0,T;\mathscr{H})$,we denote by $h_{\omega\eta} \in C(0,T;V)$ the fixed point obtained in Lemma 4.3 and let $\mathbf{u}_{\omega\eta}$ be the function defined by

$$\mathbf{u}_{\omega\eta}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}_{\omega\eta h_{\omega\eta}}(s)ds \quad \forall t \in [0,T] \quad (4.24)$$

We have $\Lambda_{\omega\eta} h_{\omega\eta} = h_{\omega\eta}$ from (4.18) and (4.24) it follows that

$$\mathbf{u}_{\omega\eta} = h_{\omega\eta} \quad (4.25)$$

Therefore, taking $h = h_{\omega\eta}$ in (4.7) and using (4.6),(4.24) and (4.25) we see that $\mathbf{u}_{\omega\eta}$ is the unique solution to the problem $\mathbf{PV}_{\omega\eta}$ satisfying the regularity expressed in (4.1) . □

Second step

In the second step, we use the displacement field $\mathbf{u}_{\omega\eta}$ obtained in Lemma 4.4 and Let $\mu \in \mathscr{C}(0,T;V')$, we consider the following variational problem.

Problem PV$\omega\mu$

Find the temperature $\theta_{\omega\mu} : [0,T] \to V$ which is solution of the following variational problem

$$\begin{aligned}(\dot{\theta}_{\omega\mu}, \mathbf{v}) + a(\theta_{\omega\mu}, \mathbf{v}) = (\mu(t) + \mathbf{q}(t), \mathbf{v}), \quad \forall \mathbf{v} \in V \\ \theta_\mu(0) = 0\end{aligned} \quad (4.26)$$

We have the following result.

Lemma 4.5. *$PV_{\omega\mu}$ has a unique solution $\theta_{\omega\mu}$ which satisfies the regularity* (4.3).

Proof. By an application of the Friedrichs-Poincaré inequality, we can find a constant $\beta' > 0$ such that

$$\int_\Omega \|\xi\|^2 dx + \frac{\beta}{k_0}\int_\Gamma \|\xi\|^2 d\gamma \geq \beta' \int_\Omega \|\xi\|^2 dx, \quad \forall \xi \in V. \tag{4.27}$$

Thus, we obtain

$$a_0(\xi,\xi) \geq c_1 \|\xi\|_V^2, \quad \forall \xi \in V. \tag{4.28}$$

Where $c_1 = k_0 \min(1,\beta')/2$, which implies that a_0 is V-elliptic. Consequently, based on classical arguments of functional analysis concerning parabolic equations, the variational equation (4.26) has a unique solution $\theta_{\omega\mu}$ satisfying (4.3). □

Third step

In the third step, we use the displacement field $\mathbf{u}_{\omega\eta}$ obtained in Lemma 4.4

We consider the following initial-value problem.

Problem PV$\omega\chi$

Find a damage $\zeta_{\omega\chi} : [0,T] \to H^1(\Omega)$ such that $\zeta_{\omega\chi}(t) \in K$ and

$$\begin{aligned}(\dot\zeta_{\omega\chi}(t), \xi - \zeta_{\omega\chi}(t))_{L^2(\Omega)} + a(\zeta_{\omega\chi}(t), \xi - \zeta_{\omega\chi}(t))\\ \geq \big(\chi(t), \xi - \zeta_{\omega\chi}(t)\big)_{L^2(\Omega)}, \quad \forall \xi \in K, \quad a.e.t \in (0,T),\end{aligned} \tag{4.29}$$

$$\zeta_{\omega\chi}(0) = 0 \tag{4.30}$$

To solve problem **PV**$\omega\chi$, we recall the following abstract result for parabolic variational inequalities,

Lemma 4.6. *There exists a unique solution $\zeta_{\omega\chi}$ of Problem* $\mathrm{PV}_{\omega\chi}$ *and it satisfies*

$$\zeta_{\omega\chi} \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)).$$

Proof. Using (3.18), (3.21) and a classical existence and uniqueness result on parabolic equations (see [2, P 124]) □

Fourth step

In the fourth step, we use the displacement field $\mathbf{u}_{\omega\eta}$ obtained in Lemma 4.4

We consider the following initial-value problem.

Problem PVω

Find a wear $\omega \in \mathscr{C}^1(0,T;\mathscr{L}^2(\Gamma_3))$ such that

$$\dot\omega = \kappa_\omega \alpha^* p_\nu(\mathbf{u}_\nu - \omega - g) \tag{4.31}$$

$$\omega(0) = \omega_0, \tag{4.32}$$

Let us now we consider the operator $\mathscr{L} : \mathscr{C}(0,T;L^2(\Gamma_3)) \to \mathscr{C}(0,T;L^2(\Gamma_3))$ defined by

$$\mathscr{L}\omega(t) = -k_1 \mathbf{v}^* \int_0^t (\sigma_\omega)_V(s)\,ds \quad \forall t \in [0,T]. \tag{4.33}$$

Lemma 4.7. *The operator $\mathscr{L}$ has a unique fixed point ω^* and it satisfies*

$$\omega^* \in C(0,T;L^2(\Gamma_3))$$

Proof. Let ω_1, $\omega_2 \in \mathscr{C}(0,T;L^2(\Gamma_3))$, and $t \in [0,T]$. We denote by $(\mathbf{u}_i, \sigma_i, \theta_i, \zeta_i)$, for $i = 1,2$ the solution to the problem $\mathbf{PV}_\omega$ for $\omega = \omega_i$ use the notation $\mathbf{u}_{\omega_i} = \mathbf{u}_i$, $\dot{\mathbf{u}}_{\omega_i} = \mathbf{v}_{\omega_i} = \mathbf{v}_i$, $\zeta_{\omega_i} = \zeta_i$, $\theta_{\omega_i} = \theta_i$ and $\sigma_{\omega_i} = \sigma_i$, where $\mathbf{u}_i = (\mathbf{u}_i^1, \mathbf{u}_i^2)$, $\zeta_i = (\zeta_i^1, \zeta_i^2)$. Moreover we denote in sequel by C various positive constants which may depend on $k_1$and $\mathbf{v}^*$. Using similar arguments that those used in the proof of the relation (4.39), to find that

$$\begin{aligned}&\int_0^t \|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_V^2 ds\\ &\leq C\big(\int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 ds + \int_0^t \|\omega_1(s) - \omega_2(s)\|_{L^2(\Gamma_3)}^2 ds\big)\end{aligned} \tag{4.34}$$

Since $\mathbf{u}_1(0) = \mathbf{u}_2(0) = \mathbf{u}_0$ and using (4.34) we obtain

$$\begin{aligned}\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 &\leq C\int_0^t \|\omega_1(s) - \omega_2(s)\|_{L^2(\Gamma_3)}^2 ds\\ &\leq C\big(\int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 ds + C\int_0^t \|\omega_1(s) - \omega_2(s)\|_{L^2(\Gamma_3)}^2 ds\big)\end{aligned} \tag{4.35}$$

Applying Gronwall inequality, we deduce that

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 \leq C\int_0^t \|\omega_1(s) - \omega_2(s)\|_{L^2(\Gamma_3)}^2 ds \tag{4.36}$$

So, by (4.34), (4.36) , it follows that

$$\int_0^t \|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_V^2 ds \leq C\int_0^t \|\omega_1(s) - \omega_2(s)\|_{L^2(\Gamma_3)}^2 ds \tag{4.37}$$

On other hand since

$$\sigma = \mathscr{A}\varepsilon(\dot{\mathbf{u}}_i) + \mathscr{G}\big(\varepsilon(\mathbf{u}_i), \zeta_i\big) + \mathscr{F}(\theta, \zeta_i) \tag{4.38}$$

For $i = 1,2$ we use the assumption (3.4)(b), (3.5), (3.6) and(3.7) to obtain for $s \in [0,T]$

$$\|\sigma_1(s) - \sigma_2(s)\|_{\mathscr{H}}^2 \leq C\big(\|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_V^2 + \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2\big) \tag{4.39}$$

We integrate the previous inequality with respect to time to deduce that

$$\begin{aligned}&\int_0^t \|\sigma_1(s) - \sigma_2(s)\|_{\mathscr{H}}^2 ds\\ &\leq C\big(\int_0^t \|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_V^2 ds + \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 ds\big)\end{aligned} \tag{4.40}$$

We substitute (4.36) and (4.37) in the previous inequality to find

$$\int_0^t \|\sigma_1(s)-\sigma_2(s)\|_{\mathscr{H}}^2 ds \le C\int_0^t \|\omega_1(s)-\omega_2(s)\|_{L^2(\Gamma_3)}^2 ds \tag{4.41}$$

The definition of the operator $\mathscr{L}$ given by (4.33) and estimate (4.37) give us

$$\|\mathscr{L}\omega_1(t)-\mathscr{L}\omega_2(t)\|_{L^2(\Gamma_3)}^2 \le C\int_0^t \|\omega_1(s)-\omega_2(s)\|_{L^2(\Gamma_3)}^2 ds \tag{4.42}$$

Reiterating this inequality n times leads to

$$\|\mathscr{L}^n\omega_1-\mathscr{L}^n\omega_2\|_{\mathscr{C}(0,T;L^2(\Gamma_3))}^2 \le \frac{C^nT^n}{n!}\|\omega_1(s)-\omega_2(s)\|_{\mathscr{C}(0,T;L^2(\Gamma_3))}^2 \tag{4.43}$$

□

Therefore,for n large enough, $\mathscr{L}^n$ is contractive operator on the Banach space $\mathscr{C}(0,T;L^2(\Gamma_3))$. The operator $\mathscr{L}$ has a unique fixed point $\omega^*\in\mathscr{C}(0,T;L^2(\Gamma_3))$.

Now we have all the ingredients to prove Theorem4.1

Proof of theorem. By taking into account the above results and the properties of the operators $\mathscr{G}$ and $\mathscr{F}$ and of the functions ψ and ϕ we may consider the operator $\Lambda:\mathscr{C}(0,T;\mathscr{H}\times V'\times L^2(\Omega))\to\mathscr{C}(0,T;\mathscr{H}\times V'\times L^2(\Omega))$ □

$$\Lambda(\eta,\mu,\chi)(t)=(\Lambda_1(\eta,\mu,\chi)(t),\Lambda_2(\eta,\mu,\chi)(t),\Lambda_3(\eta,\mu,\chi)(t)), \tag{4.44}$$

defined by

$$\Lambda_1(\eta,\mu,\chi)(t)=\mathscr{G}\big(\varepsilon(\mathbf{u}_{\omega\eta}),\zeta_{\omega\chi}\big)+\mathscr{F}(\theta_{\omega\mu},\zeta_{\omega\chi}), \tag{4.45}$$
$$\Lambda_2(\eta,\mu,\chi)(t)=\big(\psi\big(\sigma_{\omega\eta},\varepsilon(\dot{\mathbf{u}}_{\omega\eta}),\theta_{\omega\mu}\big)\big), \tag{4.46}$$
$$\Lambda_3(\eta,\mu,\chi)(t)=56\big(\phi\left(\varepsilon(\mathbf{u}_{\omega\eta}),\zeta_{\omega\chi}\right)\big), \tag{4.47}$$

Here for every $(\eta,\mu,\chi)\in\mathscr{C}(0,T;\mathscr{H}\times V'\times L^2(\Omega))$, $\mathbf{u}_\eta$, θ_μ, ζ_χ and ω represents the displacement , the temperature, the damage and the wear obtained in Lemma4.4, Lemma4.5, Lemma4.6 and Lemma4.7 respectively and

$$\sigma_\omega=\mathscr{A}\varepsilon(\dot{\mathbf{u}}_\omega)+\mathscr{G}\big(\varepsilon(\mathbf{u}_\omega),\zeta_\omega\big)+\mathscr{F}(\theta_\omega,\zeta_\omega) \tag{4.48}$$

We have the following result.

Lemma 4.8. *Let* (4.3) *be satisfied. Then for* $(\eta,\mu,\chi)\in\mathscr{C}(0,T;\mathscr{H}\times V'\times L^2(\Omega))$ *, the maping*

$$\Lambda(\eta,\mu,\chi):[0,T]\to\mathscr{H}\times V'\times L^2(\Omega)$$

has a unique element

$$(\eta^*,\mu^*,\chi^*)\in\mathscr{C}(0,T;\mathscr{H}\times V'\times L^2(\Omega))$$

such that $\Lambda(\eta^*,\mu^*,\chi^*)=(\eta^*,\mu^*,\chi^*)$.

Proof. Let $(\eta_1,\mu_1,\chi_1),(\eta_2,\mu_2,\chi_2)\in\mathscr{C}(0,T;\mathscr{H}\times V'\times L^2(\Omega))$, and $t\in[0,T]$.

We use the notation $\mathbf{u}_{\omega\eta_i}=\mathbf{u}_i,\dot{\mathbf{u}}_{\omega\eta_i}=\mathbf{v}_{\omega\eta_i}=\mathbf{v}_i,\zeta_{\omega\chi_i}=\zeta_i,\theta_{\omega\mu_i}=\theta_i$ and $\sigma_{\omega\eta_i}=\sigma_i$,for $i=1,2$.

Using (3.2) and the relations (3.5)-(3.7), we obtain

$$\begin{aligned}&\Lambda(\eta_1,\mu_1,\chi_1)(t)-\Lambda(\eta_2,\mu_2,\chi_2)(t)\|_{\mathscr{H}\times V'\times L^2(\Omega)}\\&\le L_{\mathscr{G}}\Big(\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V+\|\zeta_1(t)-\zeta_2(t)\|_{L^2(\Omega)}\Big)\\&+L_{\mathscr{F}}\int_0^t\Big(\|\sigma_1(s)-\sigma_2(s)\|_{\mathscr{H}}+L_{\mathscr{A}}\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|_V\\&+\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V+\|\theta_1(s)-\theta_2(s)\|_{L^2(\Omega)}\Big)ds\\&+M_\phi\Big(\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V+\|\zeta_1(t)-\zeta_2(t)\|_{L^2(\Omega)}\Big)\\&+L_\psi\Big(\|\sigma_1(t)-\sigma_2(t)\|_{\mathscr{H}}+\|\mathbf{v}_1(t)-\mathbf{v}_2(t)\|_V+\\&\|\theta_1(t)-\theta_2(t)\|_{L^2(\Omega)}\Big)\end{aligned} \tag{4.49}$$

Since

$$\mathbf{u}_i(t)=\int_0^t\mathbf{v}_i(s)\,ds+\mathbf{u}_0,\forall t\in[0,T], \tag{4.50}$$

we have

$$\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V\le\int_0^t\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|_V ds, \tag{4.51}$$

Applying Young's and Hölder's inequalities, (4.49) becomes, via (4.51) ,

$$\begin{aligned}&\Lambda(\eta_1,\mu_1,\chi_1)(t)-\Lambda(\eta_2,\mu_2,\chi_2)(t)\|_{\mathscr{H}\times V'\times L^2(\Omega)}\\&\le C\Big(\|\zeta_1(t)-\zeta_2(t)\|_{L^2(\Omega)}+\int_0^t\big(\|\sigma_1(s)-\sigma_2(s)\|_{\mathscr{H}}\\&+\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|_V+\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V\\&+\|\theta_1(s)-\theta_2(s)\|_{L^2(\Omega)}\big)ds\Big).\end{aligned} \tag{4.52}$$

Taking in mind that

$$\sigma_i(t)=\mathscr{A}(\varepsilon(\mathbf{v}_i(t)))+\eta_i(t),\forall t\in[0,T]. \tag{4.53}$$

it follows

$$\begin{aligned}&(\mathscr{A}\varepsilon(\dot{\mathbf{u}}_*(t)),\varepsilon(\mathbf{v})-\varepsilon(\dot{\mathbf{u}}_*(t))_{\mathscr{H}}\\&\le j(\mathbf{v}_1,\mathbf{v}_2,\omega)+j(\mathbf{v}_2,\mathbf{v}_1,\omega)-j(\mathbf{v}_1,\mathbf{v}_1,\omega)-j(\mathbf{v}_2,\mathbf{v}_2,\omega)\end{aligned} \tag{4.54}$$

So, by using (3.4), (3.19) and (3.3), we deduce that

$$\begin{aligned}&m_{\mathscr{A}}\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|_V^2\\&\le C_0^2\|\alpha\|_{L^\infty(\Gamma_3)}\Big(\|\lambda\|_{L^\infty(\Gamma_3)}+1\Big)\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|_V^2\\&+\|\eta_1(s)-\eta_2(s)\|_{\mathscr{H}}\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|_V^2\end{aligned} \tag{4.55}$$

Which, by the Gronwall inequality, implies

$$\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|_V^2\le C\|\eta_1(s)-\eta_2(s)\|_{\mathscr{H}}^2 \tag{4.56}$$

Then

$$\int_0^t \|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V ds \le C\int_0^t\int_0^s \|\eta_1(r)-\eta_2(r)\|_{\mathscr{H}} drds \le \int_0^T \|\eta_1(s)-\eta_2(s)\|_{\mathscr{H}} ds \quad (4.57)$$

For the temperature , if we take the substitution $\mu=\mu_1$, $\mu=\mu_2$ in (4.26) and subtracting the two obtained equations, we deduce by choosing $\mathbf{v}=\theta_1-\theta_2$ as test function

$$\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}+C_1\int_0^t\|\theta_1(t)-\theta_2(t)\|^2_V \leqslant \int_0^t\|\mu_1(s)-\mu_2(s)\|_{V'}\|\theta_1(s)-\theta_2(s)\|_V ds, \forall t\in[0,T], \quad (4.58)$$

Employing Hölder's and Young's inequalities, we deduce that

$$\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}+\int_0^t\|\theta_1(s)-\theta_2(s)\|^2_V ds \leqslant C\int_0^t\|\mu_1(s)-\mu_2(s)\|^2_{V'} ds, \forall t\in[0,T]. \quad (4.59)$$

We use the inclusion $L^2(\Omega)\subset V$, we get

$$\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}+\int_0^t\|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)} ds \leqslant C\int_0^t\|\mu_1(s)-\mu_2(s)\|^2_{V'} ds, \forall t\in[0,T]. \quad (4.60)$$

From this inequality, combined with Gronwall's inequality, we deduce that

$$\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}\leqslant C\int_0^t\|\mu_1(s)-\mu_2(s)\|^2_V ds \quad (4.61)$$

For the damage field, from (4.29) we deduce that

$$(\dot\zeta_1-\dot\zeta_2,\zeta_1-\zeta_2)_{L^2(\Omega)}+\mathfrak{a}_1(\zeta_1-\zeta_2,\zeta_1-\zeta_2) \leqslant(\chi_1-\chi_2,\zeta_1-\zeta_2)_{L^2(\Omega)} \quad (4.62)$$

Integrating the previous inequality with respect to time, using the initial conditions $\zeta_1(0)=\zeta_2(0)=\zeta_0$ and inequality $\mathfrak{a}_1(\zeta_1-\zeta_2,\zeta_1-\zeta_2)\geqslant 0$ to find

$$\frac{1}{2}\|\zeta_1(t)-\zeta_2(t)\|^2_{L^2(\Omega)}\leqslant\int_0^t(\chi_1(s)-\chi_2(s),\zeta_1(s)-\zeta_2(s))_{L^2(\Omega)}ds, \quad (4.63)$$

which implies

$$\|\zeta_1(t)-\zeta_2(t)\|^2_{L^2(\Omega)}\leqslant \int_0^t\|\chi_1(s)-\chi_2(s)\|^2_{L^2(\Omega)}ds+\int_0^t\|\zeta_1(s)-\zeta_2(s))\|^2_{L^2(\Omega)}ds. \quad (4.64)$$

This inequality, combined with Gronwall's inequality, leads to

$$\|\zeta_1(t)-\zeta_2(t)\|^2_{L^2(\Omega)}\leqslant C\int_0^t\|\chi_1(s)-\chi_2(s)\|^2_{L^2(\Omega)}ds, \quad \forall t\in[0,T]. \quad (4.65)$$

Applying the previous inequalities, the estimates (4.61) and (4.65), we substitute(4.52) to obtain

$$\|\Lambda(\eta_1,\mu_1,\chi_1)(t)-\Lambda(\eta_2,\mu_2,\chi_2)(t)\|^2_{V\times L^2(\Omega)}\leqslant C\int_0^T\|(\eta_1,\mu_1,\chi_1)(s)-(\eta_2,\mu_2,\chi_2)(s)\|^2_{V\times L^2(\Omega)}ds. \quad (4.66)$$

Thus, for m sufficiently large, Λ^m is a contraction on $\mathscr{C}(0,T;V\times L^2(\Omega))$, and so Λ has a unique fixed point in this Banach space. □

Existence. Let $(\eta^*,\mu^*,\chi^*)\in\mathscr{C}(0,T;\mathscr{H}\times V'\times L^2(\Omega))$ be the fixed point of Λ defined by (4.44)-(4.47) and let $h^*=h^*_{\eta^*}$ be the fixed point of the operator Λ_{η^*} given by Lemma4.2. We denote

$$\mathbf{u}_* = \mathbf{u}_{\omega\eta^*}, \theta_*=\theta_{\omega\mu^*}, \zeta_*=\zeta_{\omega\chi^*}.$$
$$\sigma_* = \mathscr{A}\varepsilon(\dot{\mathbf{u}}_*)+\mathscr{G}\big(\varepsilon(\mathbf{u}_*),\zeta_*\big)+\mathscr{F}(\theta_*,\zeta_*)$$

$\Lambda_1(\eta^*,\mu^*,\chi^*)=\eta^*$, $\Lambda_2(\eta^*,\mu^*,\chi^*)=\mu^*$ and $\Lambda_3(\eta^*,\mu^*,\chi^*)=\chi^*$, the definitions (4.45)-(4.47) show that (3.21)-(3.27) are satisfied. Next, from Lemmas 4.2, 4.4, 4.5, 4.6and 4.7, the regularity conditions (4.1)-(4.5) follow. □

Uniqueness. Let ω^* be the fixed point of the operator $\mathscr{L}$ given by (4.33). The unique solution $(\mathbf{u}_*,\sigma_*,\theta_*,\zeta_*,\omega^*)$ is a consequence of the uniqueness of the fixed point of the operator Λ defined by (4.44)-(4.47) and the unique solvability of the Problem $\mathbf{PV}_{\omega\eta h}$, $\mathbf{PV}_{\omega\mu}$, $\mathbf{PV}_{\omega\chi}$and , $\mathbf{PV}_{\omega}$ which completes the proof. □

5. References

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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