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On Steiner domination in graphs

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Abstract

The steiner dominating set is a variant of dominating set in graphs. For a non – empty set W of vertices in a connected graph G, the steiner distance $d(W)$ of W is the minimum size of a connected subgraph G containing W. Necessarily, each such subgraph is a tree and is called a steiner tree or a steiner $W-$ tree. The set of all vertices of G that lie on some steiner $W-$ tree is denoted by $S(W)$. If $S(W) = V(G)$ then W is called a steiner set for G. The steiner number $s(G)$ is the minimum cardinality of a steiner set. The minimum cardinality of a steiner dominating set is called the steiner domination number of graph. We present here some new results on steiner domination in graphs.

Keywords

Dominating set, domination number, steiner dominating set, steiner domination number.

AMS Subject Classification

05C69; 05C76.

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Contents

1. Introduction

We begin with simple, finite, connected and undirected graph G with vertex set $V(G)$ and edge set $E(G)$. For standard terminology and notation in graph theory we rely upon West [15] while the concepts related to theory of domination we refer Haynes *et al.* [3].

Definition 1.1. The distance $d(u,v)$ between two vertices u and v in a connected graph G is the length of shortest $u-v$ path in G.

Definition 1.2. The vertex connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal leaves a disconnected graph or K_1.

Definition 1.3. For a non - empty set W of vertices in connected graph G, the steiner distance $d(W)$ of W is the minimum size of a connected subgraph G containing W. Necessarily, each such subgraph is a tree and is called a steiner tree or a steiner W - tree. The set of all vertices of G that lie on some steiner W - tree is denoted by $S(W)$. If $S(W) = V(G)$ then W is called steiner set for G. The steiner number $s(G)$ is the minimum cardinality of a steiner set.

The roots of steiner tree porblem lies in steiner problem in Euclidean space while the steiner interval was developed in [7]. The concept of steiner number was initiated by Chartrand and Zhang in [2]. Pelayo [10] corrected main result in the same paper. He proved that not all steiner sets are geodetic sets and there are connected graphs whose steiner number is strictly lower than their geodetic number. Hernando *et al.* [4] have explored the relationships between steiner sets and geodetic sets and between steiner sets and monophonic sets. Many results on steiner distance were given by Chartrand *et al.* [1] as well as Santhakumaran and John [11].

Definition 1.4. A set $S \subseteq V(G)$ of vertices in graph $G = (V(G), E(G))$ is called dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. The domination number $\gamma(G)$ is the minimum cardinality of minimal dominating set of G.

The concept of dominating set is explored in various contexts in [6, 8, 9, 12]

Definition 1.5. A systematic visit of each vertex of a tree is called a tree traversal.

Definition 1.6. A vertex v is an extreme vertex of a graph G if the subgraph induced by its neighbors is a complete graph.

Definition 1.7. For a connected graph $G = (V(G), E(G))$, a set of vertices W in G is called a steiner dominating set if W is both steiner and dominating set while the steiner domination number $\gamma_s(G)$ is the minimum cardinality of a steiner dominating set of G.

The concept of steiner domination was introduced by John *et al.* [5] in which they have investigated steiner domination number for K_n, tree and $K_{m,n}$. Vaidya and Mehta [14] as well as Vaidya and Karkar [13] have obtained various results on steiner domination numbers.

In the immediate section we will state some existing results for our ready reference.

2. Existing Results

Proposition 2.1. *[5] Each extreme vertex of a connected graph G belongs to every steiner dominating set of G.*

Proposition 2.2. *[5] If G is a connected graph of order n, then* $2 \leq max\{s(G), \gamma(G)\} \leq \gamma_s(G) \leq n$

Proposition 2.3. *[5] If G is a non complete connected graph with n vertices such that it has a minimum cut-set, then* $\gamma_s(G) \leq n - \kappa(G)$.

Proposition 2.4. *[2] Let G be a connected noncomplete graph and let S be a minimum vertex cut-set such that the components of* $G - S$ *are* $G_1, G_2, \ldots, G_r$ *where* ≥ 2.

- *There exists a steiner set W of G such that* $S \cap W = \phi$.
- *Every minimum steiner set containing no vertex of S contains at least one vertex of every graph* G_i *where* $1 \leq i \leq r$.
- *No cut-vertex of G belongs to any minimum steiner set of G.*

Proposition 2.5. *[5] For the complete bipartite graph* $G = K_{m,n}$,

- $\gamma_s(G) = 2$ *if* $m = n = 1$
- $\gamma_s(G) = n$ *if* $n \geq 2, m = 1$
- $\gamma_s(G) = min\{m, n\}$ *if* $m, n \geq 2$

Proposition 2.6. *[5] Let G be a connected graph and v be a cut-vertex of G. Then every steiner dominating set of G contains at least one element from each component of* $G - v$.

Proposition 2.7. *[5] If v is a cut-vertex of a connected graph G and W is a steiner dominating set of G, then v lies in every steiner W-tree of G.*

We quickly list out following observations.

Observation 2.8. *For cycle* C_n,

$$\gamma_s(C_n) = \begin{cases} 3 & if\ n = 3 \\ 2 & if\ n = 4 \\ \left\lceil \frac{n}{3} \right\rceil & if\ n \geq 6 \end{cases}$$

Observation 2.9. *For path* P_n, $\gamma_s(P_n) = \left\lceil \frac{n}{3} \right\rceil$.

3. Some Characterizations

Theorem 3.1. *Let G be a graph with n vertices and if* $diam(G) = rad(G) = 2$ *then there exists a graph G such that* $\gamma_s(G) = 2$.

Proof: Let G be a graph with n vertices. Let $diam(G) = rad(G) = 2$. Then any two vertices u and v which are non adjacent in a graph can be reached through a vertex x where $x \in G, x \neq u, v$. Then it is sufficient to take u and v in the steiner set. Naturally remaining vertices will be dominated by either u or v. Hence $\gamma_s(G) = 2$.

Theorem 3.2. *For* $r \neq 1$, $n - r \neq 2$ *where r denotes mutually non ajacent vertices of G of order n then there exists* $(n - r)$ *regular graph G such that* $\gamma_s(G) = r$.

Proof: Let G be a graph with n vertices namely $v_1, v_2, \ldots, v_n$. Let G be $(n - r)$ regular graph such that $r \neq 1$, $n - r \neq 2$. For $v \in V(G)$, $deg(v) = n - r$. Thus v is mutually non-adjacent with r vertices. We consider r mutually non adjacent vertices in steiner set otherwise a steiner tree of size one is obtained. Let $W = \{v_1, v_2, \ldots, v_r\}$ then $n - r$ number of steiner trees of size r are obtained. Then each steiner tree contains only one internal vertex. Thus remaining $n - r$ vertices of graph G lies on some steiner tree. Thus $S(W) = V(G)$. Hence W is the steiner set.

Claim: W is minimum steiner set.
Let $v_i \notin W$,that is $W - v_i, (i = 1, 2, \ldots, r)$. The vertices $v_1, v_2, \ldots, v_r$ are mutually non adjacent vertices implying that v_i must be the internal vertex lying on some steiner W-tree. But $v_i \notin N(v_1) \cap N(v_2) \cap \ldots \cap N(v_r)$ as $v_1, v_2 \ldots, v_r$ are mutually non adjacent vertices. So to obtain a steiner tree we have to traverse through such a vertex which is adjacent to v_i and at least one vertex from $W - v_i$. The steiner tree so obtained contains two vertices as internal vertices resulting into the size of steiner tree equal to $r + 1$ which cannot be considered since remaining all steiner trees are of size r. Then v_i is not the internal vertex. Thus $v_i \notin S(W)$. Thus our supposition is wrong. Therefore $v_i \in W$. Hence W is the minimum steiner set.

Claim: W is dominating set.
As discussed above $W = \{v_1, v_2, \ldots, v_r\}$ is the minimum steiner

set. Hence r number of steiner trees are obtained containing only one internal vertex. Naturally this internal vertex is dominated by all the r vertices in the set W. Hence W is also the dominating set. Therefore W is the steiner dominating set with minimum cardinality. Thus, For $r \neq 1$, $n-r \neq 2$ then there exists $(n-r)$ regular graph G such that $\gamma_s(G) = r$.

We state the following obvious corollary.

Corollary 3.3. *If G is r regular graph such that $n-r=2$ then $\gamma_s(G) = 2$, for $r \neq 1$.*

Proof: Let G be r regular graph with n vertices. Let $n-r=2$ where $r \neq 1$. The eccentricity of vertex $v \in V(G)$ is $e(v) = 2$. Then naturally $diam(G) = rad(G) = 2$. By Theorem 3.1 $\gamma_s(G) = 2$.

Theorem 3.4. *If $\gamma(G) = 1$ then $\gamma_s(G) = s(G)$.*

Proof: Let G be any graph of order n with vertices $v_1, v_2, \ldots, v_n$. The domination number $\gamma(G) = 1$ then $\triangle(G) = n-1$. Let there are $m(m < n)$ number of vertices with degree $n-1$. The $diam(G) \leq 2$ then any two vertices in G are at a distance one or two apart. Any two vertices can be reached via some m type of vertices. Then in any steiner tree there will be only one internal vertex since there are vertices of degree $n-1$. Without loss of generality let $v_i \in W$ where W is some steiner set. Let $deg(v_i) = k$ where $m \neq k$. Let us consider vertices in steiner set which are at a distance two from v_i. We know that $s(G) \geq 2$. Let $v_j \in W$ $v_j \notin N(v_i)$ such that it is at a distance two from v_i. So v_i to v_j can be reached via some m or k type of vertex. Thus remaining $n-k-1$ number of vertices are not traversed by a steiner tree from v_i to v_j implying that these $n-k-1$ number of vertices do not belong to the set $S(W)$. Since the $diam(G) \leq 2$ so the steiner tree is obtained which is of size two and in one steiner tree only one internal vertex can be accomodated so these $n-k-1$ number of vertices will never lie on steiner tree as they are at distance two. Then minimum $n-k$ number of vertices must be in the steiner set W. Hence $|W| = n-k = s(G)$. The steiner tree contains internal vertices which are adjacent to v_i. Then this internal vertices will be naturally dominated by at least v_i. Hence $\gamma_s(G) = s(G)$

Theorem 3.5. *Let G be a connected graph of order $n \geq 3$ then $\gamma_s(G) = n-1$ if and only if G contains cut-vertex of degree $n-1$.*

Proof: Let G be a graph with $n \geq 3$ vertices. Let $\gamma_s(G) = n-1 = |W|$. The set W is steiner set as well as dominating set. Then there exists only one steiner tree of size $n-1$. This steiner tree contains only one internal vertex v. Then $deg(v) = n-1$. Thus G contains a cut-vertex of degree $n-1$.

Conversely suppose that G contains cut-vertex v of degree $n-1$. Then $\gamma(G) = 1$. If v is the cut-vertex of degree $n-1$, W is the minimum steiner set and let S be the minimum vertex cut-set then by Proposition 2.4, $S \cap W = \phi$, therefore $\{v\} \cap W = \phi$. Then $v \notin W$ and W contains at least one vertex from each component of G. Thus every steiner tree contains vertices such that its ends are in set W. Thus $S(W) = V(G) \cup \{v\}$. Then $|W| = V(G) - \{v\} = n-1 = s(G)$. Here $\gamma(G) = 1$ then by Theorem 3.4, $\gamma_s(G) = s(G) = n-1$.

Definition 3.6. Let G and H be two graphs and let n be the order of G. The corona product $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G and n copies of H and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H.

Theorem 3.7. *For any graph G, $\gamma_s(G \circ P_2) = 2n$.*

Proof: Let G be any graph with vertices $v_1, v_2, \ldots, v_n$. The corona product $G \circ P_2$, each v_i is joined to u_i's and w_i's $(i = 1, 2, \ldots, n)$. Here u_i's and w_i's are extreme vertices of v_i's. Therefore the vertices of type u_i's and w_i's must be in every steiner dominating set. There exists only one steiner tree of length $3n-1$ since there are $2n$ extreme vertices. Each v_i lie on the steiner tree. Thus $S(W) = V(G)$. Therefore $W = \{u_i, w_i\}$ is the minimum steiner set. The vertices of type v_i's are adjacent to u_i's and w_i's. Naturally $W = \{u_i, w_i\}$ also forms the dominating set. Thus $W = \{u_i, w_i\}$ is the steiner dominating set of minimum cardinality. Hence $\gamma_s(G \circ P_2) = 2n$.

Definition 3.8. Let G be a graph with q edges. A graph H is called supersubdivision of G if H is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i} (for some m_i and $1 \leq i \leq q$) in such a way that the ends of each ei are merged with the two vertices of 2-vertices part of $K_{2,mi}$ after removing the edge ei from graph G. A supersubdivision H of G is said to be an arbitrary supersubdivision of G if every edge of G is replaced by an arbitrary $K_{2,m}$ where m may vary for each edge arbitrarily.

Theorem 3.9. *Let G be any graph G then the arbitrary supersubdivision of G, the steiner domination is $\gamma_s(G) = n$.*

Proof: Let H be any graph with n vertices $v_1, v_2, v_3, \ldots, v_n$ and q edges. Let e_i denote the edge $v_{i-1}v_i$ of the graph H for $1 \leq i \leq n$. Let G be an arbitrary supersubdivision of H. That is, for $1 \leq i \leq n$ each edge e_i of the G is replaced by a complete bipartite graph K_{2,m_i} where m_i is any positive integer.

Let us partition the the graph G into q parts namely $G_1, G_2, \ldots, G_q$. Let us consider the first patition G_1 such that it contains vertices v_{i-1}, v_i and vertices of type m_i's. By proposition 2.5, $\gamma_s(G_1) = min\{2, m_i\} = 2$. That is, v_{i-1} and v_i must be in steiner dominating set. This implies that the steiner dominating set contains n vertices of type v_i. Thus $\gamma_s(G) = n$.

Conclusion

Theory of dominating sets is one of the potential areas of research. Many domination models are available in the literature. The steiner domination in graphs is one such model which depends upon steiner distance in graphs. We have contributed some characterizations and also investigated the steiner domination number of the larger graphs obtained by means of graph operations, namely, corona product and arbitrary supersubdivision of two graphs.

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