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The combined of Homotopy analysis method with new transform for nonlinear partial differential equations

Djelloul Ziane[1]*

Abstract
The idea proposed in this work is to extend the Aboodh transform method to resolve the nonlinear partial differential equations by combining them with the so-called homotopy analysis method (HAM). This method can be called homotopy analysis aboodh transform method (HAATM). The results obtained by the application to the proposed examples show that this method is easy to apply and can therefore be used to solve other nonlinear partial differential equations.

Keywords
Homotopy analysis method, Aboodh transform method, nonlinear partial differential equations.

AMS Subject Classification
26A33, 44A05, 34K37, 35F61.

[1] *Laboratory of mathematics and its applications (LAMAP), University of Oran1 Ahmed Ben Bella, Oran, 31000, Algeria.*
***Corresponding author**: [1] djeloulz@yahoo.com;
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Contents

1. Introduction

Nonlinear equations are of great importance to our contemporary world. Nonlinear phenomena have important applications in applied mathematics, physics, and issues related to engineering. Despite the importance of obtaining the exact solution of nonlinear partial differential equations in physics and applied mathematics, there is still the daunting problem of finding new methods to discover new exact or approximate solutions [5]. So we find that a lot of researchers are working to develop new methods to solve this kind of equations. These efforts have strengthened this area of research through many methods, among them we find, homotopy analysis method (HAM). This method was developed in 1992 by Liao Shijun of Shanghai Jiaotong University ([10]-[13]). Then, a new option emerged recently, includes the composition of Laplace transform, Sumudu transform, Natural transform or Elzaki transform with this method to solve linear and nonlinear ordinary partial differential equations. Among which are the homotopy analysis method coupled with Laplace transform ([1], [2], [8], [14], [18], [19]), homotopy analysis Sumudu transform method ([3], [16], [17]), homotopy Natural transform method ([9], [21]) and homotopy analysis Elzaki transform method ([20], [22]).

The objective of the present study is to combine two powerful methods, homotopy analysis method and Aboodh transform method to get a better method to solve nonlinear partial differential equations. The modified method is called homotopy analysis Aboodh transform method (HAATM). Three examples are given to re-confirm the effectiveness of this method.

The present paper has been organized as follows: In Section 2 Some basic definitions and properties of the Aboodh transform method. In section 3 We give an analysis of the proposed method. In section 4 We present three examples explaining how to apply the proposed method. Finally, the conclusion follows.

2. Basic definitions and theorems

In this section, we give some basic definitions and properties of Aboodh transform which are used further in this paper. A new transform called the Aboodh transform defined for function of exponential order, we consider functions in the set $\bar{A}$, defined by [7]

$$\bar{A}=\left\{f(t):\exists\, M,\ k_1,k_2>0,\ |f(t)|<Me^{-vt}\right\}. \quad (2.1)$$

For given function in the set $\bar{A}$, the constant M must be finite number, k_1,k_2 my be finite or infinite.

Definition 2.1. *The Aboodh transform denoted by the operator $A(\cdot)$ defined by the integral equation*

$$A[f(t)]=K(v)=\frac{1}{v}\int_0^{\infty}f(t)e^{-vt}dt,\ t\geqslant 0,k_1\leqslant v\leqslant k_2. \quad (2.2)$$

We will summarize here some results of simple functions related to Aboodh transform in the following table [7]

$f(t)$	$A[f(t)]$	$f(t)$	$A[f(t)]$
1	$\frac{1}{v^2}$	$\sin at$	$\frac{a}{v(v^2+a^2)}$
t	$\frac{1}{v^3}$	$\cos at$	$\frac{1}{v^2+a^2}$
t^n	$\frac{n!}{v^{n+2}}$	$\sinh at$	$\frac{a}{v(v^2-a^2)}$
e^{at}	$\frac{1}{v^2-av}$	$\cosh at$	$\frac{1}{v^2-a^2}$

Theorem 2.2. *Let $K(v)$ is the Aboodh transform of $f(t)$, then one has*

$$A\left[f'(t)\right]=vK(v)-\frac{f(0)}{v}, \quad (2.3)$$

$$A\left[f''(t)\right]=v^2K(v)-\frac{f'(0)}{v}-f(0), \quad (2.4)$$

$$A\left[f^{(n)}(t)\right]=v^nK(v)-\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{v^{2-n+k}}. \quad (2.5)$$

Proof. (see [7]). □

Theorem 2.3. *Let $K(x,v)$ is the Aboodh transform of $u(x,t)$, to obtain Aboodh transform of partial derivativewe use integration by parts, and then we have*

$$A\left[\frac{\partial u(x,t)}{\partial t}\right]=vK(x,v)-\frac{u(x,0)}{v}, \quad (2.6)$$

$$A\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right]=v^2K(x,v)-\frac{1}{v}\frac{\partial u(x,0)}{\partial t}-u(x,0), \quad (2.7)$$

Proof. (see [4]). □

Theorem 2.4. *Let $K(x,v)$ is the Aboodh transform of $u(x,t)$, then one has*

$$A\left[\frac{\partial^n u(x,t)}{\partial t^n}\right]=v^nK(x,v)-\sum_{k=0}^{n-1}\frac{1}{v^{2-n+k}}\frac{\partial^k u(x,0)}{\partial t^k}. \quad (2.8)$$

Proof. To demonstrate the validity of the formula (2.8), we use mathematical induction.

If $n=1$ and according to formula (2.8), we obtain

$$A\left[\frac{\partial u(x,t)}{\partial t}\right]=vK(x,v)-\frac{u(x,0)}{v}. \quad (2.9)$$

So, according to the (2.6) we note that the formula holds when $n=1$.

Assume inductively that the formula holds for n, so that

$$A\left[\frac{\partial^n u(x,t)}{\partial t^n}\right]=v^nK(x,v)-\sum_{k=0}^{n-1}\frac{1}{v^{2-n+k}}\frac{\partial^k u(x,0)}{\partial t^k}, \quad (2.10)$$

and show that it stays true at rank $n+1$. Let $\frac{\partial^n u(x,t)}{\partial t^n}=w(x,t)$ and according to (2.6) and (2.10) , we have

$$\begin{aligned}
& A\left[\frac{\partial^{n+1}u(x,t)}{\partial t^{n+1}}\right]\\
=\ & A\left[\frac{\partial w(x,t)}{\partial t}\right]=vA\left(w(x,t)\right)-\frac{w(x,0)}{v}\\
=\ & v\left[v^nK(x,v)-\sum_{k=0}^{n-1}\frac{1}{v^{2-n+k}}\frac{\partial^k u(x,0)}{\partial t^k}\right]-\frac{w(x,0)}{v}\\
=\ & v^{n+1}K(x,v)-\sum_{k=0}^{n-1}\frac{1}{v^{1-n+k}}\frac{\partial^k u(x,0)}{\partial t^k}-\frac{1}{v}\frac{\partial^n u(x,0)}{\partial t^n}\\
=\ & v^{n+1}K(x,v)-\sum_{k=0}^{n}\frac{1}{v^{1-n+k}}\frac{\partial^k u(x,0)}{\partial t^k}.
\end{aligned}$$

Thus by the principle of mathematical induction, the formula (2.8) holds for all $n\geqslant 1$. □

3. Homotopy analysis Aboodh transform method (HAATM)

We consider the following general nonlinear partial differential equation as

$$\frac{\partial^n X(\varkappa,\tau)}{\partial\tau^n}+RX(\varkappa,\tau)+NX(\varkappa,\tau)=f(\varkappa,\tau), \quad (3.1)$$

where $n=1,2,3,\ldots$, with the initial conditions

$$\left.\frac{\partial^{n-1}X(\varkappa,\tau)}{\partial\tau^{n-1}}\right|_{\tau=0}=g_{n-1}(\varkappa),\qquad n=1,2,3,\ldots \quad (3.2)$$

and $\frac{\partial^n X(x,t)}{\partial \tau^n}$ is the partial derivative of the function $X(\varkappa,\tau)$ of order n $(n = 1,2,3...)$, R is the linear partail differential operator, N represents the general nonlinear partail differential operator, and $f(\varkappa,\tau)$ is the source term.

Applying Aboodh transform on both sides of (3.1), we can get

$$A\left[\frac{\partial^n X(\varkappa,\tau)}{\partial \tau^n}\right] + A\left[RX(\varkappa,\tau) + NX(\varkappa,\tau)\right] = A[f(\varkappa,\tau)], \tag{3.3}$$

Using the property of Aboodh transform, we have the following form

$$\begin{aligned} A\left[X(\varkappa,\tau)\right] &- \sum_{k=0}^{n-1} \frac{1}{v^{2+k}} \frac{\partial^k X(\varkappa,0)}{\partial \tau^k} \\ &+ \frac{1}{v^n} A\left[RX(\varkappa,\tau) + NX(\varkappa,\tau) - f(\varkappa,\tau)\right] = 0 \end{aligned} \tag{3.4}$$

Define the nonlinear operator

$$\begin{aligned} N[\phi(\varkappa,\tau;p)] &= A\left[\,\phi(\varkappa,\tau;p)\right] - \sum_{k=0}^{n-1} \frac{1}{v^{2+k}} \frac{\partial^k \phi(\varkappa,0;p)}{\partial \tau^k} \\ &+ \frac{1}{v^n} A\left[R\phi(\varkappa,\tau;p) + N\phi(\varkappa,\tau;p) - f(\varkappa,\tau)\right] \end{aligned} \tag{3.5}$$

By means of homotopy analysis method ([10]-[13]), we construct the so-called the zero-order deformation equation

$$(1-q)A[\phi(\varkappa,\tau;p) - \phi(\varkappa,\tau;0) = phH(\varkappa,\tau)N[\phi(\varkappa,\tau;p)], \tag{3.6}$$

where p is an embedding parameter and $p \in [0,\ 1]$, $H(\varkappa,\tau) \neq 0$ is an auxiliary function, $h \neq 0$ is an auxiliary parameter, A is an auxiliary linear Aboodh operator. When $p = 0$ and $p = 1$, we have

$$\begin{cases} \phi(\varkappa,\tau;0) = X_0(\varkappa,\tau), \\ \phi(\varkappa,\tau;1) = X(\varkappa,\tau). \end{cases} \tag{3.7}$$

When P increases from 0 to 1, the $\phi(\varkappa,\tau,p)$ various from $X_0(\varkappa,\tau)$ to $X(\varkappa,\tau)$. Expanding $\phi(\varkappa,\tau;p)$ in Taylor series with respect to p, we have

$$\phi(x,t;p) = X_0(\varkappa,\tau) + \sum_{m=1}^{+\infty} X_m(\varkappa,\tau)p^m, \tag{3.8}$$

where

$$X_m(\varkappa,\tau) = \frac{1}{m!} \frac{\partial^m \phi(\varkappa,\tau;p)}{\partial p^m}\Big|_{p=0} \tag{3.9}$$

When $p = 1$, the formula (3.8) becomes

$$X(\varkappa,\tau) = X_0(\varkappa,\tau) + \sum_{m=1}^{+\infty} X_m(\varkappa,\tau). \tag{3.10}$$

Define the vectors

$$\overrightarrow{X} = \{X_0(\varkappa,\tau), X_1(\varkappa,\tau), X_2(\varkappa,\tau), \ldots, X_m(\varkappa,\tau)\}. \tag{3.11}$$

Differentiating (3.6) $m-$times with respect to p, then setting $p = 0$ and finally dividing them by $m!$, we obtain the so-called mth order deformation equation

$$A[X_m(\varkappa,\tau) - \chi_m X_{m-1}(\varkappa,\tau)] = hH(\varkappa,\tau)\Re_m(\overrightarrow{X}_{m-1}(\varkappa,\tau)), \tag{3.12}$$

where

$$\Re_m(\overrightarrow{X}_{m-1}(\varkappa,\tau)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N(\varkappa,\tau;p)}{\partial p^{m-1}}\Big|_{p=0}, \tag{3.13}$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases}$$

Applying the inverse Aboodh transform on both sides of (3.12), we can obtain

$$X_m(\varkappa,\tau) = \chi_m X_{m-1}(\varkappa,\tau) + hA^{-1}\left[H(\varkappa,\tau)\Re_m(\overrightarrow{X}_{m-1}(\varkappa,\tau))\right]. \tag{3.14}$$

The mth deformation equation (3.14) is a linear which can be easily solved. So, the solution of (3.1) can be written into the following form

$$X(\varkappa,\tau) = \sum_{m=0}^{N} X_m(\varkappa,\tau), \tag{3.15}$$

when $N \to \infty$, we can obtain an accurate approximation solution of (3.1).

For the proof of the convergence of the homotopy analysis method (see [11]).

4. Application of the HAATM

In this section, we apply the homotopy analysis method (HAM) coupled with Aboodh transform method for solving somme exemples of nonlinear partial differential equations.

Example 4.1. *First, we consider the following nonlinear gas dynamics equation*

$$X_\tau + XX_\varkappa - X(1-X) = 0,\ \tau > 0, \tag{4.1}$$

with the initial condition

$$X(\varkappa, 0) = e^{-x}. \tag{4.2}$$

In view of the HAM technique and assuming $H(\varkappa,\tau) = 1$, we construct the the so-called the zero-order deformation equation as follows

$$(1-p)A[\phi(\varkappa,\tau;p) - X_0(\varkappa,\tau)] = phN[\phi(\varkappa,\tau;p)], \tag{4.3}$$

where

$$N[\phi(\varkappa,\tau,p)] = A[\phi(\varkappa,\tau;p)] - \frac{1}{v^2}e^{-x} + \frac{1}{v}A[XX_\varkappa - X(1-X)]. \tag{4.4}$$

The series solution of Eq.(4.1) is given by (3.10). Thus, we obtain the m-th order deformation equation

$$X_m(\varkappa,\tau) = \chi_m X_{m-1}(\varkappa,\tau) + hA^{-1}[\Re_m(\overrightarrow{X}_{m-1}(\varkappa,\tau))]. \tag{4.5}$$

with

$$\begin{aligned} \Re_m(\overrightarrow{X}_{m-1}(\varkappa,\tau)) &= A[X_{m-1}(\varkappa,\tau)] - \frac{1}{v^2}(1-\chi_m)e^{-x} \\ &+ \frac{1}{v}A\left[\sum_{i=0}^{m-1} X_i (X_{m-1-i})_\varkappa\right] \\ &+ \frac{1}{v}\left[A\sum_{i=0}^{m-1} X_i X_{m-1-i} - X_{m-1}\right], \end{aligned} \tag{4.6}$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases} \tag{4.7}$$

According to (4.5) and (4.6), the formulas of the first terms is given by

$$\begin{aligned} X_1(x,\tau) &= hA^{-1}\left(\frac{1}{v}A\left[X_0(X_0)_\varkappa + (X_0)^2 - X_0\right]\right), \\ X_2(x,\tau) &= (1+h)X_1(\varkappa,\tau) \\ &+ hA^{-1}\left(\frac{1}{v}A\left[X_0(X_1)_\varkappa + X_1(X_0)_\varkappa + 2X_0X_1 - X_1\right]\right), \\ X_3(x,\tau) &= (1+h)X_2(\varkappa,\tau) \\ &+ hA^{-1}\left(\frac{1}{v}A\left[X_0(X_2)_\varkappa + X_1(X_1)_\varkappa\right]\right) \\ &+ hA^{-1}\left(\frac{1}{v}A\left[X_2(X_0)_\varkappa + 2X_0X_2 + (X_1)^2 - X_2\right]\right), \\ &\vdots \end{aligned} \tag{4.8}$$

Using the initial condition (4.2) and the iteration formulas (4.8), we obtain

$$\begin{aligned} X_0(\varkappa,\tau) &= e^{-\varkappa}, \\ X_1(\varkappa,\tau) &= (-h)e^{-x}\tau, \\ X_2(\varkappa,\tau) &= (-h)(1+h)e^{-x}\tau + h^2e^{-x}\frac{\tau^2}{2!}, \\ X_3(\varkappa,\tau) &= (-h)(1+h)^2e^{-x}\tau + 2(1+h)h^2e^{-x}\frac{\tau^2}{2!} + (-h^3)e^{-x}\frac{\tau^3}{3!}, \\ &\vdots \end{aligned} \tag{4.9}$$

The other components of the (HAATM) can be determined in a similar way. Finally, the approximate solution of Eq.(4.1) in a series form

$$\begin{aligned} X(\varkappa,\tau) &= X_0(\varkappa,\tau) + X_1(\varkappa,\tau) + X_2(\varkappa,\tau) + X_3(\varkappa,\tau) + \cdots \\ &= e^{-\varkappa} + \left[2(-h) - h^2 + (-h)(1+h)^2\right]e^{-x}\tau \\ &+ \left[h^2 + 2h^2(1+h)\right]e^{-x}\frac{\tau^2}{2!} + (-h^3)e^{-x}\frac{\tau^3}{3!} + \cdots \end{aligned} \tag{4.10}$$

Substiting $h = -1$ in (4.10), the approximate solution of (4.1), given as follows

$$X(\varkappa,\tau) = e^{-\varkappa}\left[1 + \tau + \frac{\tau^2}{2!} + \frac{\tau^3}{3!} + \cdots\right]. \tag{4.11}$$

And in the closed form, is given by

$$X(\varkappa,\tau) = e^{-\varkappa}e^{\tau} = e^{\tau-\varkappa}. \tag{4.12}$$

This result represents the exact solution of the equation (4.1) as presented in [15].

Example 4.2. *Second, we consider the inhomogeneous nonlinear Klein Gordon equation*

$$X_{\tau\tau} - X_{\varkappa\varkappa} + X^2 = \varkappa^2\tau^2, \tag{4.13}$$

with the initial conditions

$$X(\varkappa,0) = 0,\ X_\tau(\varkappa,0) = \varkappa. \tag{4.14}$$

In view of the HAM technique and assuming $H(\varkappa,\tau) = 1$, we construct the the so-called the zero-order deformation equation as follows

$$(1-p)A[\phi(\varkappa,\tau;p) - X_0(\varkappa,\tau)] = phN[\phi(\varkappa,\tau;p)], \tag{4.15}$$

where

$$N[\phi(\varkappa,\tau,p)] = A\left[\phi(\varkappa,\tau;p)\right] - \frac{1}{v^3}\varkappa + \frac{1}{v^2}A\left[-X_{\varkappa\varkappa} + X^2 - \varkappa^2\tau^2\right]. \tag{4.16}$$

The series solution of (4.13) is given by (3.10). Thus, we obtain the m-th order deformation equation

$$X_m(\varkappa,\tau) = \chi_m X_{m-1}(\varkappa,\tau) + hA^{-1}[\Re_m(\overrightarrow{X}_{m-1}(\varkappa,\tau))], \tag{4.17}$$

with

$$\begin{aligned} \Re_m(\overrightarrow{X}_{m-1}(\varkappa,\tau)) &= A\left[X_{m-1}(\varkappa,\tau)\right] - \frac{1}{v^3}(1-\chi_m)\varkappa \\ &+ \frac{1}{v^2}A\left[-(X_{m-1})_{\varkappa\varkappa}\right] \\ &+ \frac{1}{v^2}A\left[\sum_{i=0}^{m-1} X_i X_{m-1-i} - \varkappa^2\tau^2\right], \end{aligned} \tag{4.18}$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases} \tag{4.19}$$

According to (4.17) and (4.18), the formulas of the first terms is given by

$$\begin{aligned} X_1(x,\tau) &= hA^{-1}\left(\frac{1}{v^2}A\left[-(X_0)_{\varkappa\varkappa} + X_0^2 - \varkappa^2\tau^2\right]\right), \\ X_2(x,\tau) &= (1+h)X_1(\varkappa,\tau) \\ &+ hA^{-1}\left(\frac{1}{v^2}A\left[-(X_1)_{\varkappa\varkappa} + 2X_0X_1\right]\right), \\ X_3(x,\tau) &= (1+h)X_2(\varkappa,\tau) \\ &+ hA^{-1}\left(\frac{1}{v^2}A\left[-(X_2)_{\varkappa\varkappa} + 2X_0X_2 + X_1^2\right]\right), \\ &\vdots \end{aligned} \tag{4.20}$$

Using the initial condition (4.14) and the iteration formulas (4.20), we obtain

$$\begin{aligned} X_0(\varkappa,\tau) &= \varkappa\tau, \\ X_1(\varkappa,\tau) &= 0, \\ X_2(\varkappa,\tau) &= 0, \\ X_3(\varkappa,\tau) &= 0, \\ &\vdots \end{aligned} \tag{4.21}$$

Finally, the approximate solution of Eq.(4.13) is given by

$$\begin{aligned} X(\varkappa,\tau) &= X_0(\varkappa,\tau) + X_1(\varkappa,\tau) + X_2(\varkappa,\tau) + X_3(\varkappa,\tau) \\ &= \varkappa\tau + 0 + 0 + 0 + \cdots \end{aligned} \tag{4.22}$$

According to formula (3.15), we obtain

$$X(\varkappa,\tau) = \varkappa\tau. \tag{4.23}$$

This result represents the exact solution of the Klein Gordon equation (4.13) as presented in [6].

Example 4.3. *Finaly, we consider the nonlinear partial differential equation of third order*

$$X_{\tau\tau\tau} - \frac{3}{8}\left[(X_{\varkappa\varkappa})^2\right]_\varkappa = \frac{3}{2}\tau, \tag{4.24}$$

with the initial conditions

$$X(\varkappa,0) = -\frac{1}{2}\varkappa^2,\ X_\tau(\varkappa,0) = \frac{1}{3}\varkappa^3,\ X_{\tau\tau}(\varkappa,0) = 0. \tag{4.25}$$

In view of the HAM technique and assuming $H(\varkappa,\tau) = 1$, we construct the the so-called the zero-order deformation equation as follows

$$(1-p)A[\phi(\varkappa,\tau;p) - X_0(\varkappa,\tau)] = phN[\phi(\varkappa,\tau;p)], \tag{4.26}$$

where

$$\begin{aligned} N[\phi(\varkappa,\tau,p)] &= A\left[\phi(\varkappa,\tau;p)\right] + \frac{1}{2v^2}\varkappa^2 - \frac{1}{3v^3}\varkappa^3 \\ &+ \frac{1}{v^3}A\left[-\frac{3}{8}\left[(X_{\varkappa\varkappa})^2\right]_\varkappa - \frac{3}{2}\tau\right] \end{aligned} \tag{4.27}$$

The series solution of (4.24) is given by (3.10). Thus, we obtain the m-th order deformation equation

$$X_m(\varkappa,\tau) = \chi_m X_{m-1}(\varkappa,\tau) + hA^{-1}[\Re_m(\overrightarrow{X}_{m-1}(\varkappa,\tau))], \tag{4.28}$$

with

$$\Re_m(\overrightarrow{X}_{m-1}(\varkappa,\tau)) = A[X_{m-1}(\varkappa,\tau)] - (-\frac{1}{2\nu^2}\varkappa^2+\frac{1}{3\nu^3}\varkappa^3)(1-\chi_m) + \frac{1}{\nu^3}A\left[-\frac{3}{8}\sum_{i=0}^{m-1}X_{i\varkappa\varkappa}X_{(m-1-i)\varkappa\varkappa}\right] - \frac{1}{\nu^3}A\left[\frac{3}{2}\tau\right], \tag{4.29}$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases} \tag{4.30}$$

According to (4.28) and (4.29), the formulas of the first terms is given by

$$\begin{aligned} X_1(x,\tau) &= hA^{-1}\left(\frac{1}{\nu^3}A\left[-\frac{3}{8}\left[(X_{0\varkappa\varkappa})^2\right]_\varkappa - \frac{3}{2}\tau\right]\right), \\ X_2(x,\tau) &= (1+h)X_1(\varkappa,\tau) + hA^{-1}\left(\frac{1}{\nu^3}A\left[-\frac{3}{4}(X_{0\varkappa\varkappa}X_{1\varkappa\varkappa})_\varkappa\right]\right), \\ X_3(x,\tau) &= (1+h)X_2(\varkappa,\tau) + hA^{-1}\left(\frac{1}{\nu^3}A\left[-\frac{3}{8}\left(2X_{0\varkappa\varkappa}X_{2\varkappa\varkappa}+(X_{1\varkappa\varkappa})^2\right)\right]\right), \\ &\vdots \end{aligned} \tag{4.31}$$

and so on.

Consequently, while taking the initial conditions (4.25), and according to the Eqs.(4.31), the first few components of the homotopy analysis Aboodh transform method of Eq.(4.24), are derived as follows

$$\begin{aligned} X_0(x,t) &= -\frac{1}{2}x^2+\frac{1}{3}x^3\tau, \\ X_1(x,\tau) &= (-h)\varkappa\frac{\tau^5}{20}, \\ X_2(x,\tau) &= (1+h)X_1, \\ X_3(x,\tau) &= (1+h)X_2, \\ &\vdots \end{aligned} \tag{4.32}$$

The other components of the (HAATM) can be determined in a similar way. Finally, the approximate solution of Eq.(4.24) in a series form

$$\begin{aligned} X(\varkappa,\tau) &= X_0(\varkappa,\tau)+X_1(\varkappa,\tau)+X_2(\varkappa,\tau) + X_3(\varkappa,\tau)+\cdots \\ &= -\frac{1}{2}x^2+\frac{1}{3}x^3\tau+(-h)\varkappa\frac{\tau^5}{20} + (1+h)X_1+(1+h)X_2+\cdots \end{aligned} \tag{4.33}$$

Substiting $h=-1$ in (4.33), we obtain

$$\begin{aligned} X(\varkappa,\tau) &= -\frac{1}{2}x^2+\frac{1}{3}x^3\tau+\varkappa\frac{\tau^5}{20}+0+0+\cdots \\ &= -\frac{1}{2}x^2+\frac{1}{3}x^3\tau+\varkappa\frac{\tau^5}{20}. \end{aligned} \tag{4.34}$$

This result represents the exact solution of the equation (4.24) as presented in [21] in the case $\alpha = 3$.

5. Conclusion

The coupling of homotopy analysis method (HAM) and the Aboodh transform method, proved very effective to solve nonlinear partial differential equations. The proposed algorithm provides the solution in a series form that converges rapidly to the exact solution if it exists. From the obtained results, it is clear that the HAATM yields very accurate solutions using only a few iterates. The results show that the homotopy analysis Aboodh transform method (HAATM) is an appropriate method for solving nonlinear partial differential equations.

References

[1] V.G. Gupta, S. Gupta, *Applications of Homotopy Analysis Transform Method for Solving Various Nonlinear Equations,* W. Appl. Sci. J., 18(12), 1839-1846, 2012.

[2] V.G. Gupta, P. Kumar, *Approximate Solutions of Fractional Linear and Nonlinear Differential Equations Using Laplace Homotopy Analysis Method,* Int. J. of nonlinear Sci., 19(2), 113-12, 2015.

[3] V.G. Gupta, P. Kumar, *Approximate Solutions of Fractional Biological Population Model by Homotopy Analysis Sumudu Transform Method,* Int. J. of Sci. and Res., 5(5), 908-917,2016.

[4] A.K. Hassan Sedeeg, M.M. Abdelrahim Mehgoub, *Aboodh Transform of Homotopy Perturbation Method for Solving System of Nonlinear Partial Differential Equations,* Math. Theo. and Mod., 6(8), 108-113, 2016.

[5] E.M.A. Hilal, T.M. Elzaki, *Solution of Nonlinear Partial Differential Equations by New Laplace Variational Iteration Method,* J. of Fun. Spa., Vol. 2014, A. ID 790714, 5 pp, 2014.

[6] M. Hussain, M. Khan, *A Variational Iterative Method for Solving the Linear and Nonlinear Klein-Gordon Equations,* Appl. Math. Sci., 4(39), 1931-1940, 2010.

[7] S. Khalid, K.S. Aboodh, *The New Integral Transform"Aboodh Transform"*, Glo. J. of pure and appl. Math., 9(1), 35-43, 2013.

[8] M.M. Khader, S. Kumar, S. Abbasbandy, *New homotopy analysis transform method for solving the discontinued problems arising in nanotechnology,* Chin. Phys. B., 22(11), 110201 (1-5), 2013.

[9] A. Khan, M. Junaid, I. Khan, F. Ali, K. Shah, D. Khan, *Application of Homotopy Natural Transform Method to*

the Solution of Nonlinear partial differential equations, Sci. Int. (Lahore), 29(1), 297-303, 2017.

[10] S.J. Liao, *The proposed homotopy analysis technique for the solution of nonlinear problems*, Ph.D. Thesis, Shanghai Jiao Tong University, 1992.

[11] S.J. Liao, *Beyond Perturbation: Introduction to Homotopy Analysis Method,* Chapman and Hall/CRC Press, Boca Raton, 2003.

[12] S.J. Liao, *On the homotopy analysis method for nonlinear problems*, Appl. Math. Comput., 147, 499-513, 2004.

[13] S.J. Liao, *Notes on the homotopy analysis method: Some definitions and theorems,* Com. Nonlinear Sci. Num. Sim., 14, 983–997, 2009.

[14] M.S. Mohamed, F. Al-Malki, M. Al-humyani, *Homotopy Analysis Transform Method for Time-Space Fractional Gas Dynamics Equation,* Gen. Math. Notes, 24(1), 1-16, 2014.

[15] M. Matinfar, M. Saeidy, M. Mahdavi, M. Rezaei, *Variational Iteration Method for Exact Solution of Gas Dynamic Equation Using He's Polynomials,* Bulletin of Math. Anal. and Appl., 3(3), 50-55, 2011.

[16] R.K. Pandey, H.K. Mishra, *Numerical Simulation of Time-fractional Fourth Order Differential Equations via Homotopy Analysis Fractional Sumudu Transform Method,* Ame. J. of Num. Anal., 3(3), 52-64, 2015.

[17] S. Rathorea, D. Kumarb , J. Singh, S. Gupta, *Homotopy Analysis Sumudu Transform Method for Nonlinear Equations,* Int. J. Ind. Math., 4(4), A., ID IJIM-00204, 13 p, 2012.

[18] K.M. Saad, A.A. AL-Shomrani, *An Application of Homotopy Analysis Transform Method for Riccati Differential Equation of Fractional Order,* J. of Fra. Cal. and Appl., 7(1), 61-72, 2016.

[19] K. Wang, S. Liu, *Application of new iterative transform method and modified fractional homotopy analysis transform method for fractional Fornberg-Whitham equation,* J. Nonlinear Sci. Appl., 9, 2419-2433, 2016.

[20] K. Wang, S. Liu, *Application of new iterative transform method and modified fractional homotopy analysis transform method for fractional Fornberg-Whitham equation,* J. Nonlinear Sci. Appl., 9, 2419-2433, 2016.

[21] D. Ziane, M. Hamdi Cherif, *Modified Homotopy Analysis Method for Nonlinear fractional partial differential equations,* Int. J. of Anal. and Appl., 14(1), 77-87, 2017.

[22] D. Ziane, *Application of Homotopy Analysis Method Combined with Elzaki Transform for Fractional Porous Medium Equation,* J. of Appr. Theo. and Appl. Math., 6, 1-19, 2016.

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