Malaya Journal of Matematik, Vol. 6, No. 2, 390-395, 2018
https://doi.org/10.26637/MJM0602/0015

Solvability of some fractional-order three point boundary value problems

A. M. A. El-Sayed[1]* and Abd El-Salam Sh. A.[2]

Abstract
In this work, we prove the existence of at least one solution of the two fractional-order three point boundary value problems:

$$\begin{cases} D^{\beta}\, u(t) + \lambda\, a(t)\, f(u(t)) = 0,\ \beta \in (1,2],\ t \in (0,1), \\ u(0) = 0,\ \alpha\, u(\eta) = u(1),\ 0 < \eta < 1,\ 0 \le \alpha\, \eta < 1. \end{cases}$$

and

$$\begin{cases} D^{\beta}\, u(t) + \lambda\, a(t)\, f(u(t)) = 0,\ \beta \in (1,2],\ t \in (0,1), \\ u'(0) = 0,\ \alpha\, u'(\eta) = u(1),\ 0 < \eta < 1,\ 0 \le \alpha\, \eta < 1. \end{cases}$$

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Keywords
Fractional calculus; Three point boundary value problems.

AMS Subject Classification
26A33, 30E25, 34A12, 34A34, 34A37, 37C25, 45J05.

[1] *Faculty of Science, Alexandria University, Alexandria, Egypt.*
[2] *Faculty of Science, Damanhour University, Egypt.*
***Corresponding author**: [1] amasayed@alexu.edu.eg; [2]shrnahmed@sci.dmu.edu.eg
Article History: Received **06** September **2017**; Accepted **11** January **2018**

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1. Introduction

Let $L_1[a,b]$ denotes the space of all Lebesgue integrable functions on the interval $[a,b]$, $0 \le a < b < \infty$.

Definition 1.1 The fractional (arbitrary) order integral of the function $f \in L_1[a,b]$ of order $\beta \in R^+$ is defined by (see [7], [9] - [10] and [12])

$$I^{\beta}\, f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\, f(s)\, ds,$$

where $\Gamma(.)$ is the gamma function.

Definition 1.2 The Riemann-Liouville fractional-order derivative of $f(t)$ of order $\alpha \in (0,1)$ is defined as (see [7], [9] - [10] and [12])

$${}_*D^{\alpha}\, f(t) = \frac{d}{dt}\, I^{1-\alpha}\, f(t), \quad t \in [a,b].$$

Definition 1.3 The (Caputo) fractional-order derivative D^{α} of order $\alpha \in (0,1]$ of the function $g(t)$ is defined as (see [9] - [10] and [12])

$$D^{\alpha}\, g(t) = I^{1-\alpha}\, \frac{d}{dt}\, g(t), \quad t \in [a,b].$$

We consider here the fractional-order three point boundary value problems:

$$\begin{cases} D^{\beta}\, u(t) + \lambda\, a(t)\, f(u(t)) = 0,\ \beta \in (1,2],\ t \in (0,1), \\ u(0) = 0,\ \alpha\, u(\eta) = u(1),\ 0 < \eta < 1,\ 0 \le \alpha\, \eta < 1. \end{cases} \tag{1.1}$$

and

$$\begin{cases} D^{\beta}\, u(t) + \lambda\, a(t)\, f(u(t)) = 0,\ \beta \in (1,2],\ t \in (0,1), \\ u'(0) = 0,\ \alpha\, u'(\eta) = u(1),\ 0 < \eta < 1,\ 0 \le \alpha\, \eta < 1. \end{cases}$$

(1.2)

where the function $f \in C([0,1],\Re)$ and there exists a constant K_1 such that $|\frac{\partial f}{\partial u}| \leq K_1$.

The three point boundary value problem was studied by many authors, for example; in [8] the existence of at least one positive solution of the three point boundary value problem:

$$\begin{cases} u'' + a(t)\, f(u) = 0,\ t \in (0,1), \\ u(0) = 0,\ \alpha\, u(\eta) = u(1),\ 0 < \eta < 1,\ \alpha\, \eta < 1 \end{cases}$$

has been studied if f is either superlinear or sublinear by applying the fixed point theorems in cones, In [11], they concerned with determining values for λ so that the three point nonlinear second order boundary value problem:

$$\begin{cases} u''(t) + \lambda\, a(t)\, f(u(t)) = 0,\ t \in (0,1), \\ u(0) = 0,\ \alpha\, u(\eta) = u(1),\ 0 < \eta < 1,\ 0 < \alpha < \frac{1}{\eta} \end{cases}$$

has positive solutions.
And in [13] they proved the existence of nontrivial solutions for the second order three point boundary value problem:

$$\begin{cases} u'' + f(t,u) = 0,\ 0 < t < 1, \\ u'(0) = 0,\ u(1) = \alpha\, u'(\eta), \end{cases}$$

where $\eta \in (0,1), \alpha \in \Re, f \in C([0,1] \times \Re, \Re)$.
Also, The nonlocal nonlinear three point boundary value problem of fractional orders was studied by many authors, for example; in [3] they studied the nonlocal nonlinear boundary value problem of a fractional-order functional differential equation

$$\begin{cases} {}_*D^{\beta}\, u(t) + f(t, u(\phi(t))) = 0,\ t \in (0,1), \\ I^{\gamma} u(t)|_{t=0} = 0, \alpha u(\eta) = u(1), 0 < \eta < 1, 0 < \alpha \eta^{\beta-1} < 1. \end{cases}$$

Where $\beta \in (1,2)$ and $\gamma \in (0,1]$, they proved the existence of L_1-solution such that the function f satisfies the Caratheodory conditions and the growth condition.
And, in [4] they studied the nonlocal nonlinear boundary value problem of fractional-order differential equation:

$$\begin{cases} {}_*D^{\beta}\, u(t) + f(t, u(t)) = 0,\ \beta \in (1,2),\ t \in (0,1), \\ I^{\gamma} u(t)|_{t=0} = 0, \alpha u(\eta) = u(1), 0 < \eta < 1, 0 \leq \alpha \eta^{\beta-1} < 1. \end{cases}$$

they proved the existence of continuous solution such that the function f satisfies Caratheodory conditions and growth condition. Also the existence of maximal and minimal solutions with $\alpha = 0$ was studied. Also, the nonlocal conditions was studied in [1] and [5] - [6].

Now, let us recall Schauder fixed point Theorem which will be needed in the sequel.

Theorem 1.1. ***(Schauder fixed point Theorem)*** *[2]*
Let U be a convex subset of a Banach space X, and $T : U \to U$ is compact, continuous map. Then T has at least one fixed point in U.

2. Existence of solution

Here the space $C[0,1]$ denotes the space of all continuous functions on the interval $[0,1]$ with the supremum norm $||u|| = \sup_{t\in[0,1]} |u(t)|$.

To facilitate our discussion, let us first state the following assumptions:

(i) $|\dfrac{\partial f}{\partial u}| \leq K_1$,

(ii) $f \in C([0,1],\Re)$,

(iii) $a(t)$ is a function which is absolutely continuous.

Definition 2.1. *By a solution of the fractional-order three point boundary value problem (1.1) or (1.2) we mean a function $u \in C^1[0,1]$ with $u'' \in L_1[0,1]$.*

Firstly consider problem (1.1):

Theorem 2.2. *If the above assumptions (i) - (iii) are satisfied, then the three point boundary value problem (1.1) has a solution.*

Proof: Firstly: we will prove the equivalence of this problem (problem (1.1)):

$$D^{\beta}\, u(t) + \lambda\, a(t)\, f(u(t)) = 0,\ \beta \in (1,2],\ t \in (0,1), \quad (2.1)$$

$$u(0) = 0,\ \alpha\, u(\eta) = u(1),\ 0 < \eta < 1,\ 0 \leq \alpha\, \eta < 1 \quad (2.2)$$

with the integral equation:

$$\begin{aligned} u(t) &= -I^{\beta}\, \lambda\, a(t)\, f(u(t)) \\ &+ \frac{t}{1-\alpha\,\eta} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\, \lambda\, a(s)\, f(u(s))\, ds \\ &- \frac{\alpha\, t}{1-\alpha\,\eta} \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\, \lambda\, a(s)\, f(u(s))\, ds. \end{aligned} \quad (2.3)$$

Indeed; operating by $I^{\beta-1}$ on both sides of equation (2.1), we get

$$I\, u'' = -I^{\beta-1}\, \lambda\, a(t)\, f(u(t)),$$

then

$$u'(t) - C_1 = -I^{\beta-1}\, \lambda\, a(t)\, f(u(t)).$$

By integration, we get

$$u(t) = -I^{\beta}\, \lambda\, a(t)\, f(u(t)) + C_1\, t + C_2.$$

By (2.2), we get $C_2 = 0$ and

$$\begin{aligned} C_1 &= \frac{1}{1-\alpha\,\eta} \Big(\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\, \lambda\, a(s)\, f(u(s))\, ds \\ &- \alpha \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\, \lambda\, a(s)\, f(u(s))\, ds \Big). \end{aligned}$$

Therefore, the solution of problem (2.1) - (2.2) is given by the formula (2.3).
Now define the operator $T : C \to C$ by

$$\begin{aligned} Tu(t) &= -I^{\beta}\,\lambda\, a(t)\, f(u(t)) \\ &+ \frac{t}{1-\alpha\,\eta}\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\,\lambda\, a(s)\, f(u(s))\, ds \\ &- \frac{\alpha\, t}{1-\alpha\,\eta}\int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\,\lambda\, a(s)\, f(u(s))\, ds, \end{aligned}$$

and define a convex set $U \subset C[0,1]$ by $U = \{u \in C[0,1] : ||u|| \leq \frac{3\,K\,|\lambda|\,||a||}{(1-\alpha\,\eta)\,\Gamma(1+\beta)}\}$, where $K = sup_t\,|f(u(t))|$.
The operator T is a continuous operator, indeed:

$$\begin{aligned} |Tu(t_2)-Tu(t_1)| &= \Big| -\int_0^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\lambda a(s)\, f(u(s))\, ds \\ &+ \frac{t_2}{1-\alpha\eta}\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\lambda a(s) f(u(s)) ds \\ &- \frac{\alpha\, t_2}{1-\alpha\,\eta}\int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\,\lambda\, a(s)\, f(u(s))\, ds \\ &+ \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}\,\lambda\, a(s)\, f(u(s))\, ds \\ &- \frac{t_1}{1-\alpha\,\eta}\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\,\lambda\, a(s)\, f(u(s))\, ds \\ &+ \frac{\alpha\, t_1}{1-\alpha\,\eta}\int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\,\lambda\, a(s)\, f(u(s))\, ds \Big| \\ &\leq \Big| \int_0^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\,\lambda\, a(s)\, f(u(s))\, ds \\ &- \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}\,\lambda\, a(s)\, f(u(s))\, ds \Big| \\ &+ \Big| \frac{t_2}{1-\alpha\,\eta}\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\,\lambda\, a(s)\, f(u(s))\, ds \\ &- \frac{t_1}{1-\alpha\,\eta}\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\,\lambda\, a(s)\, f(u(s))\, ds \Big| \\ &+ \Big| \frac{\alpha\, t_2}{1-\alpha\,\eta}\int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\,\lambda\, a(s)\, f(u(s))\, ds \\ &- \frac{\alpha\, t_1}{1-\alpha\,\eta}\int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\,\lambda\, a(s)\, f(u(s))\, ds \Big| \\ &\leq \Big| \int_0^{t_1} \left(\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} - \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} \right) \lambda\, a(s)\, f(u(s))\, ds \\ &+ \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\,\lambda\, a(s)\, f(u(s))\, ds \Big| \\ &+ \frac{K\,|\lambda|\,||a||}{1-\alpha\,\eta}\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\, ds\, |t_2-t_1| \\ &+ \frac{\alpha\, K\,|\lambda|\,||a||}{1-\alpha\,\eta}\int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\, ds\, |t_2-t_1| \\ &\leq \frac{K\,|\lambda|\,||a||}{\Gamma(\beta)} \left(\int_0^{t_1} |(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}|\, ds \right. \\ &+ \left. \int_{t_1}^{t_2} (t_2-s)^{\beta-1}\, ds \right) \\ &+ \frac{K|\lambda|||a||}{(1-\alpha\eta)\Gamma(1+\beta)}|t_2-t_1| + \frac{\alpha K|\lambda|||a||}{(1-\alpha\eta)\Gamma(1+\beta)}|t_2-t_1| \\ &\leq \frac{K|\lambda|||a||}{\Gamma(1+\beta)}\left(2(t_2-t_1)^{\beta} + |t_2^{\beta}-t_1^{\beta}|\right) \\ &+ \frac{K|\lambda|\,||a||}{(1-\alpha\eta)\Gamma(1+\beta)}|t_2-t_1| + \frac{\alpha K|\lambda|||a||}{(1-\alpha\,\eta)\Gamma(1+\beta)}|t_2-t_1|. \end{aligned}$$

The above inequality shows that

$$|Tu(t_2) - Tu(t_1)| \to 0 \text{ as } t_2 \to t_1, \tag{2.4}$$

then Tu is uniformly continuous in $[0,1]$, and hence $T : U \to U$ is well defined.
Also;

$$\begin{aligned} |Tu(t)| &\leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\,|\lambda|\,|a(s)|\,|f(u(s)|\, ds \\ &+ \frac{t}{1-\alpha\,\eta}\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\,|\lambda|\,|a(s)|\,|f(u(s))|\, ds \\ &+ \frac{\alpha\, t}{1-\alpha\,\eta}\int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\,|\lambda|\,|a(s)|\,|f(u(s))|\, ds, \\ ||Tu|| &\leq \frac{3\,K\,|\lambda|\,||a||}{(1-\alpha\,\eta)\,\Gamma(1+\beta)}. \end{aligned} \tag{2.5}$$

Therefore from Arzela-Ascoli Theorem it is easy to show that $T : U \to U$ is compact, immediately we obtain from inequality (2.5) that $T(U)$ is uniformly bounded, while the equicontinuity of $T(U)$ follows from inequality (2.4).
Now, from Schauder fixed point Theorem (1.1), we obtain that the operator T has a fixed point in $C[0,1]$.

Now, differentiate (2.3), we obtain

$$\begin{aligned}
u'(t) &= -I^{\beta-1}\lambda\, a(t)\, f(u(t)) \\
&+ \frac{1}{1-\alpha\,\eta}\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\lambda\, a(s)\, f(u(s))\, ds \\
&- \frac{\alpha}{1-\alpha\,\eta}\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\lambda\, a(s)\, f(u(s))\, ds, \\
|u'(t)| &\leq \int_0^t \frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)}|\lambda|\,|a(s)|\,|f(u(s)|\, ds \\
&+ \frac{1}{1-\alpha\,\eta}\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}|\lambda|\,|a(s)|\,|f(u(s))|\, ds \\
&+ \frac{\alpha}{1-\alpha\,\eta}\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}|\lambda|\,|a(s)|\,|f(u(s))|\, ds, \\
||u'|| &\leq \left(\frac{1}{\Gamma(\beta)} + \frac{1+\alpha}{(1-\alpha\,\eta)\Gamma(\beta+1)}\right)K\,|\lambda|\,||a||.
\end{aligned}$$

Therefore we obtain that $u' \in C[0,1]$.

$$\begin{aligned}
u''(t) &= -\frac{d}{dt}I^{\beta-1}\lambda\, a(t)\, f(u(t)) \\
&= -(\lambda\, a(t)\, f(u(t)))\,|_{t=0}\,\frac{t^{\beta-2}}{\Gamma(\beta-1)} \\
&- I^{\beta-1}\frac{d}{dt}(\lambda\, a(t)\, f(u(t))) \\
&= -K_2\frac{t^{\beta-2}}{\Gamma(\beta-1)} - \lambda\, I^{\beta-1}\Big(a'(t)\, f(u(t)) \\
&+ \frac{\partial f}{\partial u}u'(t)\, a(t)\Big),
\end{aligned}$$

$$\begin{aligned}
\int_0^1 |u''(t)|\, dt &\leq \frac{K_2}{\Gamma(\beta)}t^{\beta-1}|_0^1 \\
&+ |\lambda|\int_0^1\int_0^t \frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)}\Big|a'(s)\, f(u(s)) \\
&+ \frac{\partial f}{\partial u}u'(s)\, a(s)\Big|ds\, dt \\
&= \frac{K_2}{\Gamma(\beta)} + |\lambda|\int_0^1\Big|a'(s)\, f(u(s)) \\
&+ \frac{\partial f}{\partial u}u'(s)\, a(s)\Big|\int_s^1 \frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)}\, dt\, ds \\
&\leq \frac{K_2}{\Gamma(\beta)} + \frac{|\lambda|}{\Gamma(\beta)}\int_0^1\Big|a'(s)\, f(u(s)) \\
&+ \frac{\partial f}{\partial u}u'(s)\, a(s)\Big|ds, \\
\|u''\|_{L_1} &\leq \frac{K_2}{\Gamma(\beta)} + \frac{|\lambda|}{\Gamma(\beta)}\left(K\|a'\|_{L_1} + K_1\|u'\|_{L_1}\|a\|\right).
\end{aligned}$$

Therefore we obtain that $u'' \in L_1[0,1]$.
To complete the equivalence of equation (2.3) with the fractional-order three point boundary value problem (2.1) - (2.2), let $u(t)$ be a solution of (2.3), differentiate it twice we get

$$\begin{aligned}
u''(t) &= -\frac{d}{dt}I^{\beta-1}\lambda\, a(t)\, f(u(t)) \\
&= -(\lambda\, a(t)\, f(u(t)))\,|_{t=0}\,\frac{t^{\beta-2}}{\Gamma(\beta-1)} \\
&- I^{\beta-1}\frac{d}{dt}(\lambda\, a(t)\, f(u(t))),
\end{aligned}$$

and operating by $I^{2-\beta}$ on both sides of the last equation, we get

$$D^\beta u(t) + \lambda\, a(t)\, f(u(t)) = 0.$$

Also it is easy to prove that conditions (2.2) are satisfied. Which proves the equivalence. ■

Secondly consider problem (1.2):

$$\begin{cases} D^\beta u(t) + \lambda a(t) f(u(t)) = 0, \beta \in (1,2], t \in (0,1), \\ u'(0) = 0, \alpha\, u'(\eta) = u(1), 0 < \eta < 1, 0 \leq \alpha\eta < 1. \end{cases} \tag{2.6}$$

Theorem 2.3. *If the above assumptions (i) - (iii) are satisfied, then the three point boundary value problem (1.2) has a solution.*

Proof: We will prove the equivalence of this problem (problem(1.2)):

$$D^\beta u(t) + \lambda a(t) f(u(t)) = 0, \beta \in (1,2], t \in (0,1), \tag{2.7}$$

$$u'(0) = 0, \alpha u'(\eta) = u(1), 0 < \eta < 1, 0 \leq \alpha\eta < 1 \tag{2.8}$$

with the integral equation:

$$\begin{aligned}
u(t) &= -I^\beta\lambda\, a(t)\, f(u(t)) \\
&+ \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\lambda\, a(s)\, f(u(s))\, ds \\
&- \alpha\int_0^\eta \frac{(\eta-s)^{\beta-2}}{\Gamma(\beta-1)}\lambda\, a(s)\, f(u(s))\, ds.
\end{aligned} \tag{2.9}$$

Indeed; as in theorem (2.2) Equation (2.7) can be reduced to an equivalent integral equation:

$$u(t) = -I^\beta\lambda\, a(t)\, f(u(t)) + C_1\, t + C_2.$$

By differentiating the last equation, we get

$$u'(t) = -I^{\beta-1}\lambda\, a(t)\, f(u(t)) + C_1.$$

By (2.8), we get $C_1 = 0$ and

$$\begin{aligned}
C_2 &= \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\lambda\, a(s)\, f(u(s))\, ds \\
&- \alpha\int_0^\eta \frac{(\eta-s)^{\beta-2}}{\Gamma(\beta-1)}\lambda\, a(s)\, f(u(s))\, ds.
\end{aligned}$$

Therefore, the solution of problem (2.7) - (2.8) is given by the formula (2.9).
Now define the operator $T : C \to C$ by

$$\begin{aligned} Tu(t) &= -I^{\beta}\,\lambda\,a(t)\,f(u(t)) \\ &+ \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\,\lambda\,a(s)\,f(u(s))\,ds \\ &- \alpha \int_0^{\eta} \frac{(\eta-s)^{\beta-2}}{\Gamma(\beta-1)}\,\lambda\,a(s)\,f(u(s))\,ds. \end{aligned}$$

and define a convex set $U \subset C[0,1]$ by $U = \{u \in C[0,1] : ||u|| \le \frac{(2+\alpha\,\beta)}{\Gamma(1+\beta)} K\,|\lambda|\,||a||\}$, where $K = sup_t\,|f(u(t))|$.
The operator T is a continuous operator, indeed:

$$\begin{aligned} |Tu(t_2)-Tu(t_1)| &= \left| -\int_0^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\,\lambda\,a(s)\,f(u(s))\,ds \right. \\ &+ \left. \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}\,\lambda\,a(s)\,f(u(s))\,ds \right| \\ &\le \left| \int_0^{t_1} \left(\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} - \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} \right) \right. \\ &\times\ \lambda\,a(s)\,f(u(s))\,ds \\ &+ \left. \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\,\lambda\,a(s)\,f(u(s))\,ds \right| \\ &\le \frac{K\,|\lambda|\,||a||}{\Gamma(\beta)} \left(\int_0^{t_1} \left| (t_2-s)^{\beta-1} \right. \right. \\ &- \left. \left. (t_1-s)^{\beta-1} \right| ds + \int_{t_1}^{t_2} (t_2-s)^{\beta-1}\,ds \right) \\ &\le \frac{K\,|\lambda|\,||a||}{\Gamma(1+\beta)} \left(2\,(t_2-t_1)^{\beta} + |t_2^{\beta}-t_1^{\beta}| \right). \end{aligned}$$

The above inequality shows that

$$|Tu(t_2) - Tu(t_1)| \to 0 \ \text{as}\ t_2 \to t_1, \tag{2.10}$$

then Tu is uniformly continuous in $[0,1]$, and hence $T : U \to U$ is well defined.
Also;

$$\begin{aligned} |Tu(t)| &\le \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\,|\lambda|\,|a(s)|\,|f(u(s))|\,ds \\ &+ \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\,|\lambda|\,|a(s)|\,|f(u(s))|\,ds \\ &+ \alpha \int_0^{\eta} \frac{(\eta-s)^{\beta-2}}{\Gamma(\beta-1)}\,|\lambda|\,|a(s)|\,|f(u(s))|\,ds, \\ ||Tu|| &\le \frac{(2+\alpha\,\beta)}{\Gamma(1+\beta)}\,K\,|\lambda|\,||a||. \end{aligned} \tag{2.11}$$

Therefore from Arzela-Ascoli Theorem it is easy to show that $T : U \to U$ is compact, immediately we obtain from inequality (2.11) that $T(U)$ is uniformly bounded, while the equicontinuity of $T(U)$ follows from inequality (2.10).
Now, from Schauder fixed point Theorem (1.1), we obtain that the operator T has a fixed point in $C[0,1]$.

Now, differentiate (2.9), we obtain

$$\begin{aligned} u'(t) &= -I^{\beta-1}\,\lambda\,a(t)\,f(u(t)), \\ |u'(t)| &\le \int_0^t \frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)}\,|\lambda|\,|a(s)|\,|f(u(s))|\,ds, \\ ||u'|| &\le \frac{K\,|\lambda|\,||a||}{\Gamma(\beta)}. \end{aligned}$$

Therefore we obtain that $u' \in C[0,1]$.

$$\begin{aligned} u''(t) &= -\frac{d}{dt}\,I^{\beta-1}\,\lambda\,a(t)\,f(u(t)) \\ &= -(\lambda\,a(t)\,f(u(t)))\,|_{t=0}\ \frac{t^{\beta-2}}{\Gamma(\beta-1)} \\ &- I^{\beta-1}\,\frac{d}{dt}\,(\lambda\,a(t)\,f(u(t))) \\ &= -K_2\,\frac{t^{\beta-2}}{\Gamma(\beta-1)} \\ &- \lambda\,I^{\beta-1}\left(a'(t)\,f(u(t)) + \frac{\partial f}{\partial u}\,u'(t)\,a(t) \right), \end{aligned}$$

$$\begin{aligned} \int_0^1 |u''(t)|\,dt &\le \frac{K_2}{\Gamma(\beta)}\,t^{\beta-1}|_0^1 \\ &+ |\lambda| \int_0^1 \int_0^t \frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)} \left| a'(s)\,f(u(s)) \right. \\ &+ \left. \frac{\partial f}{\partial u}\,u'(s)\,a(s) \right| ds\,dt \\ &= \frac{K_2}{\Gamma(\beta)} + |\lambda| \int_0^1 \left| a'(s) f(u(s)) \right. \\ &+ \left. \frac{\partial f}{\partial u} u'(s) a(s) \right| \int_s^1 \frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)}\,dt\,ds \\ &\le \frac{K_2}{\Gamma(\beta)} + \frac{|\lambda|}{\Gamma(\beta)} \int_0^1 \left| a'(s) f(u(s)) \right. \\ &+ \left. \frac{\partial f}{\partial u} u'(s) a(s) \right| ds, \\ \|u''\|_{L_1} &\le \frac{K_2}{\Gamma(\beta)} + \frac{|\lambda|}{\Gamma(\beta)} \left(K\|a'\|_{L_1} + K_1\|u'\|_{L_1}\|a\| \right). \end{aligned}$$

Therefore we obtain that $u'' \in L_1[0,1]$.
To complete the equivalence of equation (2.9) with the fractional-order three point boundary value problem (2.7) - (2.8), let $u(t)$ be a solution of (2.9), differentiate it twice we get

$$\begin{aligned} u''(t) &= -\frac{d}{dt}\,I^{\beta-1}\,\lambda\,a(t)\,f(u(t)) \\ &= -(\lambda\,a(t)\,f(u(t)))\,|_{t=0}\ \frac{t^{\beta-2}}{\Gamma(\beta-1)} \\ &- I^{\beta-1}\,\frac{d}{dt}\,(\lambda\,a(t)\,f(u(t))), \end{aligned}$$

and operating by $I^{2-\beta}$ on both sides of the last equation, we get

$$D^{\beta}\,u(t)\,+\,\lambda\,a(t)\,f(u(t))\,=\,0.$$

Also it is easy to prove that conditions (2.8) are satisfied. Which proves the equivalence.■

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ISSN(P):2319 − 3786

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ISSN(O):2321 − 5666

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