Malaya Journal of Matematik, Vol. 6, No. 1 , 28-33, 2018

https://doi.org/10.26637/MJM0601/0004

Stability for impulsive implicit Hadamard fractional differential equations

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Abstract

In this paper, we analyze the uniqueness and stability for implicit fractional differential equations with impulsive conditions involving the hadamard derivative of fractional order α. An illustrative example is also presented.

Keywords

Hadamard fractional derivative, Implicit Fractional Differential equations, Impulsive condition, UH (Ulam-Hyers) stability and generalized UH (Ulam-Hyers) stability.

AMS Subject Classification

34A09, 34A08, 37J25.

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Article History: Received **24** October **2017**; Accepted **09** December **2017**

Contents

1. Introduction

In the last few decades, fractional differential equations (FDE's) have gained considerably more attention and attracted by many researchers in fields of such as physics, mechanics, chemistry, aerodynamics and the electrodynamics of complex media. Several papers are introduced many form of fractional differential equations and also discussed different results on existence and uniqueness for fractional differential equations, see [3, 6, 7, 10–12].It comes from the fact that they have been proved to be valuable tools in the mathematical models for systems [13, 16, 17, 20].

They concerned the UH (Ulam-Hyers) stability and generalized UH (Ulam-Hyers) stability for a implicit fractional differential equations (IFDE's)with hadamard derivative [8, 9, 14, 15, 18, 19, 24, 25]. In 2010 Deliang Qian, Ranchao Wu and Yanfen Lu discussed the stability analysis of fractional differential system with Riemann-Liouville derivative to applying the Mittag-Leffler function and generalized Gronwall inequality.

Recently, H.L.Tidke and R.P.Mahajan [21] investigated the existence, uniqueness of nonlinear implicit fractional differential equations involving Riemann-Liouville derivative by Banach fixed point theorem. In [1] the authors S Abbas, W Albarakati, M Benchohra, JJ Trujillo have considered the Ulam Stabilities for partial hadamard fractional integral equations by appliqueing the Schauder's fixed-point theorem. The reader to refer the papers of Benchohra et al.[4, 5], S. Abbas, M. Benchohra et al. [2] and see [22, 23, 26, 27].

Motivated from recent research papers treating the varies derivatives for implicit fractional differential equations, here we study the existence of solutions for the implicit fractional order differential equations with Hadamard derivative and impulsive conditions of the form

$$
\begin{gathered}
{}^{H}D^{\alpha}x(t)=f(t,x(t),{}^{H}D^{\alpha}x(t)), \\
t\in J':=J\backslash\{t_0,...,t_m\}, J=[1,T], 0<\alpha\leq 1.
\end{gathered} \tag{1.1}
$$

$$x(t_k^+)=x(t_k^-)+y_k,\ k=1,2,...,m \quad y_k\in\mathbb{R} \tag{1.2}$$

$$x(1)=x_0, \tag{1.3}$$

where ${}^{H}D^{\alpha}$ is the Hadamard fractional derivative, $f:J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a given function and $x_0\in\mathbb{R}$ and t_k satisfy $1=t_0<t_1<...<t_m<t_{m+1}=T$.

In section 2 deals with definitions and basic results and in section 3, the existence and stability results of implicit impulsive hadamard fractional differential equations are studied. In section 4, an application is provided in support of main results.

2. Preliminaries

In this section the most commonly used basic definitions and results are stated.

Definition 2.1. *The fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by*

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds,$$

provided the right side is pointwise defined on $(0,\infty)$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. *The fractional derivative of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by*

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha]+1$, provided the right side is pointwise defined on $(0,\infty)$.

Definition 2.3. *The Hadamard fractional integral of order q for a continuous function g is defined as*

$${}_H J^q g(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s}\right)^{q-1} \frac{g(s)}{s} ds, \quad q > 0.$$

Definition 2.4. *The Hadamard derivative of fractional order q for a continuous function $g : [1,\infty) \to \mathbb{X}$ is defined as*

$${}_H D^q g(t) = \frac{1}{\Gamma(n-q)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-q-1} \frac{g(s)}{s} ds,$$
$$n-1 < q < n,\ n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.5. *The **Problem (1.1) is UH (Ulam-Hyers) stable** if $\exists$ a $\varphi_M > 0$ such that for each $\varepsilon > 0$ and for each solution $x \in C(J,\mathbb{R})$ of the inequality $|{}^H D^{\alpha} x(t) - f(t, x(t), {}^H D^{\alpha} x(t))| \le \varepsilon, t \in J$ $\exists$ a solution $y \in C(J,\mathbb{R})$ of the problem (1.1) with $|x(t) - y(t)| \le \varphi_M \varepsilon$.*

Definition 2.6. *The **Problem (1.1) is Generalized UH (Ulam-Hyers) stable** if $\exists$ a $\varphi_M \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\varphi_M(0) = 0$, such that for each $\varepsilon > 0$ and for each solution $x \in C(J,\mathbb{R})$ of the inequality $|{}^H D^{\alpha} x(t) - f(t, x(t), {}^H D^{\alpha} x(t))| \le \varepsilon, t \in J$ $\exists$ a solution $y \in C(J,\mathbb{R})$ of the problem (1.1) with $|x(t) - y(t)| \le \varphi_M(\varepsilon)$.*

Clearly, UH (Ulam-Hyers) stable $\Rightarrow$ Generalized UH (Ulam-Hyers) stable.

Definition 2.7. *The **Problem (1.1) is UHR (Ulam-Hyers-Rassias) stable** with respect to $\zeta \in C(J,\mathbb{R})$ if $\exists$ a $\varphi_M > 0$ such that for each $\varepsilon > 0$ and for each solution $x \in C(J,\mathbb{R})$ of the inequality $|{}^H D^{\alpha} x(t) - f(t, x(t), {}^H D^{\alpha} x(t))| \le \varepsilon \zeta(t), t \in J$ $\exists$ a solution $y \in C(J,\mathbb{R})$ of the problem (1.1) with $|x(t) - y(t)| \le \varphi_M \varepsilon \zeta(t)$.*

Definition 2.8. *The **Problem (1.1) is Generalized UHR (Ulam-Hyers-Rassias) stable** with respect to $\zeta \in C(J,\mathbb{R})$ if $\exists$ a $\varphi_{M,\zeta} > 0$ such that for each solution $x \in C(J,\mathbb{R})$ of the inequality $|{}^H D^{\alpha} x(t) - f(t, x(t), {}^H D^{\alpha} x(t))| \le \zeta(t), t \in J$ $\exists$ a solution $y \in C(J,\mathbb{R})$ of the problem (1.1) with $|x(t) - y(t)| \le \varphi_{M,\zeta} \zeta(t)$.*

Here also clearly, UHR (Ulam-Hyers-Rassias) stable $\Rightarrow$ Generalized UHR (Ulam-HyersRassias) stable.

Definition 2.9. *The Mittag-Leffler type is defined by*

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \alpha \in \mathbb{C}, \mathbb{R}(\alpha) > 0$$

and the general Mittag-Leffler type is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \alpha, \beta \in \mathbb{C}, \mathbb{R}(\alpha) > 0, \mathbb{R}(\beta) > 0$$

The simple generalization of the exponent functions

$$E_{\alpha,1}(z) = E_{\alpha}(z)$$
$$E_1(z) = E_{1,1}(z) = e^z$$
$$E_2(z) = cosh\sqrt{z} \quad \textit{and} \quad E_2(-z^2) = cos z \quad \textit{(Cosine Function)}$$

Lemma 2.10. *If*

$$u(t) \le a(t) + b(t) \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{u(s)}{s} ds \quad \textit{for any} \quad t \in [1,T]$$

where all the functions are not negative and continuous. The constant $\alpha > 0$, b is a bounded and monotonic increasing function on $[1,T]$, then

$$u(t) \le a(t) + \int_1^t \left(\sum_{n=1}^{\infty} \frac{(b(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\log \frac{t}{s}\right)^{n\alpha-1} a(s) \right) \frac{ds}{s}]$$

3. Main Results

The formula of solutions for equation $(1.1)-(1.3)$ should be

$$x(t) = \begin{cases} f\left(t, x_0 + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} x(s) \frac{ds}{s}, x(t)\right) \text{ for } t \in (1, t_1] \\ f\left(t, x_0 + y_1 + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} x(s) \frac{ds}{s}, x(t)\right) \text{ for } t \in (t_1, t_2] \\ \vdots \\ f\left(t, x_0 + \sum_{i=1}^{m} y_i + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} x(s) \frac{ds}{s}, x(t)\right) \\ \qquad \text{for } t \in (t_m, T] \end{cases} \tag{3.1}$$

Let a function $f(t,u,v): J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, w are continuous.

$$x(t)=x_0+\sum_{i=1}^{m}y_i+_H I^{\alpha}g(t) \quad \text{for } t\in J=[1,T] \tag{3.2}$$

where $g\in C(J,\mathbb{R})$ satisfies the functional equation $g(t)=f(t,x_0+\sum_{i=1}^{m}y_i+_H I^{\alpha}g(t),g(t))$ and ${}_HI^{\alpha}g(t)=\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}g(s)\frac{ds}{s}$.

To prove the main result we need the following assumptions :

(A_1) The function $f: J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, w are continuous.

(A_2) There exist constants $K_1>0$ and $0<K_2<1$ such that

$$||f(t,u_1,v_1)-f(t,u_2,v_2)||\le K_1||u_1-u_2||+K_2||v_1-v_2||$$

for any $u_1,u_2,v_1\&v_2\in\mathbb{X}$ and $t\in J$.

(A_3) The function $\zeta\in C(J,\mathbb{R})$ is increasing and there exists $\lambda_\zeta>0$ such that $I^{\alpha}\zeta(t)\le\lambda_\zeta\zeta(t), t\in J$

Theorem 3.1. *Assume the $(A_1)-(A_2)$ are holds. If*

$$\sum_{i=1}^{m}y_i+\frac{K_1(\log T)^{\alpha}}{\Gamma(\alpha+1)}+K_2<1 \tag{3.3}$$

then there exists a unique solution of the problem $(1.1)-(1.3)$ on J

Proof. Define the operator $M: C(J,\mathbb{R})\times\mathbb{R}\to C(J,\mathbb{R})$.

$$\begin{aligned}(Mx)(t)&=f\Big(t,x_0+\sum_{i=1}^{m}y_i\\&\quad+\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}x(s)\frac{ds}{s},x(t)\Big)\\&\quad\text{for } t\in J\end{aligned} \tag{3.4}$$

Let $w_1,w_2\in C(J,\mathbb{R})$, we have

$$\begin{aligned}&|(Mw_1)(t)-(Mw_2)(t)|\\&\le\sum_{i=1}^{m}|y_i|+\frac{k_1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}|w_1(s)-w_2(s)|\frac{ds}{s}\\&\quad+k_2|w_1(t)-w_2(t)|\\&\le\sum_{i=1}^{m}|y_i|+\left(\frac{k_1}{\Gamma(\alpha)}\int_1^t(\log t)^{\alpha-1}\frac{ds}{s}+k_2\right)||w_1-w_2||_\infty\\&\le\sum_{i=1}^{m}|y_i|+\left(\frac{k_1(\log T)^{\alpha}}{\Gamma(\alpha+1)}+k_2\right)||w_1-w_2||_\infty\end{aligned}$$

by (3.3), the operator M is a contraction.

By using the lemma 9 in [5], If ${}^HD^{\alpha}y(t)=g(t)$ then ${}_HI^{\alpha}\left({}^HD^{\alpha}y(t)\right)={}_HI^{\alpha}g(t)$ so we have $y(t)=y_0+_HI^{\alpha}g(t)$. Hence, by Banach contraction principle M has a unique fixed point which is a unique solution of the problem (1.1). ☐

Theorem 3.2. *Assume the $(A_1)-(A_2)$, and equation (3.3) are holds. Then the problem (1.1) is UH (Ulam-Hyers) stable.*

Proof. Let us $x\in C(J,\mathbb{R})$ of the inequality

$$|{}^HD^{\alpha}x(t)-f(t,x(t),{}^HD^{\alpha}x(t))|\le\varepsilon, t\in J \tag{3.5}$$

Denoted by $y\in C(J,\mathbb{R})$ the unique solution of the problem

$$\begin{aligned}&{}^HD^{\alpha}y(t)=f(t,y(t),{}^HD^{\alpha}y(t)),\\&\qquad t\in J':=J\backslash\{t_0,...,t_m\}, J=[1,T], 0<\alpha\le 1.\\&y(t_k^+)=y(t_k^-)+y_k,\ k=1,2,...,m \quad y_k\in\mathbb{R}\\&y(1)=x_0,\end{aligned}$$

By lemma 9 in [5], we obtain

$$y(t)=x_0+\sum_{i=1}^{m}y_i+\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}g_y(s)\frac{ds}{s}$$

where $g_y\in C(J,\mathbb{R})$ satisfies the functional equation $g_y(t)=f(t,y_0+\sum_{i=1}^{m}y_i+_HI^{\alpha}g_y(t),g_y(t))$ and ${}_HI^{\alpha}g_y(t)=\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}g_y(s)\frac{ds}{s}$, we get

$$\begin{aligned}&|x(t)-x_0-\sum_{i=1}^{m}y_i-\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}g_x(s)\frac{ds}{s}|\\&\le G-\sum_{i=1}^{m}|y_i|+\frac{\varepsilon(\log T)^{\alpha}}{\Gamma(\alpha+1)}\end{aligned} \tag{3.6}$$

where $g_x\in C(J,\mathbb{R})$ satisfies the functional equation $g_x(t)=f(t,x_0+\sum_{i=1}^{m}y_i+_HI^{\alpha}g_x(t),g_x(t))$ and ${}_HI^{\alpha}g_x(t)=\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}g_x(s)\frac{ds}{s}$. For $t\in J$, we have

$$\begin{aligned}|(x(t)-y(t)|&=|x(t)-x_0-\sum_{i=1}^{m}y_i-\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}g_x(s)\frac{ds}{s}\\&\quad+\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}|(g_x(s)-g_y(s))|\frac{ds}{s}|\end{aligned}$$

where

$$\begin{aligned}&g_x(t)=f(t,x(t),g_x(t)),\\&g_y(t)=f(t,y(t),g_y(t)),\end{aligned}$$

By (A_2) we have

$$\begin{aligned}|g_x(t)-g_y(t)|&=|f(t,x(t),g_x(t))-f(t,y(t),g_y(t))|\\&\le K_1|x(t)-y(t)|+K_2|g_x(t)-g_y(t)|\\&\le\frac{K_1}{1-K_2}|x(t)-y(t)|\end{aligned}$$

Therefore

$$
\begin{aligned}
&|(x(t)-y(t)| \\
&= \sum_{i=1}^{m}|y_i| + \frac{\varepsilon(\log T)^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad + \frac{K_1}{(1-K_2)\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}|(g_x(s)-g_y(s))|\frac{ds}{s} \\
&\leq \sum_{i=1}^{m}|y_i| + \frac{\varepsilon(\log T)^{\alpha}}{\Gamma(\alpha+1)}\int_1^t \Bigg(\sum_{i=1}^{\infty}\left(\frac{K_1}{1-K_2}\right)^i \\
&\qquad \frac{1}{\Gamma(i\alpha)}(\log\frac{t}{s})^{i\alpha-1}\frac{\varepsilon(\log T)^{\alpha}}{\Gamma(\alpha+1)}\Bigg)\frac{ds}{s} \\
&\leq \sum_{i=1}^{m}|y_i| + \frac{\varepsilon(\log T)^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad + \left(1+\sum_{i=1}^{\infty}\left(\frac{K_1}{1-K_2}\right)^i \frac{1}{\Gamma(i\alpha)}\frac{(\log T)^{i\alpha}}{i\alpha}\right) \\
&\leq \sum_{i=1}^{m}|y_i| + \frac{\varepsilon(\log T)^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad + \left(1+\sum_{i=1}^{\infty}\left(\frac{K_1}{1-K_2}\right)^i \frac{(\log T)^{i\alpha}}{\Gamma(i\alpha+1)}\right) \\
&\leq \sum_{i=1}^{m}|y_i| + \frac{\varepsilon(\log T)^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad + \left(1+\sum_{i=1}^{\infty}\left(\frac{K_1(\log T)^{\alpha}}{(1-K_2)\Gamma(i\alpha+1)}\right)^i\right) \\
&\leq \sum_{i=1}^{m}|y_i| + \frac{\varepsilon(\log T)^{\alpha}}{\Gamma(\alpha+1)} + E_{\alpha}\left(\frac{K_1}{1-K_2}(\log T)^{\alpha}\right)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
|(x(t)-y(t)| &\leq \sum_{i=1}^{m}|y_i| + \frac{\varepsilon(\log T)^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad + E_{\alpha}\left(\frac{K_1}{1-K_2}(\log T)^{\alpha}\right) := \varphi_M\varepsilon
\end{aligned}
\tag{3.7}
$$

So, the problem (1.1) is UH (Ulam-Hyers) stable. By Putting $\varphi(\varepsilon)=\varphi\varepsilon, \varphi(0)=0$ yields that the problem (1.1) is generalized UH (Ulam-Hyers) stable. □

Theorem 3.3. *Assume that $(A_1),(A_2)$ and (A_3) are holds. Then the problem (1.1) is UHR stable.*

Proof. Let $z\in C(J,\mathbb{R})$.

$$
|{}^{H}D^{\alpha}y(t)-f(t,y(t),{}^{H}D^{\alpha}y(t))| \leq \varepsilon\zeta(t), t\in J \tag{3.8}
$$

Let $x\in C(J,\mathbb{R})$ the unique solution of the problem (1.1).

$$
\begin{aligned}
&{}^{H}D^{\alpha}x(t) = f(t,x(t),{}^{H}D^{\alpha}x(t)), \\
&\qquad\qquad t\in J' := J\backslash\{t_0,...,t_m\}, J=[1,T], 0<\alpha\leq 1. \\
&x(t_k^+) = x(t_k^-)+y_k,\ k=1,2,...,m \quad y_k\in\mathbb{R} \\
&x(1) = y(0),
\end{aligned}
$$

we have

$$
x(t) = y(0)+\sum_{i=1}^{m}y_i + {}_{H}I^{\alpha}g_x(t) \tag{3.9}
$$

where $g_x\in C(J,\mathbb{R})$ satisfies the functional equation $g_x(t)=f(t,x(0)+\sum_{i=1}^{m}y_i+{}_{H}I^{\alpha}g_x(t),g_x(t))$ and ${}_{H}I^{\alpha}g_x(t)=\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}g_x(s)\frac{ds}{s}$. then we have

$$
\begin{aligned}
&|y(t)-y(0)-\sum_{i=1}^{m}y_i-\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}g_y(s)\frac{ds}{s}| \\
&\leq \sum_{i=1}^{m}y_i+\frac{\varepsilon}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}\zeta(s)ds \\
&\leq \sum_{i=1}^{m}y_i+\varepsilon\lambda_{\zeta}\zeta(t)
\end{aligned}
\tag{3.10}
$$

where $g_y\in C(J,\mathbb{R})$ satisfies the functional equation $g_y(t)=f(t,y(0)+\sum_{i=1}^{m}y_i+{}_{H}I^{\alpha}g_y(t),g_y(t))$ and ${}_{H}I^{\alpha}g_y(t)=\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}g_y(s)\frac{ds}{s}$. we have

$$
\begin{aligned}
&|y(t)-x(t)| \\
&= |y(t)-y(0)-\sum_{i=1}^{m}y_i-\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}g_x(s)\frac{ds}{s}| \\
&= |y(t)-y(0)-\sum_{i=1}^{m}y_i-\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}g_y(s)\frac{ds}{s} \\
&\quad +\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}(g_y(s)-g_x(s))\frac{ds}{s}| \\
&\leq |y(t)-y(0)-\sum_{i=1}^{m}y_i-\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}g_y(s)\frac{ds}{s}| \\
&\quad +\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}(|g_y(s)-g_x(s)|)\frac{ds}{s}
\end{aligned}
\tag{3.11}
$$

where

$$
\begin{aligned}
g_x(t) &= f(t,x(t),g_x(t)), \\
g_y(t) &= f(t,y(t),g_y(t)),
\end{aligned}
$$

By (A_2) we have

$$
\begin{aligned}
|g_x(t)-g_y(t)| &= |f(t,x(t),g_x(t))-f(t,y(t),g_y(t))| \\
&\leq K_1|x(t)-y(t)|+K_2|g_x(t)-g_y(t)| \\
&\leq \frac{K_1}{1-K_2}|x(t)-y(t)|
\end{aligned}
$$

$$
\begin{aligned}
&|y(t)-x(t)| \\
&\le \sum_{i=1}^{m} y_i + \varepsilon\lambda_\zeta\zeta(t) \\
&\quad + \frac{K_1}{(1-K_2)\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}|x(s)-y(s)|\frac{ds}{s} \\
&\le \sum_{i=1}^{m} y_i + \varepsilon\lambda_\zeta\zeta(t) \\
&\quad + \frac{K_1||y-x||_\infty}{(1-K_2)\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}\zeta(s)\frac{ds}{s} \\
&\le \sum_{i=1}^{m} y_i + \varepsilon\lambda_\zeta\zeta(t) + \frac{K_1(\log T)^\alpha\varepsilon\lambda_\zeta^2\zeta(t)}{(1-K_2)\Gamma(\alpha+1)} \\
&= \sum_{i=1}^{m} y_i + \left[1+\frac{K_1(\log T)^\alpha\varepsilon\lambda_\zeta}{(1-K_2)\Gamma(\alpha+1)}\right]\varepsilon\lambda_\zeta\zeta(t)
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
&||y(t)-x(t)||_\infty\Big(\sum_{i=1}^{m} y_i \\
&\quad + \left[\left\{1+\frac{K_1(\log T)^\alpha\varepsilon\lambda_\zeta}{(1-K_2)\Gamma(\alpha+1)}\right\}\lambda_\zeta\right]\Big) \le \varepsilon\lambda_\zeta\zeta(t)
\end{aligned} \tag{3.13}
$$

Then the equation becomes,

$$
\begin{aligned}
&||y(t)-x(t)||_\infty \\
&\le \frac{\varepsilon\lambda_\zeta\zeta(t)}{\left(\sum_{i=1}^{m} y_i + \left[\left\{1+\frac{K_1(\log T)^\alpha\varepsilon\lambda_\zeta}{(1-K_2)\Gamma(\alpha+1)}\right\}\lambda_\zeta\right]\right)}
\end{aligned}
$$

$$
\begin{aligned}
&|y(t)-x(t)| \le \Big(\sum_{i=1}^{m} y_i + \Big[\Big\{1 \\
&\quad + \frac{K_1(\log T)^\alpha\varepsilon\lambda_\zeta}{(1-K_2)\Gamma(\alpha+1)}\Big\}\lambda_\zeta\Big]\Big)^{-1}\varepsilon\lambda_\zeta\zeta(t) := \varphi_M\varepsilon\zeta(t)
\end{aligned} \tag{3.14}
$$

then the problem (1.1) is UHR stable. □

4. Example

4.1 Example

Consider the implicit fractional order differential equations with Hadamard derivative and impulsive conditions of the form

$$
{}^H D^{\frac{1}{2}} x(t) = \frac{1}{10}\left(\frac{1}{1+|x(t)|+|{}^H D^{\frac{1}{2}}x(t)|}\right), \tag{4.1}
$$

$$
t \in J' := J\backslash\{t_0,\ldots,t_m\}, J=[1,T], 0<\alpha\le 1.
$$

$$
x(t_k^+) = x(t_k^-) + \frac{1}{6},\ k=1,2,\ldots,m \tag{4.2}
$$

$$
x(1) = 1, \tag{4.3}
$$

where $f(t,x(t),{}^H D^\alpha x(t)) = \frac{1}{10}\left(\frac{1}{1+|x(t)|+|{}^H D^\alpha x(t)|}\right)$
For any $u_1,u_2,v_1,v_2 \in \mathbb{R}$

$$
|f(t,u_1,v_1)-f(t,u_2,v_2)| \le \frac{1}{10}(|u_1-u_2|+|v_1-v_2|)
$$

Hence the assumptions (A_2) is satisfied with $K_1=K_2=\frac{1}{10}$

$$
\sum_{i=1}^{m} y_i + \frac{K_1(\log T)^\alpha}{(1-K_2)\Gamma(\alpha+1)} = \frac{2}{9\sqrt{\pi}} < 1
$$

Then by theorem 3.1 and UH (Ulam-Hyers) stable from theorem 3.2 the equation (4.1)-(4.3) as unique solution.

4.2 Example

Consider the implicit impulsive fractional order differential equations with Hadamard derivative of the form

$$
{}^H D^{\frac{3}{2}} x(t) = \frac{1}{10}\left(\frac{1}{1+|x(t)|+|{}^H D^{\frac{3}{2}}x(t)|}\right), \tag{4.4}
$$

$$
t \in J' := J\backslash\{t_0,\ldots,t_m\}, J=[1,T], 0<\alpha\le 1.
$$

$$
x(t_k^+) = x(t_k^-) + \frac{1}{6},\ k=1,2,3,\ldots,m \tag{4.5}
$$

$$
x(1) = x_0, \tag{4.6}
$$

where $f(t,x(t),{}^H D^\alpha x(t)) = \frac{1}{10}\left(\frac{1}{1+|x(t)|+|{}^H D^\alpha x(t)|}\right)$
For any $u_1,v_1,u_2,v_2 \in \mathbb{R}$

$$
|[f(t,u_1,v_1)-f(t,u_2,v_2)]| \le \frac{1}{12}(|u_1-u_2|+|v_1-v_2|)
$$

Hence A_2 is satisfied with $K_1=K_2=\frac{1}{12}$.

Let $\varphi(t) = (\log t)^{\frac{3}{2}}$.

$$
\begin{aligned}
{}_H I^\alpha\varphi(t) &= \frac{1}{\Gamma(\frac{3}{2})}\int_1^t (\log\frac{t}{s})^{\frac{3}{2}-1}(\log t)^{\frac{3}{2}}\frac{ds}{s} \\
&\le \frac{2}{\sqrt{\pi}}\int_1^t (\log\frac{t}{s})^{\frac{3}{2}-1}\frac{ds}{s} \\
&\le \frac{4\omega(t)}{\sqrt{\pi}} \\
{}_H I^\alpha\varphi(t) &\le \frac{4}{\sqrt{\pi}}(\log t)^{\frac{3}{2}} = \lambda_\varphi\varphi(t)
\end{aligned}
$$

Thus the condition (A_3) is satisfied with $\varphi(t)=(\log t)^{\frac{3}{2}}$ and $\lambda_\varphi = \frac{4}{\sqrt{\pi}}$ it follows, from theorem 3.1 and UHR stable from theorem 3.3, the (4.4)-(4.6) has a unique solution.

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ISSN(P):2319 − 3786

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ISSN(O):2321 − 5666

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