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# Some inequalities for the Kirchhoff index of graphs

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**Abstract**

Let $G$ be a simple connected graph of order $n$, sequence of vertex degrees $d_1 \geq d_2 \geq \cdots \geq d_n > 0$ and Laplacian eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$. With $\Pi_1 = \Pi_1(G) = \prod_{i=1}^{n} d_i^2$ we denote the multiplicative first Zagreb index of graph, and $Kf(G) = n\sum_{i=1}^{n-1} \frac{1}{\mu_i}$ the Kirchhoff index of $G$. In this paper we determine several lower and upper bounds for $Kf$ depending on some of the graph parameters such as number of vertices, maximum degree, minimum degree, and number of spanning trees or multiplicative Zagreb index.

**Keywords**

Kirchhoff index, Laplacian eigenvalues (of graph), vertex degree.

**AMS Subject Classification**

05C12, 05C50

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## Contents



## 1. Introduction

Let $G = (V,E)$ be a simple connected graph with $n$ vertices and $m$ edges with vertex degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n > 0$, and $\Delta = d_1$, $\Delta_1 = d_2$, $\delta = d_n$, $\delta_1 = d_{n-1}$. If vertices $i$ and $j$ are adjacent we write $i \sim j$. Further, let $\mathbf{A}$ be the adjacency matrix of $G$ and $\mathbf{D} = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ the diagonal matrix of its vertex degrees. Then $\mathbf{L} = \mathbf{D} - \mathbf{A}$ is the Laplacian matrix of $G$. Eigenvalues of $\mathbf{L}$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$, form the so-called Laplacian spectrum of $G$. Following identities are valid for $\mu_i$ (see [1])

$$\sum_{i=1}^{n-1} \mu_i = 2m \quad \text{and} \quad \sum_{i=1}^{n-1} \mu_i^2 = \sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i = M_1 + 2m,$$

where $M_1 = M_1(G) = \sum_{i=1}^{n} d_i^2$ is the first Zagreb index introduced in [2]. More about this degree–based topological index one can be found in [3–8].

It is well known that a connected graph $G$ of order $n$ has

$$t = t(G) = \frac{1}{n}\prod_{i=1}^{n-1} \mu_i$$

spanning trees.

In [9] (see also [10]) a multiplicative variant of the first Zagreb index, named the first multiplicative Zagreb index, $\Pi_1$, was introduced. It is define as

$$\Pi_1 = \Pi_1(G) = \prod_{i=1}^{n} d_i^2.$$

In [11], Klein and Randić, introduced the notion of resistance distance, $r_{ij}$. It is defined as the resistance between the nodes $i$ and $j$ in an electrical network corresponding to the graph $G$ in which all edges are replaced by unit resistors. The sum of resistance distances of all pairs of vertices of a graph $G$ is named as the Kirchhoff index, i.e.

$$Kf(G) = \sum_{i<j} r_{ij}.$$

There are several equivalent ways to define the resistance distance. Gutman and Mohar [12] (see also [13]) proved that the Kirchhoff index can be obtained from the non-zero eigenvalues of the Laplacian matrix:

$$Kf(G) = n\sum_{i=1}^{n-1} \frac{1}{\mu_i}.$$

Among various indices in mathematical chemistry, those based on the effective resistance, $r_{ij}$, such as the Kirchhoff index and its generalizations, have received a lot of attention in the literature, as it turned out that they play an important role in solving problems in different scientific disciplines, such as molecular chemistry, spectral graph theory, network theory, etc. (see, for example, [14–21]).

Considering the fact that obtaining the exact and easy to compute formula for the Kirchhoff index is not always possible, it is useful to know approximating expressions, i.e. upper and lower bounds and corresponding extremal graphs. In this paper we report several lower and upper bounds for $Kf(G)$ of a connected (molecular) graph in terms of some structural graph parameters, such as the number of vertices (atoms), maximum vertex degree (valency), minimal vertex degree, and graph invariants such as number of spanning trees, $t$, and multiplicative first Zagreb index, $\Pi_1$.

## 2. Preliminaries

In this section we recall some inequalities for the Kirchhoff index, and some analytic inequalities for real number sequences that are of interest for the subsequent considerations.

Let $G$ be a simple connected graph with $n \geq 2$ vertices. In [21] the following inequality was proved

$$Kf(G) \geq -1 + (n-1)\sum_{i=1}^{n}\frac{1}{d_i}, \tag{2.1}$$

with equality if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \cong K_{1,n-1}$, or $G \in \Gamma_d$.

The following lower bound for $Kf(G)$ that depends on number of vertices, $n$, the maximum degree, $\Delta$ and the number of spanning trees, $t$, was determined in [14]:

$$Kf(G) \geq \frac{n}{1+\Delta} + n(n-2)\left(\frac{\Delta+1}{nt}\right)^{\frac{1}{n-2}}, \tag{2.2}$$

with equality if and only if $G \cong K_n$, or $G \cong K_{1,n-1}$.

Let $a = (a_i)$, and $b = (b_i)$, $i = 1,2,\dots,n$, be two positive real number sequences with the properties $0 < r_1 \leq a_i \leq R_1 < +\infty$ and $0 < r_2 \leq b_i \leq R_2 < +\infty$. In [22] the following inequality was proved

$$\left| n\sum_{i=1}^{n}a_ib_i - \sum_{i=1}^{n}a_i\sum_{i=1}^{n}b_i \right| \leq n^2(R_1-r_1)(R_2-r_2)\alpha(n), \tag{2.3}$$

where

$$\alpha(n) = \frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\right) = \frac{1}{4}\left(1-\frac{(-1)^{n+1}+1}{2n^2}\right).$$

Let $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ be real number sequence. In [23] it was proved

$$a_1+a_2+\cdots+a_n - n\sqrt[n]{a_1a_2\cdots a_n} \geq \left(\sqrt{a_1}-\sqrt{a_n}\right)^2, \tag{2.4}$$

with equality if and only if $a_2 = a_3 = \cdots = a_{n-1} = \sqrt{a_1a_n}$, and

$$\frac{a_1+a_2+\cdots+a_n}{n\sqrt[n]{a_1a_2\cdots a_n}} \geq \frac{\left(\sqrt{\frac{a_1}{a_n}}+\sqrt{\frac{a_n}{a_1}}\right)^{\frac{2}{n}}}{2^{\frac{2}{n}}}, \tag{2.5}$$

with equality if and only if $a_2 = a_3 = \cdots = a_{n-1} = \frac{a_1+a_2}{2}$.

Let $a = (a_i)$, $i = 1,2,\dots,n$, be positive real number sequence. In [24] (see also [25]) the following was proved

$$\begin{aligned} n\left(\frac{1}{n}\sum_{i=1}^{n}a_i - \left(\prod_{i=1}^{n}a_i\right)^{\frac{1}{n}}\right) &\leq n\sum_{i=1}^{n}a_i - \left(\sum_{i=1}^{n}\sqrt{a_i}\right)^2 \\ &\leq n(n-1)\left(\frac{1}{n}\sum_{i=1}^{n}a_i - \left(\prod_{i=1}^{n}a_i\right)^{\frac{1}{n}}\right), \end{aligned} \tag{2.6}$$

with equalities if and only if $a_1 = a_2 = \cdots = a_n$.

Before we proceed, let us define one special class of $d$-regular graphs $\Gamma_d$ (see [20]). Let $N(i)$ be a set of all neighbors of the vertex $i$, i.e. $N(i) = \{k \,|\, k \in V, k \sim i\}$, and $d(i,j)$ the distance between vertices $i$ and $j$. Denote by $\Gamma_d$ a set of all $d$-regular graphs, $1 \leq d \leq n-1$, with diameter 2 and $|N(i)\cap N(j)| = d$ for $i \nsim j$.

## 3. Main results

In the following theorem we establish upper bound for $Kf(G)$ in terms of number of spanning trees, number of vertices, $n$, and parameter $k$, where $k$ is an arbitrary real number such that $\mu_{n-1} \geq k > 0$.

**Theorem 3.1.** *Let $G$ be a simple connected graph with $n \geq 3$ vertices. Then, for any real $k$ with the property $\mu_{n-1} \geq k > 0$, holds*

$$Kf(G) \leq n(n-1)(nt)^{-\frac{1}{n-1}} + n(n-1)^2\alpha(n-1)\frac{\left(\sqrt{n}-\sqrt{k}\right)^2}{nk}, \tag{3.1}$$

*with equality if and only if $k = n$ and $G \cong K_n$.*

*Proof.* For $n := n-1$, $a_i = b_i = \frac{1}{\sqrt{\mu_i}}$, $i = 1,2,\dots,n-1$, $R_1 = R_2 = 1/\sqrt{\mu_{n-1}}$, $r_1 = r_2 = 1/\sqrt{\mu_1}$, the inequality (2.3) transforms into

$$\begin{aligned} (n-1)\sum_{i=1}^{n-1}\frac{1}{\mu_i} - \left(\sum_{i=1}^{n-1}\frac{1}{\sqrt{\mu_i}}\right)^2 &\leq \\ &\leq (n-1)^2\alpha(n-1)\left(\frac{1}{\sqrt{\mu_{n-1}}}-\frac{1}{\sqrt{\mu_1}}\right)^2. \end{aligned}$$

Since $0 < \mu_1 \leq n$ and $\mu_{n-1} \geq k > 0$, it follows

$$\begin{aligned} (n-1)\sum_{i=1}^{n-1}\frac{1}{\mu_i} &\leq \left(\sum_{i=1}^{n-1}\frac{1}{\sqrt{\mu_i}}\right)^2 + \\ &+ (n-1)^2\alpha(n-1)\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{n}}\right)^2. \end{aligned} \tag{3.2}$$

For $n := n-1$, $a_i = \frac{1}{\mu_i}$, $i = 1,2,\dots,n-1$, left hand side of inequality (2.6) becomes

$$\sum_{i=1}^{n-1}\frac{1}{\mu_i} - (n-1)\left(\prod_{i=1}^{n-1}\frac{1}{\mu_i}\right)^{\frac{1}{n-1}} \leq (n-1)\sum_{i=1}^{n-1}\frac{1}{\mu_i} - \left(\sum_{i=1}^{n-1}\frac{1}{\sqrt{\mu_i}}\right)^2,$$

i.e.

$$\left(\sum_{i=1}^{n-1}\frac{1}{\sqrt{\mu_i}}\right)^2 \le (n-2)\sum_{i=1}^{n-1}\frac{1}{\mu_i}+(n-1)(nt)^{-\frac{1}{n-1}}. \quad (3.3)$$

From (3.2) and (3.3) we obtain

$$(n-1)\sum_{i=1}^{n-1}\frac{1}{\mu_i} \le (n-2)\sum_{i=1}^{n-1}\frac{1}{\mu_i}+(n-1)(nt)^{-\frac{1}{n-1}}+ +(n-1)^2\alpha(n-1)\frac{\left(\sqrt{n}-\sqrt{k}\right)^2}{nk},$$

i.e.

$$\sum_{i=1}^{n-1}\frac{1}{\mu_i}\le (n-1)(nt)^{-\frac{1}{n-1}}+(n-1)^2\alpha(n-1)\frac{\left(\sqrt{n}-\sqrt{k}\right)^2}{nk},$$

wherefrom we get (3.1).

Equality in (3.3) holds if and only if $\mu_1=\cdots=\mu_{n-1}$, therefore equality in (3.1) holds if and only if $k=n$ and $G\cong K_n$.

□

In the next theorem we establish lower bound for $Kf(G)$ depending on structural graph parameters $n$, $\Delta$, $\Delta_2$, $\delta$ and the number of spanning trees $t$.

**Theorem 3.2.** *Let $G$ be a simple connected graph with $n\ge 3$ vertices. Then*

$$Kf(G)\ge \frac{n}{1+\Delta}+n(n-2)\left(\frac{\Delta+1}{nt}\right)^{\frac{1}{n-2}}+n\left(\frac{1}{\sqrt{\delta}}-\frac{1}{\sqrt{\Delta_2}}\right)^2, \quad (3.4)$$

*with equality if and only if $G\cong K_n$, or $G\cong K_{1,n-1}$, or $G\cong K_{\frac{n}{2},\frac{n}{2}}$*

*Proof.* According to (2.4) we have that

$$a_2+a_3+\cdots+a_{n-1}-(n-2)(a_2a_3\cdots a_{n-1})^{\frac{1}{n-2}}\ge(\sqrt{a_2}-\sqrt{a_{n-1}})^2.$$

For $a_i=\dfrac{1}{\mu_{n-i-1}}$, $i=2,\dots,n-1$, the above inequality transforms into

$$\sum_{i=2}^{n-1}\frac{1}{\mu_i}-(n-2)\left(\prod_{i=2}^{n-1}\frac{1}{\mu_i}\right)^{\frac{1}{n-2}}\ge\left(\frac{1}{\sqrt{\mu_{n-1}}}-\frac{1}{\sqrt{\mu_2}}\right)^2$$

i.e.

$$\sum_{i=1}^{n-1}\frac{1}{\mu_i}\ge\frac{1}{\mu_1}+(n-2)\left(\prod_{i=2}^{n-1}\frac{\mu_1}{\mu_i}\right)^{\frac{1}{n-2}}+\left(\frac{1}{\sqrt{\mu_{n-1}}}-\frac{1}{\sqrt{\mu_2}}\right)^2. \quad (3.5)$$

Obviously, equality in (3.5), i.e. (3.4), is attained if $G\cong K_n$. Therefore suppose that $G\ne K_n$. In that case $\mu_2\ge\Delta_2$ (see [26]) and $\mu_{n-1}\le\delta$ (see [27]), and the inequality (3.5) transforms into

$$\sum_{i=1}^{n-1}\frac{1}{\mu_i}\ge\frac{1}{\mu_1}+(n-2)\left(\frac{\mu_1}{nt}\right)^{\frac{1}{n-2}}+\left(\frac{1}{\sqrt{\delta}}-\frac{1}{\sqrt{\Delta_2}}\right)^2.$$

Now, consider the function $g(x)=\frac{1}{x}+(n-2)\left(\frac{x}{nt}\right)^{\frac{1}{n-2}}$. It was proved that it is monotone increasing for $x\ge 1+\Delta$ and $x\ge (nt)^{\frac{1}{n-1}}$ (see [14]). Since $\mu_1\ge 1+\Delta$ (see [28]) and $\mu_1\ge (nt)^{\frac{1}{n-1}}$, we have that

$$\sum_{i=1}^{n-1}\frac{1}{\mu_i}\ge\frac{1}{1+\Delta}+(n-2)\left(\frac{1+\Delta}{nt}\right)^{\frac{1}{n-2}}+\left(\frac{1}{\sqrt{\delta}}-\frac{1}{\sqrt{\Delta_2}}\right)^2,$$

wherefrom we obtain (3.4).

Equality in (3.5) holds if and only if $\mu_2=\cdots=\mu_{n-1}$, hence equality in (3.4) holds if and only if $G\cong K_n$, or $G\cong K_{1,n-1}$, or $G\cong K_{\frac{n}{2},\frac{n}{2}}$ (see [29]).

□

Similarly, the following result can be proved.

**Theorem 3.3.** *Let $G$ be a simple connected graph with $n\ge 3$ vertices. Then*

$$Kf(G)\ge\frac{n}{1+\Delta}+n(n-2)\left(\frac{1+\Delta}{nt}\right)^{\frac{1}{n-2}}\left(\frac{\Delta_2+\delta}{2\sqrt{\Delta_2\delta}}\right)^{\frac{2}{n-2}}. \quad (3.6)$$

*Equality holds if and only if $G\cong K_n$, or $G\cong K_{1,n-1}$ or $G\cong K_{\frac{n}{2},\frac{n}{2}}$.*

**Remark 3.4.** *Since $\left(\frac{\Delta_2+\delta}{2\sqrt{\Delta_2\delta}}\right)^{\frac{2}{n-2}}\ge 1$, the inequality (3.6) is stronger than (2.2).*

In the following theorem we determine lower bound for $Kf(G)$ in terms of number of vertices, $n$, maximum degree, $\Delta$, minimum degree, $\delta$, and topological index $\Pi_1$.

**Theorem 3.5.** *Let $G$ be a simple connected graph with $n\ge 2$ vertices. Then*

$$Kf(G)\ge -1+n(n-1)\frac{(\Delta+\delta)^{\frac{2}{n}}}{(2\sqrt{\Delta\delta})^{\frac{2}{n}}}(\Pi_1)^{-\frac{1}{2n}}. \quad (3.7)$$

*Equality holds if and only if $G\cong K_n$, or $G\cong K_{\frac{n}{2},\frac{n}{2}}$, or $G\in\Gamma_d$.*

*Proof.* For $a_i=\frac{1}{d_i}$, $i=1,2,\dots,n$, $a_1=\frac{1}{\delta}$, $a_n=\frac{1}{\Delta}$, the inequality (2.5) transforms into

$$\frac{\sum_{i=1}^n\frac{1}{d_i}}{n\left(\prod_{i=1}^n\frac{1}{d_i}\right)^{\frac{1}{n}}}\ge\frac{\left(\sqrt{\frac{\Delta}{\delta}}+\sqrt{\frac{\delta}{\Delta}}\right)^{\frac{2}{n}}}{2^{\frac{2}{n}}}, \quad (3.8)$$

i.e.

$$\sum_{i=1}^{n} \frac{1}{d_i} \geq \frac{n(\Delta+\delta)^{\frac{2}{n}}}{(4\Delta\delta)^{\frac{1}{n}}} (\Pi_1)^{-\frac{1}{2n}}. \tag{3.9}$$

From (3.9) and (2.1) we obtain (3.7).

Equality in (3.8) holds if and only if $d_2 = \cdots = d_{n-1} = \frac{2d_1 d_n}{d_1+d_n}$, i.e. if and only if $d_1 = d_2 = \cdots = d_{n-1} = d_n$. Equality in (2.1) is attained if and only if $G \cong K_n$, or $G \cong K_{1,n-1}$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \in \Gamma_d$, hence equality in (3.7) holds if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \in \Gamma_d$. □

By a similar procedure as in case of Theorem 3.5, the following results can be proved.

**Theorem 3.6.** *Let G be a simple connected graph with $n \geq 3$ vertices. Then*

$$Kf(G) \geq \frac{n-1-\Delta}{\Delta} + (n-1)^2 \frac{(\Delta_1+\delta)^{\frac{2}{n-1}}}{(4\Delta_1\delta)^{\frac{1}{n-1}}} \left(\frac{\Pi_1}{\Delta^2}\right)^{-\frac{1}{2(n-1)}}.$$

*Equality holds if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \cong K_{1,n-1}$, or $G \in \Gamma_d$.*

**Theorem 3.7.** *Let G be a simple connected graph with $n \geq 3$ vertices. Then*

$$Kf(G) \geq \frac{n-1-\delta}{\delta} + (n-1)^2 \frac{(\Delta+\delta_1)^{\frac{2}{n-1}}}{(4\Delta\delta_1)^{\frac{1}{n-1}}} \left(\frac{\Pi_1}{\delta^2}\right)^{-\frac{1}{2(n-1)}}.$$

*Equality holds if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \in \Gamma_d$.*

**Theorem 3.8.** *Let G be a simple connected graph with $n \geq 4$ vertices. Then*

$$\begin{aligned} Kf(G) \quad &\geq \quad \frac{(n-1)(\Delta+\delta)-\Delta\delta}{\Delta\delta} + \\ &+ \quad (n-1)(n-2)\frac{(\Delta_1+\delta_1)^{\frac{2}{n-2}}}{(4\Delta_1\delta_1)^{\frac{1}{n-2}}} \left(\frac{\Pi_1}{\Delta^2\delta^2}\right)^{-\frac{1}{2(n-2)}}. \end{aligned}$$

*Equality holds if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \cong K_{1,n-1}$, or $G \in \Gamma_d$.*

**Theorem 3.9.** *Let G be a simple connected graph with $n \geq 2$ vertices. Then*

$$Kf(G) \geq -1 + n(n-1)(\Pi_1)^{-\frac{1}{2n}} + (n-1)\frac{\left(\sqrt{\Delta}-\sqrt{\delta}\right)^2}{\Delta\delta}. \tag{3.10}$$

*Equality holds if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \in \Gamma_d$.*

*Proof.* For $a_i = \frac{1}{d_i}$, $i = 1,2,\ldots,n$, $a_1 = \frac{1}{\delta}$, $a_n = \frac{1}{\Delta}$, the inequality (2.4) transforms into

$$\sum_{i=1}^{n} \frac{1}{d_i} - n\left(\prod_{i=1}^{n} \frac{1}{d_i}\right)^{\frac{1}{n}} \geq \left(\frac{1}{\sqrt{\delta}} - \frac{1}{\sqrt{\Delta}}\right)^2, \tag{3.11}$$

i.e.

$$\sum_{i=1}^{n} \frac{1}{d_i} \geq n(\Pi_1)^{-\frac{1}{2n}} + \frac{\left(\sqrt{\Delta}-\sqrt{\delta}\right)^2}{\Delta\delta}. \tag{3.12}$$

Finally, from (2.1) and (3.12) we arrive at (3.10).

Equality in (3.11) holds if and only if $d_2 = \cdots = d_{n-1} = \sqrt{\Delta\delta}$, i.e. if and only if $\Delta = d_1 = \cdots = d_n = \delta$. Equality in (2.1) is attained if and only if $G \cong K_n$, or $G \cong K_{1,n-1}$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \in \Gamma_d$, therefore equality in (3.7) holds if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \in \Gamma_d$. □

In a similar way as in case of Theorem 3.9, the following statements can be proved.

**Theorem 3.10.** *Let G be a simple connected graph with $n \geq 3$ vertices. Then*

$$\begin{aligned} Kf(G) \quad &\geq \quad \frac{n-1-\Delta}{\Delta} + (n-1)^2\left(\frac{\Pi_1}{\Delta^2}\right)^{-\frac{1}{2(n-1)}} + \\ &+ \quad (n-1)\frac{\left(\sqrt{\Delta_1}-\sqrt{\delta}\right)^2}{\Delta_1\delta}. \end{aligned}$$

*Equality holds if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \cong K_{1,n-1}$, or $G \in \Gamma_d$.*

**Theorem 3.11.** *Let G be a simple connected graph with $n \geq 3$ vertices. Then*

$$\begin{aligned} Kf(G) \quad &\geq \quad \frac{n-1-\delta}{\delta} + (n-1)^2\left(\frac{\Pi_1}{\delta^2}\right)^{-\frac{1}{2(n-1)}} + \\ &+ \quad (n-1)\frac{\left(\sqrt{\Delta}-\sqrt{\delta_1}\right)^2}{\Delta\delta_1}. \end{aligned}$$

*Equality holds if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \in \Gamma_d$.*

**Theorem 3.12.** *Let G be a simple connected graph with $n \geq 4$ vertices. Then*

$$\begin{aligned} Kf(G) \geq &\frac{(n-1)(\Delta+\delta)-\Delta\delta}{\Delta\delta} + \\ &+ (n-1)(n-2)\left(\frac{\Pi_1}{\Delta^2\delta^2}\right)^{-\frac{1}{2(n-2)}} + \\ &+ (n-1)\frac{\left(\sqrt{\Delta_1}-\sqrt{\delta_1}\right)^2}{\Delta_1\delta_1}. \end{aligned}$$

*Equality holds if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \cong K_{1,n-1}$, or $G \in \Gamma_d$.*

**Remark 3.13.** *Lower bounds for $Kf(G)$ given by (3.7) and (3.10) depend on the same parameters $n$, $\Delta$, $\delta$ and topological index $\Pi_1$. Equalities are achieved under the same conditions, i.e. if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \in \Gamma_d$. However, these bounds are not comparable. Thus, for example, for $G \cong K_{1,n-1}$ the inequality (3.7) is stronger than (3.10), but for $G \cong P_n$ the inequality (3.10) is stronger than (3.7) for $n \geq 5$. The same applies when compare inequalities from Theorems 3.6, 3.7 and 3.8 with those given in Theorems 3.10, 3.11 and 3.12.*

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