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A computational technique for the solution of high-order fractional Volterra integro-differential equations

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Abstract

The optimal q-homotopy analysis method has been employed in order to solve high-order Volterra integro-differential equations featuring time-fractional derivative. Then, in order to illustrate the simplicity and ability of the suggested approach, some specific and clear examples have been given. All numerical calculations in this manuscript have been carried out with *Mathematica*.

Keywords

Nonlinear fractional integro-differential equation, Optimal q-homotopy analysis method, Caputo derivative.

AMS Subject Classification

26A33, 30E25, 34A12, 34A34, 34A37, 37C25, 45J05.

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Contents

1. Introduction

We have suggested the optimal q-homotopy analysis method to find a solution for the following high-order fractional Volterra integro-differential equations (FVIDEs):

$$D^{\mu}v(x)-\mu\int_0^x k(x,t)G[v(t)]dt=g(x),$$
$$0<x<b,\ m-1<\mu\leq m,\ m\in Z^+ \tag{1.1}$$

subjected to the following condition:

$$v(0)=\gamma_0,\ v^{(i)}(0)=\gamma_i,\ v(b)=\theta_0,\ v^{(i)}(b)=\theta_i, \tag{1.2}$$

where D^{μ} is the fractional derivative in the Caputo sense, $k(x,t)$ and $g(x)$ are given and can be approximated by Taylor polynomials, $v(x)$ is the solution to be determined, $G[v(x)]$ is any nonlinear function, γ_0, γ_i, θ_0, $\theta_i\big(i=2r$ $(r\in Z^+, 1\leq r<\lfloor\frac{m}{2}\rfloor)\big)$ and μ are real constants, $v^{(i)}(*)$ denotes the value for i-order derivative of $v(x)$ at $*$, g is given and can be approximated by Taylor polynomials.

The present research has been conducted in order to use the homotopy analysis method (HAM) by Liao [1] and further to use its application in order to solve nonlinear partial differential equations featuring time fractional derivative. There exists a particular auxiliary parameter h in HAM which provides us with a simple approach to adjust and control the convergence region and rate of convergence of the series solution. Furthermore, through using the so-called h-curve, it is easily possible to determine the valid regions of h to obtain a convergent series solution. El-Tawil and Huseen [2] proposed a method called q-homotopy analysis method (q-HAM) which is considered a more general method of HAM. There exists an auxiliary parameter n and h in q-HAM such that in the case of $n=1$, the standard homotopy analysis method can be obtained.

Recently, a new method called optimal q-homotopy analysis method (Oq-HAM) has been introduced and further developed by Huseen et al. [3]. The advantage of this method in comparison with HAM and q-HAM is that in this method it is not necessary to determine the h-curve.

There are some more books related to fractional calculus for interested readers [4, 5]. It should be noted that there are no accurate analytical solutions for most fractional Volterra integro-differential equations . Consequently, for such equations we have to employ some direct and iterative methods. Researchers have used various methods to solve Volterra integro-differential equations in recent years. Some familiar methods as follows: variational iteration method [6, 7], homotopy perturbation method [8–10], Adomian's decomposition method [11], homotopy analysis method [12] and collocation method [13–15].

The organization of this paper is as follows: we have presented some basic idea of the optimal Oq-homotopy analysis method in section 2. In section 3 the convergence of the suggested method is explained. Following that, in section 4, the application of Oq-HAM to the high-order fractional Volterra integro-differential equations is illustrated, and some numerical examples are presented. Finally, in section 5, some conclusions regarding the proposed method are drawn.

2. Description the optimal q-homotopy analysis method

To describe the essential ideas of the Oq-HAM for PDEs featuring time-fractional derivative, consider

$$N[v(x)] = f(x), \tag{2.1}$$

In which N is linear and nonlinear operator, x and t signify the independent variables, and $v(x)$ is an indeterminate function and D^{μ} denotes that Caputo fractional of order $l-1<\mu\leq l$. At first construct the zero-order deformation equation can be written as:

$$(1-mq)L[\phi(x,t;q)-v_0(x)]q\,hH(x)\left(N[\phi(x,t;q)]-f(x)\right), \tag{2.2}$$

where

- $m>1$,
- $q\in[0,\,\frac{1}{m}]$ is the embedding parameter,
- $h\neq 0$ is an supportive parameter,
- $H(x)\neq 0$ is an supportive function,
- L is an supportive linear operator
- $v_0(x)$ is an primary speculation.

Obviously, when $q=0$ and $q=\frac{1}{m}$, Eq.(2.2) turns to:

$$\phi(x,t;0)=v_0(x),\quad \phi\left(x,t;\frac{1}{m}\right)=v(x), \tag{2.3}$$

respectively. Thus, q increases from 0 to $\frac{1}{m}$, the solution $\phi(x,t;q)$ varies from the primary speculation $v_0(x)$ to the solution $v(x)$. If $v_0(x)$, L, h and $H(x)$ are selected suitable, solution of Eq.(2.3) exists for $q\in\left[0,\,\frac{1}{m}\right]$.
Now consider Taylor series expression of $\phi(x,t;q)$ with regard to q in

$$\varphi(x,t;q)=\sum_{n=0}^{\infty}v_n(x)q^n, \tag{2.4}$$

where

$$\varphi_n(x)=\frac{1}{n!}\frac{\partial^n\varphi_n(x,t;q)}{\partial q^n}\bigg|_{q=0}. \tag{2.5}$$

It is supposed that the supportive linear operator, the primary speculation, the supportive parameter h and the supportive function $H(x)$ are selected such that the series (2.4) is convergent when $q\to\frac{1}{m}$, then the approximate solution (2.4) can be represented as:

$$v(x)=\varphi\left(x,t;\frac{1}{m}\right)=\sum_{n=0}^{\infty}v_n(x)\left(\frac{1}{m}\right)^n. \tag{2.6}$$

Then we can define the vector

$$\vec{v}_m(t)=\{v_0(x),v_1(x),v_2(x),\ldots,v_m(x)\}. \tag{2.7}$$

From (2.2), n times with regard to q, then setting $q=0$, the nth-order deformation equation [1] is achieved as

$$L[v_n(x)-\chi_n v_{n-1}v(x)]=hH(x)\mathscr{R}_n\left(\vec{v}_{n-1}(x)\right), \tag{2.8}$$

with initial conditions

$$v_n^{(k)}(x)=0,\quad k=0,1,2,3,\ldots,n-1, \tag{2.9}$$

where

$$\mathscr{R}_n\left(\vec{v}_{n-1}(x)\right)=\frac{1}{(n-1)!}\frac{\partial^{n-1}N(x,t;q)}{\partial q^{n-1}}\bigg|_{q=0}-\left(f(x)-\frac{\chi_n}{m}f(x)\right). \tag{2.10}$$

and

$$\chi_n=\begin{cases}0, & n\leq 1,\\ m, & n>1.\end{cases} \tag{2.11}$$

Operating the Rimann-Liouvile integral operator I^{μ} on both side of (2.8):

$$v_n(x)=\chi_n v_{n-1}(x)-\chi_n\left(\sum_{k=0}^{l-1}v_{n-1}^{(k)}(x,0^+)\frac{t^k}{k!}\right)+hH(x)I^{\mu}\mathscr{R}_n\left(\vec{v}_{n-1}(x)\right). \tag{2.12}$$

Remark 2.1. *The $v_n(x)$ for $n\geq 1$ is decreed by the linear equation (2.8) with boundary conditions that result from the initial problem. As a result of the existence of the factor $\left(\frac{1}{m}\right)^n$, there will be more chance for the occurrence of convergence or even we can achieve faster convergence in comparison with the standard HAM.*

Remark 2.2. *In theory, Liao [1] and Yabushita et al. [16] suggested optimization method to find out the optimal convergence control parameters by minimum of the square residual error integrated in the whole region having physical meaning. Assume $\Delta(h)$ denote the square residual error of the governing equation (2.1) and expressed as:*

$$\Delta(h) = \int_{\Omega} (N[u_n(t)])^2 \, d\Omega, \tag{2.13}$$

where

$$u_M(t) = \sum_{i=0}^{M} u_i(t). \tag{2.14}$$

The optimal value of the auxiliary parameter h is given by a nonlinear algebraic equation

$$\frac{d}{dh}\Delta(h) = 0. \tag{2.15}$$

We apply square residual error (2.13), integrated in the entire region of interest Ω, at the order of approximation M.

3. Test example

The optimal q-HAM will be employed to solve high-order fractional Volterra integro-differential equations. All of the plots and computations for these equations have been carried out with *Mathematica*.

Example 3.1. *Let us now Consider the Volterra integro-differential equation [18]:*

$$D^{\mu} v(x) + \int_0^x v(t)dt = 1, \; 0 \le x \le 1, \; 0 \le \mu \le 1, \tag{3.1}$$

with the initial condition

$$v(0) = 0. \tag{3.2}$$

From Eq. (2.12) and Eqs. (3.1-3.2), the result will be:

$$\begin{aligned}
v_0(x) =& 0,\\
v_1(x) =& -\frac{hx^{\mu}}{\Gamma(\mu+1)},\\
v_2(x) =& h^2 x^{\mu}\left(-\frac{1}{\Gamma(\mu+1)} - \frac{x^{\mu+1}}{\Gamma(2\mu+2)}\right) - \frac{hmx^{\mu}}{\Gamma(\mu+1)},\\
v_3(x) =& h^2 x^{\mu}\left(x^{\mu+1}\left(-\frac{2h+m}{\Gamma(2\mu+2)} - \frac{hx^{\mu+1}}{\Gamma(3\mu+3)}\right) - \frac{h+m}{\Gamma(\mu+1)}\right) + \\
& m\left(h^2 x^{\mu}\left(-\frac{1}{\Gamma(\mu+1)} - \frac{x^{\mu+1}}{\Gamma(2\mu+2)}\right) - \frac{hmx^{\mu}}{\Gamma(\mu+1)}\right),\\
& \dots .
\end{aligned}$$

Then, considering the first four sentences with $m = 1$, as estimates of solution for Eq.(3.1) is given by

$$\begin{aligned}
v(x) \approx & -\frac{hx^{\mu}}{\Gamma(\mu+1)} + h^2 x^{\mu}\left(-\frac{1}{\Gamma(\mu+1)} - \frac{x^{\mu+1}}{\Gamma(2\mu+2)}\right) - \\
& \frac{hmx^{\mu}}{\Gamma(\mu+1)} + h^2 x^{\mu} \times \\
& \left(x^{\mu+1}\left(-\frac{2h+m}{\Gamma(2\mu+2)} - \frac{hx^{\mu+1}}{\Gamma(3\mu+3)}\right) - \frac{h+m}{\Gamma(\mu+1)}\right) \\
& + n\left(h^2 x^{\mu}\left(\frac{-1}{\Gamma(\mu+1)} - \frac{x^{\mu+1}}{\Gamma(2\mu+2)}\right) - \frac{hnx^{\mu}}{\Gamma(\mu+1)}\right).
\end{aligned} \tag{3.3}$$

The solution that we have found is equivalent to the exact solution in a closed form $v(x) = sin(x)$, which is the same third order term approximate solution for Eq. (3.1)-(3.2). In Table 1, we can see the approximate solutions for $\mu = 1.0$, $h = -1$ and $m = 1$, which are derived for different values of x. With regarding to remark 2.2 for (2.10), when $\mu = 1$, $m = 1$, *$0 \le x \le 1$, the optimal value of h is determined by solving nonlinear algebraic equation 2.2 has its minimum value at $h = -0.986376$. Fig. 2 shows the h-curve at the third-order of approximation of the O-qHAM for various values of $0 \le x \le 1$ with $m = 1$ and $\mu = 1$ fixed.*

Table 1. Approximate result of test example 3.1.

x	u_{VHPIM}	u_{Oq-HAM}	*Exact*	*Absolute* error
0.0	0.0	0.0	0.0	0.e
0.2	0.198669	0.198669	0.198669	2.53827e-9
0.4	0.389418	0.389419	0.389418	3.24358e-7
0.6	0.564642	0.564648	0.564642	5.5266e-6
0.8	0.717356	0.717397	0.717356	0.000041242e
1.0	0.841470	0.841667	0.841471	0.000195682e

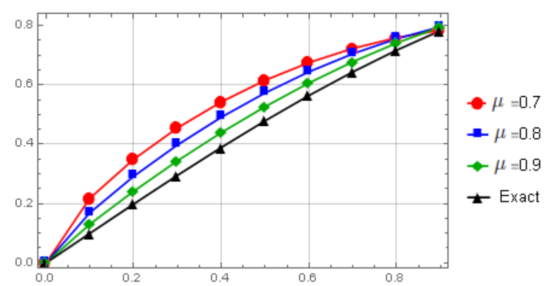

Figure 1. Comparison of the fourth order approximate solution (3.1) with exact solution for different value of μ

Example 3.2. *Let us now Consider the Volterra integro-differential equation [18]:*

$$D^{\mu} v(x) - \int_0^x (x-t)v(t)dt = 1, \; 0 \le x \le 1, \; 1 \le \mu \le 2, \tag{3.4}$$

with the initial condition

$$v(0) = 0, \qquad v'(0) = 0 \, . \tag{3.5}$$

From Eq. (2.12) and Eqs. (3.4)-(3.5), the result will be:

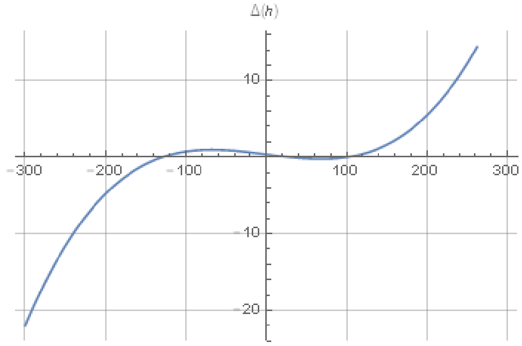

Figure 2. the h-curve at the third-order of approximation of the O-qHAM.

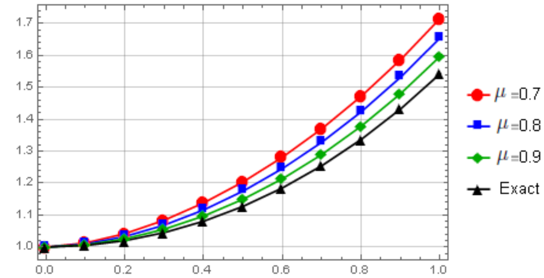

Figure 3. Comparison of the fourth order approximate solution (3.4) with exact solution for different value of μ

Table 2. Approximate result of test example 3.2.

x	u_{VHPIM}	u_{Oq-HAM}	*Exact*	*Absolute* error
0.0	1.0	1.0	1.0	0.e
0.2	1.020066941	1.02007	1.02007	0.e
0.4	1.081085602	1.08107	1.08107	0.e
0.6	1.185642306	1.18547	1.18547	8.88178e-15
0.8	1.338637450	1.33743	1.33743	5.0604e-13
1.0	1.548685515	1.54308	1.54308	1.1519e-11

$$
\begin{aligned}
v_0(x) &= 1,\\
v_1(x) &= -\frac{hx^{\mu}\left((\mu+1)(\mu+2)+x^2\right)}{\Gamma(\mu+3)},\\
v_2(x) &= h^2x^{\mu}\left(\frac{x^{\mu+2}\left(2(\mu+2)(2\mu+3)+x^2\right)}{\Gamma(2\mu+5)}-\right.\\
&\left.\frac{\mu^2+3\mu+x^2+2}{\Gamma(\mu+3)}\right)-\\
&\frac{hmx^{\mu}\left((\mu+1)(\mu+2)+x^2\right)}{\Gamma(\mu+3)},\\
&\dots .
\end{aligned}
$$

Then, the third order term approximate solution for Eq.(3.4) is given by

$$
\begin{aligned}
v(x) \approx 1 &- \frac{hx^{\mu}\left((\mu+1)(\mu+2)+x^2\right)}{\Gamma(\mu+3)}-\\
&\frac{hmx^{\mu}\left((\mu+1)(\mu+2)+x^2\right)}{\Gamma(\mu+3)}+\\
&h^2x^{\mu}\left(\frac{x^{\mu+2}\left(2(\mu+2)(2\mu+3)+x^2\right)}{\Gamma(2\mu+5)}-\right.\\
&\left.\frac{\mu^2+3\mu+x^2+2}{\Gamma(\mu+3)}\right). \qquad (3.6)
\end{aligned}
$$

The solution that we have found is equivalent to the exact solution in a closed form $v(x) = cosh(x)$,which is the same third order term approximate solution for Eq. (3.4,3.5).
In table 2, we can see the approximate solutions for $\mu = 2.0$ which are derived for different values of x. With regarding to remark 2.2 for (3.6), when $\mu = 2$, $m = 1$, $0 \le x \le 1$, the optimal value of h is determined by solving nonlinear algebraic equation 2.2 has its minimum value at $h = -1.00062$. Fig. 4 shows the h-curve at the third-order of approximation of the O-qHAM for various values of $0 \le x \le 1$ with $m = 1$ and $\mu = 2$ fixed.

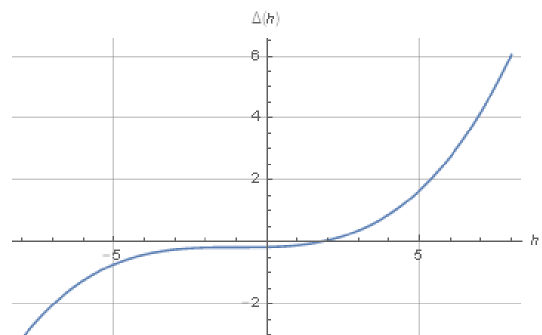

Figure 4. the h-curve at the third-order of approximation of the O-qHAM.

Example 3.3. *Let us now Consider the Volterra integro-differential equation [18]:*

$$
D^{\mu}v(x) - \int_0^x v(t)dt = -1,\ 0 \le x \le 1,\ 2 \le \mu \le 3, \qquad (3.7)
$$

with the initial condition

$$
v(0) = v'(0) = 0, \qquad v''(0) = -1\ . \qquad (3.8)
$$

From Eq. (2.12) and Eqs. (3.7)-(3.8), the result will be:

$$v_0(x) = 1 + x - \frac{x^2}{2},$$
$$v_1(x) = \frac{hx^{\mu}(\mu+2)(\mu+3)}{\Gamma(\mu+4)}\left(\mu+1+x^3-(\mu+3)x^2-x\right),$$
$$v_3(x) = \frac{hmx^{\mu}}{\Gamma(\mu+4)}\big((\mu+1)(\mu+2)(\mu+3)+x^3- (\mu+3)x^2-(\mu+2)(\mu+3)x\big)+ \frac{(\mu+4)h^3x^{2\mu+1}}{3\Gamma(2\mu+9)}\Big((\mu+5)(\mu+6)(\mu+7)(x^5- x^6+12x^2)-(\mu+2)(\mu+3)(2\mu+3)(2\mu+5)\times (2\mu+7)(48(\mu+1)+24(\mu+3)x)+ \mu+3\mu+52\mu+7\left(\mu(\mu+4)\left(2x^4- 2(\mu(\mu+7)+18)x^3\right)+6\right)\Big),$$
$$\ldots.$$

Then, the fourth order term approximate solution for Eq.(3.7)

Table 3. Approximate result of test example 3.3.

x	u_{VHPIM}	u_{Oq-HAM}	*Exact*	*Absolute* error
0.0	1.0	1.0	1.0	0.e
0.1	1.094187910	1.09468	1.09484	0.000162418e
0.2	1.115612006	1.17747	1.17874	0.00126408e
0.3	1.136046057	1.24671	1.25086	0.00414319e
0.4	1.15542339	1.30096	1.31048	0.00951983e
0.5	1.285059327	1.33902	1.35701	0.0179881e

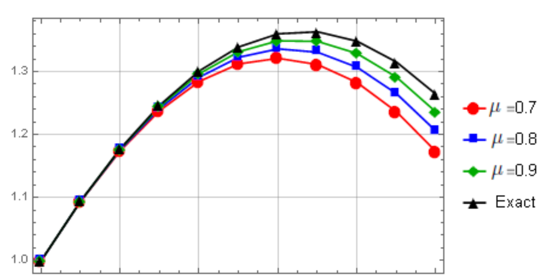

Figure 5. Comparison of the fourth order approximate solution (3.7) with exact solution for different value of μ

is given by

$$v(x) \approx 1 + x - \frac{x^2}{2}, \frac{hx^{\mu}(\mu+2)(\mu+3)}{\Gamma(\mu+4)}\left((\mu+1)+x^3-(\mu+3)x^2- x\right)+\frac{hmx^{\mu}}{\Gamma(\mu+4)}\big((\mu+1)(\mu+2)(\mu+3)+x^3- (\mu+3)x^2-(\mu+2)(\mu+3)x\big)+ \frac{(\mu+4)h^3x^{2\mu+1}}{3\Gamma(2\mu+9)}\Big((\mu+5)(\mu+6)(\mu+7)(x^5- x^6+12x^2)-(\mu+2)(\mu+3)(2\mu+3)(2\mu+5)\times (2\mu+7)(48(\mu+1)+24(\mu+3)x)+ \mu+3\mu+52\mu+7\left(\mu(\mu+4)\left(2x^4- 2(\mu(\mu+7)+18)x^3\right)+6\right)\Big). \tag{3.9}$$

The solution found here for $\mu = 1$, is consistent with the accurate solution $v(x) = cos(x) + sin(x)$, which is the same third order term approximate solution for Eq. (3.7)-(3.8).
In table 3, we can see the approximate solutions for $\mu = 3.0$ which are derived for different values of x.

With regarding to remark 2.2 for (3.9), when $\mu = 3$, $m = 1$, $0 \le x \le 1$, the optimal value of h is determined by solving nonlinear algebraic equation 2.2 has its minimum value at $h = -67.9259$. Fig. 6 shows the h-curve at the third-order of approximation of the O-qHAM for various values of $0 \le x \le 1$ with $m = 1$ and $\mu = 3$ fixed.

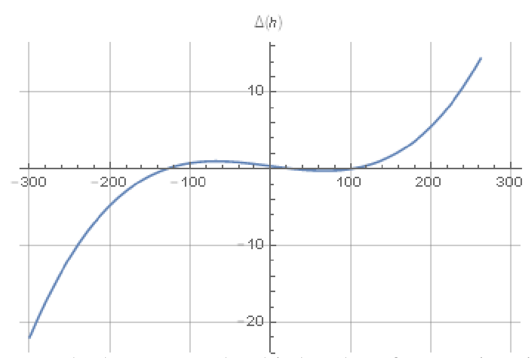

Figure 6. the h-curve at the third-order of approximation of the O-qHAM.

4. Conclusion

We have successfully applied Oq-HAM to obtain series solution of the high-order fractional Volterra integro-differential equations. The result indicate that a few iteration of Oq-HAM will result in some useful solutions.

Finally, it should be added that the suggested approach has the potentials to be applied in solving other similar nonlinear problems in partial differential equations of fractional order.

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