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Increasing the order of convergence for iterative methods in Banach space under weak conditions

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Abstract
We study the method considered in Xiao and Yin (2015), for solving systems of nonlinear equations, modified suitably to include the nonlinear equations in Banach spaces. The novelty of this study lies in the fact that our conditions are weaker than the conditions used in earlier studies. This way we extend the applicability of the method. Numerical examples are also given in this study where earlier results cannot apply to solve equations but our results can apply.

Keywords
Newton-type method, radius of convergence, local convergence, restricted convergence domains.

AMS Subject Classification
65D10, 65D99, 65J20, 49M15, 74G20, 41A25.

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Contents

1. Introduction

Let $H:\Omega\subseteq\mathscr{B}_1\longrightarrow\mathscr{B}_2$ be a continuous Fréchet-differentiable operator between the Banach spaces $\mathscr{B}_1$ and $\mathscr{B}_2$ and Ω be a convex set. In this study, we consider the problem of approximating a solution x^* of nonlinear equation

$$H(x)=0. \tag{1.1}$$

We consider the following method from [19] for increasing the order of convergence of iterative methods to solve (1.1):

$$\begin{aligned} y_n &= x_n-\alpha H'(x_n)^{-1}H(x_n)\\ z_n &= \varphi(x_n,y_n)\\ x_{n+1} &= z_n-A_nH(z_n), \end{aligned} \tag{1.2}$$

where $x_0\in\Omega$ is an initial point $\alpha\in\mathbb{R}-\{0\}$, $\varphi:\Omega\times\Omega\longrightarrow X$ is a continuous operator and $A_n:=[\frac{1}{\alpha}H'(y_n)^{-1}+(1-\frac{1}{\alpha})H'(x_n)^{-1}]$. Let $U(a,\rho),\bar{U}(a,\rho)$ stand respectively for the open and closed balls in $\mathscr{B}_1$ with center $a\in\mathscr{B}_1$ and of radius $\rho>0$.

The study of convergence of iterative algorithms is usually centered into two categories: semi-local and local convergence analysis. The semi-local convergence is based on the information around an initial point, to obtain conditions ensuring the convergence of these algorithms, while the local convergence is based on the information around a solution to find estimates of the computed radii of the convergence balls. Local results are important since they provide the degree of difficulty in choosing initial points.

Finding solution of the equation (1.1) is an important problem in mathematics due to its wide applications. So improving the order of convergence of iterative method for solving (1.1) is also an important problem in mathematics. In [19] the existence of the Fréchet derivative of H of order up to the fourth was used for the convergence analysis of method (1.2). This assumption on the higher order Fréchet derivatives of the operator H restricts the applicability of method (1.2). For example consider the following;

EXAMPLE 1.1. *Let $\mathscr{B}_1=\mathscr{B}_2=C[0,1],\Omega=\bar{B}(x^*,1)$. Consider the nonlinear integral equation of the mixed Hammerstein-*

type [1, 2, 6–9, 12] defined by

$$x(s)=\int_0^1 G(s,t)(x(t)^{3/2}+\frac{x(t)^2}{2})dt,$$

where the kernel G is the Green's function defined on the interval $[0,1]\times[0,1]$ *by*

$$G(s,t)=\begin{cases}(1-s)t, & t\le s\\ s(1-t), & s\le t.\end{cases}$$

The solution $x^*(s)=0$ *is the same as the solution of equation (1.1), where* $H:C[0,1]\longrightarrow C[0,1])$ *is defined by*

$$H(x)(s)=x(s)-\int_0^1 G(s,t)(x(t)^{3/2}+\frac{x(t)^2}{2})dt.$$

Notice that

$$\|\int_0^1 G(s,t)dt\|\le\frac{1}{8}.$$

Then, we have that

$$H'(x)y(s)=y(s)-\int_0^1 G(s,t)(\frac{3}{2}x(t)^{1/2}+x(t))dt,$$

so since $H'(x^*(s))=I$,

$$\|H'(x^*)^{-1}(H'(x)-H'(y))\|\le\frac{1}{8}(\frac{3}{2}\|x-y\|^{1/2}+\|x-y\|).$$

One can see that, higher order than one derivatives of H do not exist in this example.

Our goal is to weaken the assumptions in [19] and apply the method for solving equation (1.1) in Banach spaces, so that the applicability of the method (1.2) can be extended. This approach can be applied on other iterative methods [1–19]. Notice that earlier studies [1–19] also use hypotheses on higher order than two derivatives of H although these derivatives do not appear in the method. We also provide computable radius of convergence error bounds on the distances $\|x_n-x^*\|$ and a uniqueness result based on Lipschitz-type conditions not given in [19] or related methods [1–18].

The rest of the paper is organized as follows. In Section 2 we present the local convergence analysis. We also provide a radius of convergence, computable error bounds and uniqueness result. Special cases and numerical examples are given in the last section.

2. Local Convergence analysis

We introduce some functions and parameters for the local convergence analysis of method (1.2). Let $w_0:[0,+\infty)\longrightarrow[0,+\infty)$ be a continuous and non-decreasing function satisfying $w_0(0)=0$. Define the parameter r_0 by

$$r_0=\sup\{t\ge 0: w_0(t)<1\}. \tag{2.1}$$

Let also $\alpha\in\mathbb{R}-\{0\}$, $p\ge 2$, $w:[0,r_0)\longrightarrow[0,+\infty)$, $v:[0,r_0)\longrightarrow[0,+\infty)$ and $\psi:[0,r_0)\longrightarrow[0,+\infty)$ be continuous and nondecreasing functions, so that $w(0)=0$. Define functions g_1,h_1,g_2 and h_2 on the interval $[0,r_0)$ by

$$g_1(t)=\frac{\int_0^1 w((1-\theta)t)d\theta+|1-\alpha|\int_0^1 v(\theta t)d\theta}{1-w_0(t)},$$

$$h_1(t)=h_1(t)-1,$$

$$g_2(t)=\psi(t)t^{p-1}$$

and

$$h_2(t)=g_2(t)-1.$$

Suppose that

$$|1-\alpha|v(0)<1 \tag{2.2}$$

and

$$h_2(t)\longrightarrow+\infty \text{ or some positive number } \beta \text{ as } t\longrightarrow r_0^-. \tag{2.3}$$

We have that $h_1(0)=|1-\alpha|v(0)-1<0$ and $h_1(t)\to+\infty$ as $t\to r_0^-$ by (2.1). By applying the intermediate value theorem on the interval $[0,r_0]$ for function h_1, we deduce that function h_1 has a zero in the interval $(0,r_0)$. Denote by r_1 the smallest such zero. Moreover, we get that $h_2(0)=-1$ and $h_2(t)\longrightarrow+\infty$ or $\beta>0$ as $t\longrightarrow r_0^-$. Denote by r_2 the smallest zero of function h_2 on the interval $(0,r_0)$. Furthermore, define parameter $\bar{r}_0$ by

$$\bar{r}_0=\max\{t\in[0,r_0]: g_1(t)t<1\}. \tag{2.4}$$

Finally, define functions p,g_3 and h_3 on the interval $[0,\bar{r}_0)$ by

$$p(t)=\frac{1}{|\alpha|}\frac{w_0(t)+w_0(g_1(t)t)}{(1-w_0(t))(1-w_0(g_1(t)t))}+\frac{1}{1-w_0(t)},$$

$$g_3(t)=(1+p(t)\int_0^1 v(\theta g_2(t)t)d\theta)\psi(t)t^{p-1}$$

and

$$h_3(t)=g_3(t)-1.$$

We obtain that $h_3(0)=-1<0$ and $h_3(t)\longrightarrow+\infty$ as $t\longrightarrow\bar{r}_0^-$. Denote by r_3 the smallest zero of h_3 in the interval $(0,\bar{r}_0)$. Define the radius of convergence r by

$$r=\min\{r_i\}\ i=1,2,3. \tag{2.5}$$

Then, we have for each $t\in[0,r)$

$$0\le g_i(t)<1,\ i=1,2,3.\ \text{and}\ \ 0\le p(t). \tag{2.6}$$

Next, the local convergence analysis of method (1.2) is shown using the preceding notation.

THEOREM 2.1. *Let $H:\Omega\subset\mathscr{B}_1\to\mathscr{B}_2$ be a continuously Fréchet-differentiable operator, $\varphi:\Omega\times\Omega\longrightarrow\mathscr{B}_1$ be a continuous operator, $\alpha\in\mathbb{R}-\{0\}$ and $p>1$. Suppose:*

there exist $x^*\in\Omega$ and function $w_0:[0,+\infty)\longrightarrow[0,+\infty)$ continuous and non-decreasing with $w_0(0)=0$ such that

$$H(x^*)=0,\quad H'(x^*)^{-1}\in L(\mathscr{B}_2,\mathscr{B}_1), \tag{2.7}$$

and

$$\|H'(x^*)^{-1}(H'(x)-H'(x^*))\|\le w_0(\|x-x^*\|),\ \text{for each}\ x\in\Omega; \tag{2.8}$$

Set $\Omega_0=\Omega\cap U(x^*,r_0)$. There exist functions $w:[0,r_0)\longrightarrow[0,+\infty)$, $v:[0,r_0)\longrightarrow[0,+\infty)$, $\psi:[0,r_0)\longrightarrow[0,+\infty)$ continuous and nondecreasing with $w(0)=0$ such that for each $x,y\in\Omega_0$

$$\|H'(x^*)^{-1}(H'(x)-H'(y)\|\le w(\|x-y\|), \tag{2.9}$$

$$\|H'(x^*)^{-1}H'(x)\|\le v(\|x-x^*\|), \tag{2.10}$$

$$\|z-x^*\|=\|\varphi(x,y)-x^*\|\le\psi(\|x-x^*\|)\|x-x^*\|^p, \tag{2.11}$$

(2.2), (2.3) hold and

$$\bar{U}(x^*,r)\subseteq\Omega, \tag{2.12}$$

where $z=\varphi(x,y)$, the convergence radius r is given by (2.5) and r_0 is defined in (2.1). Then, the sequence $\{x_n\}$ generated for $x_0\in U(x^*,r)-\{x^*\}$ by method (1.2) is well defined in $U(x^*,r)$, remains in $U(x^*,r)$ for each $n=0,1,2,\dots$ and converges to x^*. Moreover, the following estimates hold

$$\|y_n-x^*\|\le g_1(\|x_n-x^*\|)\|x_n-x^*\|\le\|x_n-x^*\|<r, \tag{2.13}$$

$$\|z_n-x^*\|\le g_2(\|x_n-x^*\|)\|x_n-x^*\|\le\|x_n-x^*\| \tag{2.14}$$

and

$$\|x_{n+1}-x^*\|\le g_3(\|x_n-x^*\|)\|x_n-x^*\|\le\|x_n-x^*\|, \tag{2.15}$$

where the functions $g_i, i=1,2,3$ are defined previously. Furthermore, if there exists $R\in[r,r_0]$ such that

$$\int_0^1 w_0(\theta R)d\theta<1, \tag{2.16}$$

then the limit point x^* is the only solution of equation $H(x)=0$ in $\Omega_1=\Omega\cap\bar{U}(x^*,R)$.

Proof. We present a proof based on mathematical induction. By hypothesis $x_0\in U(x^*,r)-\{x^*\}$, (2.1) and (2.8), we have that

$$\|H'(x^*)^{-1}(H'(x_0)-H'(x^*)\|\le w_0(\|x_0-x^*\|)\le w_0(r)<1. \tag{2.17}$$

The Banach Lemma on invertible operators [2, 4, 15] and (2.7) guarantee that $H'(x_0)^{-1}\in L(\mathscr{B}_2,\mathscr{B}_1)$ and

$$\|H'(x_0)^{-1}H'(x^*)\|\le\frac{1}{1-w_0(\|x_0-x^*\|)}. \tag{2.18}$$

The points y_0 and z_0 are also well defined by the first substep of method (1.2) for $n=0$. We can write by (2.7) that

$$H(x_0)=H(x_0)-H(x^*)=\int_0^1 H'(x^*+\theta(x_0-x^*))(x_0-x^*)d\theta. \tag{2.19}$$

Notice that $\|x^*+\theta(x_0-x^*)-x^*\|=\theta\|x_0-x^*\|<r$, so $x^*+\theta(x_0-x^*)\in U(x^*,r)$ for each $\theta\in[0,1]$. In view of (2.10) and (2.19), we get that

$$\|H'(x^*)^{-1}H'(x_0)\|\le\int_0^1 v(\theta\|x_0-x^*\|)d\theta\|x_0-x^*\|. \tag{2.20}$$

We can also write by the first substep of method (1.2) that

$$y_0-x^*=x_0-x^*-H'(x_0)^{-1}H(x_0)+(1-\alpha)H'(x_0)^{-1}H(x_0). \tag{2.21}$$

Then, using (2.5), (2.6) (for $i=1$), (2.7), (2.9), (2.18), (2.20) and (2.21), we obtain in turn that

$$\begin{aligned}
&\|y_0-x^*\|\\
\le\ &\|x_0-x^*-H'(x_0)^{-1}H'(x_0)\|+|1-\alpha|\|H'(x_0)^{-1}H'(x^*)\|\\
\le\ &\|H'(x_0)^{-1}H'(x^*)\|\|\int_0^1 H'(x^*)^{-1}(H'(x^*+\theta(x_0-x^*))\\
&-H'(x_0))(x_0-x^*)d\theta\| \qquad (2.22)\\
&+|1-\alpha|\|H'(x_0)^{-1}H'(x^*)\|\|H'(x^*)^{-1}H(x_0)\|\\
\le\ &\frac{\int_0^1 w((1-\theta)\|x_0-x^*\|)d\theta\|x_0-x^*\|}{1-w_0(\|x_0-x^*\|)}\\
&+|1-\alpha|\frac{\int_0^1 v(\theta\|x_0-x^*\|)d\theta\|x_0-x^*\|}{1-w_0(\|x_0-x^*\|)}\\
=\ &g_1(\|x_0-x^*\|)\|x_0-x^*\|\le\|x_0-x^*\|<r, \qquad (2.23)
\end{aligned}$$

which shows (2.13) for $n=0$ and $y_0\in B(x^*,r)$. Moreover, by (2.5), (2.6) (for $i=2$) and (2.11) we get that

$$\begin{aligned}
&\|z_0-x^*\|\\
=\ &\|\varphi(x_0,y_0)-x^*\|\le\psi(\|x_0-x^*\|)\|x_0-x^*\|^p\\
\le\ &g_2(\|x_0-x^*\|)\|x_0-x^*\|\le\|x_0-x^*\|<r, \qquad (2.24)
\end{aligned}$$

which shows (2.14) for $n=0$ and $z_0 \in U(x^*,r)$. Next, we must show that A_0 is well defined. As in (2.18) for $x_0=y_0$ we get that $H'(y_0)^{-1} \in L(\mathscr{B}_2,\mathscr{B}_1)$,

$$\begin{aligned} & \|H'(y_0)^{-1}H'(x^*)\| \\ \leq & \frac{1}{1-w_0(\|y_0-x^*\|)} \\ \leq & \frac{1}{1-w_0(g_1(\|x_0-x^*\|)\|x_0-x^*\|)}, \end{aligned} \tag{2.25}$$

so A_0 and x_1 are well defined. We also have by (2.7), (2.18) and (2.25) that

$$\begin{aligned} & \|A_0H'(x^*)\| \\ = & \|\frac{1}{\alpha}(H'(y_0)^{-1}-H'(x_0)^{-1})H'(x^*)+H'(x_0)^{-1}H'(x^*)\| \\ = & \|\frac{1}{\alpha}[H'(y_0)^{-1}H'(x^*)][H'(x^*)^{-1}(H'(x_0)-H'(x^*)) \\ & +H'(x^*)^{-1}(H'(x^*)-H'(y_0))]H'(x_0)^{-1}H'(x^*) \\ & +H'(x_0)^{-1}H'(x^*)\| \\ \leq & \frac{w_0(\|x_0-x^*\|)+w_0(g_1(\|x_0-x^*\|)\|x_0-x^*\|)}{|\alpha|(1-w_0(g_1(\|x_0-x^*\|)\|x_0-x^*\|))(1-w_0(\|x_0-x^*\|))} \\ & +\frac{1}{1-w_0(\|x_0-x^*\|)} = p(\|x_0-x^*\|). \end{aligned} \tag{2.26}$$

Furthermore, using (2.5), (2.6) (for $i=3$) (2.20) (for $x_0=z_0$), (2.26) and the last substep of method (1.2) for $n=0$, we obtain that

$$\begin{aligned} & \|x_1-x^*\| \\ \leq & \|z_0-x^*-A_0F(z_0)\| \\ \leq & \|z_0-x^*\|+\|A_0F'(x^*)\|\|F'(x^*)^{-1}F(z_0)\| \\ \leq & \|z_0-x^*\|+p(\|x_0-x^*\|\int_0^1 v(\theta\|z_0-x^*\|)d\theta)\|z_0-x^*\| \\ = & (1+p(\|x_0-x^*\|) \\ & \times\int_0^1 v(\theta g_2(\|x_0-x^*\|)\|x_0-x^*\|)d\theta)\|x_0-x^*\| \\ = & g_3(\|x_0-x^*\|)\|x_0-x^*\| \leq \|x_0-x^*\| < r, \end{aligned} \tag{2.27}$$

which shows (2.15) for $n=0$ and $x_1 \in U(x^*,r)$. The induction is completed if, we replace x_0,y_0,z_0,x_1 by x_k,y_k,z_k,x_{k+1} in the preceding estimates. Then, from the estimates

$$\|x_{k+1}-x^*\| \leq c\|x_k-x^*\| < r, \tag{2.28}$$

where $c=g_3(\|x_0-x^*\|) \in [0,1)$, we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*,r)$. Finally to show the uniqueness part, let $T=\int_0^1 H'(x^*+\theta(y^*-x^*))d\theta$ where $y^* \in \Omega_2$ with $H(y^*)=0$. Using (2.9), we obtain that

$$\begin{aligned} \|H'(x^*)^{-1}(T-H'(x^*))\| & \leq \int_0^1 w_0(\theta\|x^*-y^*\|)d\theta \\ & \leq \int_0^1 w_0(\theta R^*)d\theta < 1, \end{aligned} \tag{2.29}$$

Hence, we have that $T^{-1} \in L(\mathscr{B}_2,\mathscr{B}_1)$. Then, from the identity $0=H(y^*)-H(x^*)=T(y^*-x^*)$, we conclude that $x^*=y^*$. □

REMARK 2.2. *(a) In the case when $w_0(t)=L_0t, w(t)=Lt$ and $\Omega_0=\Omega$, the radius $r_A=\frac{2}{2L_0+L}$ was obtained by Argyros in [2] as the convergence radius for Newton's method under condition (2.7)-(2.9). Notice that the convergence radius for Newton's method given independently by Rheinboldt [15] and Traub [18] is given by*

$$\rho = \frac{2}{3L} < r_A.$$

Let us consider, as an example, the function $H(x)=e^x-1$ with $x^*=0$. Set $\Omega=B(0,1)$. Then, we have that $L_0=e-1<L=e^{\frac{1}{L_0}}$, so $\rho=0.3827<r_A=0.324947231$. Moreover, the new error bounds [2] are:

$$\|x_{n+1}-x^*\| \leq \frac{L}{1-L_0\|x_n-x^*\|}\|x_n-x^*\|^2,$$

whereas the old ones [5, 7]

$$\|x_{n+1}-x^*\| \leq \frac{L}{1-L\|x_n-x^*\|}\|x_n-x^*\|^2.$$

Clearly, the new error bounds are more precise, if $L_0<L$. Clearly, we do not expect the radius of convergence of method (1.2) given by r_3 to be larger than r_A.

(b) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method(GMREM), the generalized conjugate method(GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [1–5].

(c) The results can be also be used to solve equations where the operator H' satisfies the autonomous differential equation [2–4]:

$$H'(x)=P(H(x)),$$

where P is a known continuous operator. Since $H'(x^*)=P(H(x^*))=P(0)$, we can apply the results without actually knowing the solution x^*. Let as an example $H(x)=e^x-1$. Then, we can choose $P(x)=x+1$ and $x^*=0$.

(d) It is worth noticing that method (1.2) are not changing if, we use the new instead of the old conditions [19]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC)

$$\xi = \frac{ln\frac{\|x_{n+2}-x^*\|}{\|x_{n+1}-x^*\|}}{ln\frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}}, \quad \text{for each } n=1,2,\ldots$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \frac{ln\frac{\|x_{n+2}-x_{n+1}\|}{\|x_{n+1}-x_n\|}}{ln\frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}}, \quad \text{for each } n=0,1,2,\ldots.$$

(e) *In view of (2.4) and the estimate*

$$\begin{aligned}\|H'(x^*)^{-1}H'(x)\| &= \|H'(x^*)^{-1}(H'(x)-H'(x^*))+I\| \\ &\leq 1+\|H'(x^*)^{-1}(H'(x)-H'(x^*))\| \leq 1+w_0(\|x-x^*\|)\end{aligned}$$

condition (2.6) can be dropped and can be replaced by

$$v(t)=1+w_0(t)$$

or

$$v(t)=1+w_0(r_0),$$

since $t\in[0,r_0)$.

(f) *Condition (2.2) can be dropped as follows: Define parameter* R_0 *by*

$$R_0=g_1(r_4)r_4,\ \ r_4=\min\{r_2,r_3\}. \tag{2.30}$$

Moreover, replace (2.2), (2.13), respectively by

$$\bar{U}(x^*,R^*)\subseteq\Omega, \tag{2.31}$$

and

$$\|y_n-x^*\|\leq g_2(r_4)r_4=R_0, \tag{2.32}$$

where

$$R^*=\max\{R_0,r_4\}.$$

Then, the conclusions of Theorem 2.1 hold with these modifications.

(g) *Let us choose* $\alpha=1$ *and* $\varphi(x,y)=y-F'(y)^{-1}F(y)$. *Then, we have in (2.23) with* x_k *replaced by* y_k

$$\|\varphi(x_k,y_k)-x^*\|\leq\frac{\int_0^1 w((1-\theta)\|y_k-x^*\|)d\theta\|y_k-x^*\|}{1-w_0(g_1(\|x_k-x^*\|)\|x_k-x^*\|)},$$

so we can choose $p=1$ *and*

$$\psi(t)=\frac{\int_0^1 w((1-\theta)g_1(t)t)d\theta g_1(t)}{1-w_0(t)}.$$

3. Numerical Examples

We present two examples in this section. We choose $\alpha=1, p=1$ and ψ as in Remark 2.2 (g) in both examples.

EXAMPLE 3.1. *Let* $\mathscr{B}_1=\mathscr{B}_2=\mathbb{R}^3, D=\bar{U}(0,1), x^*=(0,0,0)^T$. *Define function* H *on* D *for* $w=(x,y,z)^T$ *by*

$$H(w)=(e^x-1,\frac{e-1}{2}y^2+y,z)^T.$$

Then, the Fréchet-derivative is given by

$$H'(v)=\begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1\end{bmatrix}.$$

Using (2.5)–(2.7), we can choose $w_0(t)=L_0t, w(t)=e^{\frac{1}{L_0}}t, v(t)=e^{\frac{1}{L_0}}, L_0=e-1.$

Then, the radius of convergence r is given by

$$r_1=r_2=0.3827,\ r_3=0.0432=r.$$

EXAMPLE 3.2. *Returning back to the motivational example given at the introduction of this study, we can choose (see also Remark 2.2 (5) for function v)* $w_0(t)=w(t)=\frac{1}{8}(\frac{3}{2}\sqrt{t}+1)$ *and* $v(t)=1+w_0(r_0), r_0\simeq 4.7354$. *Then, the radius of convergence r is given by*

$$r=r_1=r_2=2.6302, r_3=4.7311=r.$$

We choose $r=1$, *to also satisfy (2.12).*

4. Conclusion

We use Lipschitz-type conditions and hypotheses only on the first Fréchet-derivative to provide a local convergence analysis for method (1.2) in a Banach space setting. Our analysis includes computable radius of convergence. error bounds and a uniqueness result not given in [19] or earlier similar works [1–18] using higher than one order Fréchet derivatives although these derivatives do not appear in the methods. Hence, we extend the applicability of these methods under weaker conditions.

References

[1] S. Amat, S. Busquier, Advances in iterative methods for nonlinear equations, SEMS SIMAI springer series, Switzerlands, 2016.

[2] I.K. Argyros, Computational theory of iterative methods. Series: Studies in Computational Mathematics, 15, Editors: C.K.Chui and L. Wuytack, Elsevier Publ. Co. New York, U.S.A, 2007.

[3] I.K.Argyros and F.Szidarovszky, The theory and Applications of Iterative methods,CRC Press,Boca Raton Florida,USA,1993.

[4] I.K. Argyros and H. Ren, Improved local analysis for certain class of iterative methods with cubic convergence, Numerical Algorithms, 59(2012), 505-521.

[5] I. K. Argyros and A. A. Magreñán, Local convergence analysis of proximal Gauss-Newton method for penalized nonlinear least square problems, Appl. Math. Comput.V. 241, 15(2014), 401–408

[6] I.K.Argyros, Yeol Je Cho and S. George, Local convergence for some third-order iterative methods under weak conditions, J. Korean Math. Soc. 53 (2016), No. 4, pp. 781–793.

[7] I.K.Argyros and S. George, Ball convergence of a sixth order iterative method with one parameter for solving equations under weak conditions, SSN 0008-0624, Calcolo, DOI 10.1007/s10092-015-0163-y.

[8] I. K. Argyros and A. A. Magreñán, Iterative methods and their dynamics with applications, CRC Press, New York, USA, 2017.

[9] A. Cordero, J. Hueso, E. Martinez, J. R. Torregrosa, A modified Newton-Jarratt's composition, Numer. Algor. 55, (2010), 87-99.

[10] A. Cordero, J. R. Torregrosa, Variants of Newton's method for functions of several variables, Appl.Math. Comput. 183,(2006), 199-208.

[11] A. Cordero, J. R. Torregrosa, Variants of Newton's method using fifth order quadrature formulas, Appl.Math. Comput. 190,(2007), 686-698.

[12] G.M Grau-Sanchez, A.Grau, M .Noguera, On the computational efficiency index and some iterative methods for solving systems of non-linear equations, J. Comput. Appl Math., 236 (2011), 1259-1266.

[13] J. M. Gutiérrez and M. A. Hernández, Newton's method under weak Kantorovich conditions, IMA J. Numer. Anal., 20 (2000), 521–532.

[14] M. A. Noor and M. Waseem, Some iterative methods for solving a system of nonlinear equations, Computers and Mathematics with Applications, 57, (2009), 101–106.

[15] W.C. Rheinboldt,An adaptive continuation process for solving systems of nonlinear equations, In: Mathematical models and numerical methods (A.N.Tikhonov et al. eds.) pub.3, (1977), 129-142 Banach Center, Warsaw Poland.

[16] J.R. Sharma, P.K. Gupta, An efficient fifth order method for solving systems of nonlinear equations, Comput. Math. Appl. 67, (2014), 591–601.

[17] F. A. Shah and M. A. Noor, Some numerical methods for solving nonlinear equations by using decomposition technique, Appl. Math. Comput. 251(2015), 378–386.

[18] J.F.Traub, Iterative methods for the solution of equations, AMS Chelsea Publishing, 1982.

[19] X. Y. Xiao and H. W. Yin, Increasing the order of convergence for iterative methods to solve nonlinear equations, Calcolo, (2015), DOI 10.1007/ s10092-015-0149-9.

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