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A variational form with Legendre series for linear integral equations

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Abstract

In this work, we seek the approximate solution of integral equations by truncation Legendre series approximation using a variational form for the equation. this one is reduced to a linear system where the solution of this latter gives the Legendre coefficients and thereafter the solution of the equation.The convergence and the error analysis of this method are discussed. Finally, we compare our numerical results by others.

Keywords

linear integral equations, Legendre polynomials, collocation methods, Sloan approximation.

AMS Subject Classification

45A05, 45B05, 45D05.

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Contents

1. Introduction

The domain of integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has various applications in many areas of science such as mathematical biology, geophysics, and physical problems. Many initial and boundary value problems associated with ordinary differential equation and partial differential equation can be transformed into problems of solving some approximate integral equations [1, 3, 4].

In particular the Fredholm integral equation of the second kind is one of the most practical ones, so obtaining the solution with high accuracy by a convenient and useful method with high efficiency such as quadrature collocation, Galerkin and interpolation is very important [7, 8]. In this paper the author propose a method of polynomial series accompanied with the variational form for the solution of such integral equations, certainly this technical leads us to one of the best approximations.

Noting that, the typical form of the Fredholm integral equations is given by

$$\varphi(x) - \int_{\Omega} k(x,t)\varphi(t)dt = f(x), \qquad x \in \Omega \tag{1}$$

Here, both, the function $k(x,t)$ called the kernel, and the function $f(x)$ are given and continuous functions on $\Omega \times \Omega$ and Ω respectively, the function $\varphi(x)$ is to be determined as continuous function on Ω. Depending on the domain $\Omega = [a,x]$ or $[a,b]$ the equation (1) describes the Volterra integral equation or Fredholm integral equation, respectively, and can be put in the form of a linear functional equation

$$\varphi(x) - A\varphi(x) = f(x),$$

with the linear mappings A given by

$$A\varphi(x) = \int_{\Omega} k(x,t)\varphi(t)dt.$$

For the solution of the equation (1) in the complete function spaces, usually take it $L^2(\Omega)$, we choose a sequence of finite dimensional subspaces V_n, $n \geq 1$, having n basis functions $\{L_1, L_2, \ldots, L_n\}$ with dimension of $V_n = n$.

Seeking the approximate function $\varphi_n \in V_n$ of the function φ given by

$$\varphi_n(x) = \sum_{j=1}^{n} \alpha_j L_j(x), \tag{2}$$

where the expression (2) describes the truncated Legendre series of the solution of the equation (1), with the functions $\{L_k\}_{0\leq j\leq n}$ represent the Legendre polynomials and $\{\alpha_k\}_{0\leq j\leq n}$the coefficients to be determined. In other words, we can write

$$\varphi_n(x) - \int_{\Omega} k(x,t)\varphi_n(t)dt = f(x). \tag{3}$$

For the solution of this equation, we construct a variational form of the equation (3) using the method of Galerkin. In other words, we seek to determine a function $\psi \in L^2(\Omega)$ such that

$$\langle \varphi_n(x), \psi(x)\rangle - \left\langle \int_{\Omega} k(x,t)\varphi_n(t)dt, \psi(x)\right\rangle = \langle f(x), \psi(x)\rangle,$$

where

$$\langle \varphi(x), \psi(x)\rangle = \int_{\Omega} \varphi(x)\psi(x)dx$$

2. Solution with collocation methods on one-dimensional space [0,1]

Choose a selection of distinct points $x_0,\ x_2, ...x_{n+1} \in [0,1]$ such that

$$0 = x_0 < x_1 < x_2 < .... < x_{n+1} = 1.$$

The basis is created from the hat function

$$\psi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & \text{if } x_{i-1} \leq x \leq x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i}, & \text{if } x_i \leq x \leq x_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

$$\sum_{j=1}^{n} \alpha_j \langle L_j(x), \psi_i(x)\rangle - \sum_{j=1}^{n} \alpha_j \left\langle \int_0^1 k(x,t)L_j(t)dt, \psi_i(x)\right\rangle$$
$$= \langle f(s), \psi_i(x)\rangle,$$

or still for $i, j = 1,2,...,n$ we obtain

$$\sum_{j=1}^{n} \alpha_j \left(\langle L_j(x), \psi_i(x)\rangle - \left\langle \int_0^1 k(x,t)L_j(t)dt, \psi_i(x)\right\rangle \right)$$
$$= \langle f(x), \psi_i(x)\rangle, \tag{4}$$

The equation (4) leads us to determine the coefficients $\{\alpha_1,\ \alpha_2,\ ...,\alpha_n\}$ solution of the linear system

$$\sum_{j=1}^{n} \alpha_j \int_0^1 L_j(x)\psi_i(x)dx - \int_0^1 \left(\int_0^1 k(x,t)L_j(t)dt \right) \psi_i(x)dx$$
$$= \int_0^1 f(x)\psi_i(x)dx. \tag{5}$$

Define the matrices

$$L = (L_{ij}) = \int_0^1 \psi_i(x)L_j(x)dx$$

and

$$K = (K_{ij}) = \int_0^1 \psi_i(x) \left(\int_0^1 k(x,t)L_j(t)dt \right) dx.$$

If the $\det(L-K) \neq 0$, we can ensure that, there exists a solution of the linear system (5) and consequently the approximate solution $\varphi_n(s)$ as a linear combination

$$\varphi_n(x) = \sum_{j=1}^{n} \alpha_j L_j(x),$$

for which

$$\varphi_n(x_j) - \int_0^1 k(x_j,t)\varphi_n(t)dt = f(x_j), \quad j = 1,2,...,n.$$

In fact, The linear system may be written in matrix

$$(L-K)\alpha = F, \tag{6}$$

where $\alpha = (\alpha_1,\ \alpha_2,\ ...,\alpha_n)^T$ and $F = \left(\int_0^1 f(x)\psi_1(x)dx, \int_0^1 f(x)\psi_2(x)dx\ ,..., \int_0^1 f(x)\psi_n(x)dx \right)^T$. For the determinant of the system (6) is different from zero $\det(L-K) \neq 0$, then it has a unique solution

$$\alpha = (\alpha_1,\ \alpha_2,\ ...,\alpha_n)^T = (L-K)^{-1}F.$$

The corresponding approximate solution

$$\varphi_n(x) = \sum_{j=1}^{n} \alpha_k L_k(x),$$

3. Legendre polynomials

The nth Legendre polynomials $L_n(x)$ is defined by $L_0(x) = 1$ and the following recursion

$$(n+1)L_{n+1}(x) - (2n+1)xL_n(x) + nL_{n-1}(x) = 0.$$

Noting that, the Legendre polynomial $L_n(x)$ is polynomials with rational coefficients

$$\begin{aligned} L_0(x) &= 1 \\ L_1(x) &= x \\ L_2(x) &= \frac{3}{2}x^2 - \frac{1}{2} \\ L_3(x) &= \frac{5}{2}x^3 - \frac{3}{2}x \\ L_4(x) &= \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8} \\ L_5(x) &= \frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8} \end{aligned}$$

and also it is a solution of the differential equations

$$\left((1-x^2)y'\right)' + n(n+1)y = 0 \Leftrightarrow (1-x^2)y'' - 2xy' + n(n+1)y = 0.$$

4. Sloan iterate convergence procedure

we define the operator projection P_nA as

$$P_nA\varphi_n(x) = A_n\varphi_n = \int_\Omega k_n(t,x)\varphi_n(t)dt,$$

so, with the solution φ_n of the equation $\varphi_n - A_n\varphi_n = f_n$, we construct the Sloan approximation as

$$\widetilde{\varphi}_n = f - A\varphi_n \tag{7}$$

where it is easy to see that $\widetilde{\varphi}_n$ is the projection of the approximate solution φ_n into V_n. Noting that if the equation (7) verified the Banach theorem with the application the expression (7) we can give the error bound

$$\|\varphi - \widetilde{\varphi}_n\| \le \|A\| \, \|\varphi - \varphi_n\|.$$

This shows that, the convergence of $\widetilde{\varphi}_n$ to the exact solution φ is faster than φ_n to φ.

5. Illustrating Examples

Example 1. Consider the linear integral equation of Volterra

$$\varphi(x) - \int_0^1 tx\varphi(t)dt = \frac{2}{3}x, \quad 0 \le x,t \le 1,$$

where the function $f(x)$ is chosen so that the exact solution is given by

$$\varphi(x) = x.$$

The approximate solution $\varphi_n(x)$ of $\varphi(x)$ is obtained by the truncated Legendre series method.

Table 1. We present the exact and the approximate solutions of the equation in the example 1 in some arbitrary points, the error for $N = 10$ is calculated and compared with the ones treated in [2].

Values of x	Exact solution φ	Approx solution φ_n	Error	Error [2]
0.000000	0.000000	0.000000	3.48e-17	3.95e-05
0.200000	0.200000	0.200000	2.77e-17	3.95e-05
0.400000	0.400000	0.400000	1.11e-16	3.95e-05
0.600000	0.600000	0.600000	2.22e-16	3.95e-05
0.800000	0.800000	0.800000	1.11e-16	3.95e-05
1.000000	1.000000	1.000000	0.00e+00	3.95e-05

Example 2. Consider the linear integral equation of Fredholm

$$\varphi(x) + \int_0^\pi (\cos t - \cos x)\varphi(t)dt = \sin x, \quad 0 \le x,t \le \pi,$$

where the function $f(x)$ is chosen so that the exact solution is given by

$$\varphi(x) = \sin x + \frac{4}{2-\pi^2}\cos x + \frac{2\pi}{2-\pi^2}.$$

The approximate solution $\varphi_n(x)$ of $\varphi(x)$ is obtained by the truncated Legendre series method.

Table 2. We present the exact and the approximate solutions of the equation in the example 3 in some arbitrary points, the error for $N = 8$ is calculated and compared with the ones treated in [3].

Values of x	Exact solution φ	Approx solution φ_n	Error	Error [3]
0.000000	-1.306697e+000	-1.306872e+000	1.75e-04	5.2e-01
0.785398	-4.507167e-001	-4.508725e-001	1.55e-04	2.1e-02
1.570796	2.015882e-001	2.014807e-001	1.07e-04	3.6e-01
2.356194	2.681065e-001	2.680475e-001	5.90e-05	1.1e-01
3.141593	-2.901271e-001	-2.901661e-001	3.90e-05	7.5e-01

Example 3. Consider the linear integral equation of Fredholm

$$\varphi(x) - \int_0^1 (\sqrt{t} + \sqrt{x})\varphi(t)dt = 1 + x, \quad 0 \le x,t \le 1,$$

where the function $f(x)$ is chosen so that the exact solution is given by

$$\varphi(x) = -\frac{129}{70} - \frac{141}{35}\sqrt{x} + x.$$

The approximate solution $\varphi_n(x)$ of $\varphi(x)$ is obtained by the truncated Legendre series method.

Table 3. We present the exact and the approximate solutions of the equation in the example 4 in some arbitrary points, the error for $N = 10$ is calculated and compared with the ones treated in [3].

Values of x	Exact solution φ	Approx solution φ_n	Error	Error [3]
0.000000	-1.842857e+000	-1.851789e+000	8.93e-03	6.9e-01
0.250000	-3.607143e+000	-3.620560e+000	1.34e-02	8.2e-02
0.500000	-4.191487e+000	-4.207458e+000	1.59e-02	3.7e-02
0.750000	-4.581702e+000	-4.599243e+000	1.75e-02	6.4e-02
1.000000	-4.871429e+000	-4.890259e+000	1.88e-02	9.5e-02

Example 4. Consider the linear integral equation of Fredholm

$$\varphi(x)+\frac{1}{3}\int_0^1 \exp(2x-\frac{5}{3}t)\varphi(t)dt=\exp(2x+\frac{1}{3}), \quad 0\le x,t\le 1,$$

where the function $f(x)$ is chosen so that the exact solution is given by

$$\varphi(x)=\exp(2x).$$

The approximate solution $\varphi_n(x)$ of $\varphi(x)$ is obtained by the truncated Legendre series method.

Table 4. We present the exact and the approximate solutions of the equation in the example 4 in some arbitrary points, the error for $N=10$ is calculated and compared with the ones treated in [9].

Values of x	Exact solution φ	Approx solution φ_n	Error	Error [9]
0.000000	1.000000e+000	1.000000e+000	1.94e-09	2.6e-05
0.200000	1.491825e+000	1.491825e+000	2.90e-09	3.9e-05
0.400000	2.225541e+000	2.225541e+000	4.32e-09	5.8e-05
0.6.00000	3.320117e+000	3.320117e+000	6.45e-09	1.0e-04
0.800000	4.953032e+000	4.953032e+000	9.62e-09	3.5e-04
1.000000	7.389056e+000	7.389056e+000	1.43e-08	1.9e-03

Example 5. Consider the linear integral equation of Fredholm

$$\varphi(x)-\int_{-1}^1 \exp(-t)\varphi(t)dt=\exp(x)-2, \quad 0\le x,t\le 1,$$

where the function $f(x)$ is chosen so that the exact solution is given by

$$\varphi(x)=\exp(x).$$

The approximate solution $\varphi_n(x)$ of $\varphi(x)$ is obtained by the truncated Legendre series method.

Table 5. We present the exact and the approximate solutions of the equation in the example 1 in some arbitrary points, the error for $N=10$ is calculated.

Values of x	Exact solution φ	Approx solution φ_n	Error
-1.000000	3.678794e-001	3.678794e-001	3.885781e-016
-0.600000	5.488116e-001	5.488116e-001	1.110223e-016
-0.200000	8.187308e-001	8.187308e-001	1.110223e-016
0.200000	1.221403e+000	1.221403e+000	2.220446e-016
0.600000	1.822119e+000	1.822119e+000	2.220446e-016
1.000000	2.718282e+000	2.718282e+000	0.000000e+000

6. Conclusion

A numerical method for solving linear integral equations, based on the truncated Legendre series of the solution and the variational form is presented. We note that the approximate solution $\varphi_n(x)$ will be measurably close to the solution $\varphi(x)$ on the entire interval $[0,1]$. The efficiency of this method is tested by solving some examples for which the exact solution is known. This allows us to estimate the exactness with our numerical results and compare those with another results treated by another authors [2, 3, 5, 9].

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References

[1] S. Abbasbandy, E. Shivanian, A new analytical technique to solve Fredholm's integral equations, in Numer Algor 56 (2011) 27–43.

[2] N. Beny, P. Nelson, Adaptive Method for the Numerical Solution of Fredholm Integral Equations of the Second Kind, Part I Regular Kernels, in Numerical Solution of Singular Integral Equations (A Gerasoulis and R Vichnevetsky Eds) 1984 pp 68-72.

[3] A. Chakrabarti, S.C. Martha, Approximate solutions of Fredholm integral equations of the second kind, in Applied Mathematics and Computation 211 (2009) 459–466.

[4] R. Kress, Linear Integral Equations, Springer-Verlag, Berlin, Heidelberg, 1989.

[5] K. Maleknejad, N. Aghazadeh, Numerical solution of Volterra integral equations of the second kind with convolution kernel by using Taylor-series expansion method, Appl. Math. Comput. 161 (2005) 915–922.

[6] Mirzaee, Numerical Solution for Volterra Integral Equations of the First Kind via Quadrature Rule, Applied Mathematical Sciences, Vol. 6, 2012, no. 20, 969 - 974.

[7] M. Nadir, Solving Fredholm integral equations with application of the four Chebyshev polynomials, in Journal of Approximation Theory and Applied Mathematics 4 pp 37-44 (2014).

[8] M. Nadir, A. Rahmoune, Modified Method for Solving Linear Volterra Integral equations of the Second Kind Using Simpson's Rule, in International Journal Mathematical Manuscripts (IJMM) 1, (2) 2007 133-140.

[9] S. Yalçınbaş, M Aynigül, Hermite series solutions of linear Fredholm integral equations, in Mathematical and Computational Applications, Vol. 16, No. 2, pp. 497-506, 2011.

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