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Global dynamics of $(1,2)-$ type systems of difference equations

Muhammad Naeem Qureshi[1] and Abdul Qadeer Khan[2]*

Abstract
We study the global dynamics of following $(1,2)-$ type systems of difference equations:

$$x_{n+1}=\frac{\eta y_{n-1}}{1+\mu x_{n-2}^{p}},\ y_{n+1}=\frac{\mu x_{n-1}}{1+\eta y_{n-2}^{p}},$$

$$x_{n+1}=\frac{\eta y_{n-1}}{1+\mu y_{n-2}^{p}},\ y_{n+1}=\frac{\mu x_{n-1}}{1+\eta x_{n-2}^{p}},$$

where $\eta,\ \mu,\ p$ and initial conditions $x_l,\ y_l,\ l=-2,-1,0$ are non-negative real numbers. Several numerical simulations are provided to support obtained results.

Keywords
$(1,2)-$ type systems of difference equations; equilibrium point; stability; rate of convergence

AMS Subject Classification
39A10, 40A05.

[1,2] *Department of Mathematics, University of Azad Jammu & Kashmir, Muzaffarabad 13100, Pakistan.*
***Corresponding author**: [1] nqureshi58@gmail.com; [2]abdulqadeerkhan1@gmail.com
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Contents

1. Introduction

Difference equations and systems of rational difference equations play a vital role in the development of different sciences ranging from life to decision sciences. This made the study of qualitative behavior of difference equations an active area of research (see [1–17] and references cited therein). For instance, Touafek and Elsayed [18, 19] investigated the behavior of following systems of difference equations:

$$x_{n+1}=\frac{y_n}{x_{n-1}(\pm 1\pm y_n)},\ y_{n+1}=\frac{x_n}{y_{n-1}(\pm 1\pm x_n)},$$

and

$$x_{n+1}=\frac{x_{n-3}}{\pm 1\pm x_{n-3}y_{n-1}},\ y_{n+1}=\frac{y_{n-3}}{\pm 1\pm y_{n-3}x_{n-1}}.$$

Kalabuŝić *et. al.* [20] investigated the behavior of following systems of difference equations:

$$x_{n+1}=\frac{\alpha_1+\beta_1 x_n}{A_1+y_n},\ y_{n+1}=\frac{\gamma_2 y_n}{A_2+B_2 x_n+y_n}.$$

Kurbanli *et. al.* [21] investigated the behavior of following system of difference equation:

$$x_{n+1}=\frac{x_{n-1}}{y_n x_{n-1}+1},\ y_{n+1}=\frac{y_{n-1}}{x_n y_{n-1}+1}.$$

El-Owaidy *et. al.* [22] studied the behavior of following difference equations:

$$x_{n+1}=\frac{\alpha x_{n-1}}{\beta+\gamma y_{n-2}^{p}},$$

with positive parameters as well as initial conditions.

Recently, Gümüş and Soykan [23] investigated the behavior of following system of difference equations:

$$u_{n+1}=\frac{\alpha u_{n-1}}{\beta+\gamma v_{n-2}^{p}},\ v_{n+1}=\frac{\alpha_1 v_{n-1}}{\beta_1+\gamma_1 u_{n-2}^{p}},$$

where $\alpha,\ \beta,\ \gamma,\ \alpha_1,\ \beta_1,\ \gamma_1, p$ and $u_{-2},\ u_{-1},\ u_0,\ v_{-2},\ v_{-1},\ v_0$ are positive real numbers. Motivated from above said work, this paper deals with the study of global dynamics of following $(1,2)-$type systems of difference equations:

$$u_{n+1}=\frac{\alpha v_{n-1}}{\beta+\gamma u_{n-2}^p},\ v_{n+1}=\frac{\alpha_1 u_{n-1}}{\beta_1+\gamma_1 v_{n-2}^p}, \tag{1.1}$$

$$u_{n+1}=\frac{\alpha v_{n-1}}{\beta+\gamma v_{n-2}^p},\ v_{n+1}=\frac{\alpha_1 u_{n-1}}{\beta_1+\gamma_1 v_{n-2}^p}, \tag{1.2}$$

where $\alpha,\ \beta,\ \gamma,\ \alpha_1,\ \beta_1,\ \gamma_1,\ p$ and $u_{-2},\ u_{-1},\ u_0,\ v_{-2},\ v_{-1},\ v_0$ are positive real numbers. It is noted that using following transformations:

$$u_n=\left(\frac{\beta\beta_1}{\gamma\gamma_1}\right)^{\frac{1}{p}}x_n,\ v_n=\left(\frac{\beta\beta_1}{\gamma\gamma_1}\right)^{\frac{1}{p}}y_n$$

systems (1.1) and (1.2) then becomes

$$x_{n+1}=\frac{\eta y_{n-1}}{1+\mu x_{n-2}^p},\ y_{n+1}=\frac{\mu x_{n-1}}{1+\eta y_{n-2}^p}, \tag{1.3}$$

$$x_{n+1}=\frac{\eta y_{n-1}}{1+\mu y_{n-2}^p},\ y_{n+1}=\frac{\mu x_{n-1}}{1+\eta x_{n-2}^p}, \tag{1.4}$$

where

$$\eta=\frac{\alpha}{\beta},\ \mu=\frac{\alpha_1}{\beta_1}.$$

2. Main Finding

This section deals with the study of main results. Before giving the following Theorems regarding the local stability about $O(0,0)$, we construct corresponding linearized form of systems (1.3) and (1.4). The corresponding Jacobian matrix of system (1.3) about $(\bar{x},\bar{y})$ is

$$J_{(\bar{x},\bar{y})}=\begin{pmatrix} 0 & 0 & -\frac{\eta\mu p\bar{y}\bar{x}^{p-1}}{(1+\mu\bar{x}^p)^2} & 0 & \frac{\eta}{1+\mu\bar{x}^p} & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & \frac{\mu}{1+\eta\bar{y}^p} & 0 & 0 & 0 & -\frac{\eta\mu p\bar{x}\bar{y}^{p-1}}{(1+\eta\bar{y}^p)^2}\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Similarly, the Jacobian matrix of system (1.4) about $(\bar{x},\bar{y})$ is

$$J_{(\bar{x},\bar{y})}=\begin{pmatrix} 0 & 0 & 0 & 0 & \frac{\eta}{1+\mu\bar{y}^p} & -\frac{\eta\mu p\bar{y}^p}{(1+\mu\bar{y}^p)^2}\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & \frac{\mu}{1+\eta\bar{x}^p} & -\frac{\eta\mu p\bar{x}^p}{(1+\eta\bar{x}^p)^2} & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

2.1 Existence of equilibrium and local stability

Theorem 2.1. *For all parameter values η and μ, systems (1.3) and (1.4) have a unique equilibrium point $O(0,0)$.*

Both the above Jacobian matrices have the same eigenvalues at O: $\lambda_{1,2}$. Consequently we have the following result:

Theorem 2.2. *(i) For system (1.3) following hold:*

(i.1) O is locally asymptotically stable if $\eta<1$ and $\mu<1$;

(i.2) O is unstable if $\eta>1$ or $\mu>1$.

(ii) For system (1.4) following hold:

(ii.1) O is locally asymptotically stable if $\eta<1$ and $\mu<1$;

(ii.2) O is unstable if $\eta>1$ or $\mu>1$.

Proof. $(i.1)$. The linearized system of (1.3) about O is

$$\varpi_{n+1}=J_{(0,0)}\varpi_n,$$

where

$$\varpi_n=\begin{pmatrix} x_n\\ x_{n-1}\\ x_{n-2}\\ y_n\\ y_{n-1}\\ y_{n-2}\end{pmatrix},\ J_{(0,0)}=\begin{pmatrix} 0&0&0&0&\eta&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&\mu&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\end{pmatrix}.$$

The roots of characteristic equation of $J_{(0,0)}$ about O are

$$\kappa_{1,2}=0,\ \kappa_{3,4}=\pm\sqrt[4]{\eta\mu},\ \kappa_{5,6}=\pm\iota\sqrt[4]{\eta\mu}.$$

If $\eta<1$ and $\mu<1$ then all eigenvalues of $J_{(0,0)}$ lie in $|\kappa|<1$. Hence the proof.

$(i.2)$. It is easy to see that if $\eta>1$ or $\mu>1$ then O of system (1.3) is unstable.

(ii). Similarly one can prove (ii). □

Now, we will study the global dynamics of systems (1.3) and (1.4) about the equilibrium point $O(0,0)$.

2.2 Global stability about equilibrium $O(0,0)$

Theorem 2.3. *(i) O of system (1.3) is globally asymptotically stable if $\eta<1$ and $\mu<1$.*

(ii) O of system (1.4) is globally asymptotically stable if $\eta<$ and $\mu<1$.

Proof. (i) In view of Theorem 2.2, it suffices to prove that

$$\lim_{n\to\infty}(x_n,y_n)=(0,0).$$

It is evident from (1.3) that

$$0\le x_{n+1}=\frac{\eta y_{n-1}}{1+\mu x_{n-2}^p}<\eta y_{n-1},$$

and

$$0 \leq y_{n+1} = \frac{\mu x_{n-1}}{1+\eta y_{n-2}^p} < \mu x_{n-1}.$$

Induction then implies that

$$x_{4n-1} < (\eta\mu)^n x_{-1} \text{ and } x_{4n} < (\eta\mu)^n x_0,$$

and

$$y_{4n-1} < (\eta\mu)^n y_{-1} \text{ and } y_{4n} < (\eta\mu)^n y_0.$$

Thus for $\eta < 1$ and $\mu < 1$,

$$\lim_{n\to\infty}(x_n, y_n) = (0,0).$$

Similarly part (ii) can be proved. □

2.3 Prime period two-solutions

Theorem 2.4. *System (1.3) and (1.4) has no prime period-two solutions.*

Proof. Assuming

$$\cdots,(a,b),(c,d),(a,b),(c,d),\cdots,$$

prime period two solutions of the system (1.3) such that $a,\ b,\ c,\ d \neq 0$ and $a \neq c,\ b \neq d$. Then

$$a = \frac{\eta b}{1+\mu c^p},\ b = \frac{\mu a}{1+\eta d^p}, \tag{2.1}$$

and

$$c = \frac{\eta d}{1+\mu a^p},\ d = \frac{\mu c}{1+\eta b^p}. \tag{2.2}$$

A calculation then leads to:

$$(a+c)^2 - 4ac = 0,$$

and

$$(b+d)^2 - 4bd = 0,$$

But it is contrary to our assumption and therefore system (1.3) has no prime period-two solutions. □

2.4 Rate of convergence

Consider

$$\varpi_{n+1} = [G + D(n)]\,\varpi_n, \tag{2.3}$$

where $G \in C^{k\times k}$ is a constant matrix, and $D : \mathbb{Z}^+ \to C^{k\times k}$ is a matrix function satisfying

$$\|D(n)\| \to 0 \tag{2.4}$$

as $n \to \infty$.

Proposition 2.5. *[24] If ϖ_n is a solution of (2.3) such that (2.4) holds. Then following holds:*

(i) Either $\varpi_n = 0,\ \forall\ n > N$ or $\lim_{n\to\infty}(\|\varpi_n\|)^{1/n}$ exists and is equal to the modulus of one of the eigenvalues of matrix C.

(ii) Either $\varpi_n = 0,\ \forall\ n > N$ or $\lim_{n\to\infty}\frac{\|\varpi_{n+1}\|}{\|\varpi_n\|}$ exists and is equal to the modulus of one of the eigenvalues of matrix C.

The following Theorem give the rate of convergence of systems (1.3) and (1.4).

Theorem 2.6. *(i) If conditions (i) of Theorem 2.3 hold then error vector $\varepsilon_n = \begin{pmatrix}\varepsilon_n^1\\ \varepsilon_{n-1}^1\\ \varepsilon_{n-2}^1\\ \varepsilon_n^2\\ \varepsilon_{n-1}^2\\ \varepsilon_{n-2}^2\end{pmatrix}$ of every solution of system (1.3) about O satisfies the both asymptotic relations:*

$$\lim_{n\to\infty}(\|\varepsilon_n\|)^{\frac{1}{n}} = |\kappa_{1,2}J_O|,$$

$$\lim_{n\to\infty}\frac{\|\varepsilon_{n+1}\|}{\|\varepsilon_n\|} = |\kappa_{1,2}J_O|,$$

where $|\kappa_{1,2}J_O|$ is equal to one of the eigenvalues of J_O evaluated at O.

(ii) If conditions (ii) of Theorem 2.3 hold then error vector $\varepsilon_n = \begin{pmatrix}\varepsilon_n^1\\ \varepsilon_{n-1}^1\\ \varepsilon_{n-2}^1\\ \varepsilon_n^2\\ \varepsilon_{n-1}^2\\ \varepsilon_{n-2}^2\end{pmatrix}$ of every solution of system (1.4) about O satisfies the both asymptotic relations:

$$\lim_{n\to\infty}(\|\varepsilon_n\|)^{\frac{1}{n}} = |\kappa_{1,2}J_O|,$$

$$\lim_{n\to\infty}\frac{\|\varepsilon_{n+1}\|}{\|\varepsilon_n\|} = |\kappa_{1,2}J_O|,$$

where $|\kappa_{1,2}J_O|$ is equal to one of the eigenvalues of J_O evaluated at O.

Proof. (i) Let $\{(x_n, y_n)\}$ be any solution of system (1.3) such that $\lim_{n\to\infty} x_n = \bar{x}$, and $\lim_{n\to\infty} y_n = \bar{y}$. Then

$$
\begin{aligned}
x_{n+1}-\bar{x} &= \frac{\eta y_{n-1}}{1+\mu x_{n-2}^{p}}-\frac{\eta \bar{y}}{1+\mu \bar{x}^{p}} \\
&= -\frac{\eta \mu \bar{y}(x_{n-2}^{p}-\bar{x}^{p})}{\left(1+\mu x_{n-2}^{p}\right)\left(1+\mu \bar{x}^{p}\right)(x_{n-2}-\bar{x})}(x_{n-2}-\bar{x})+\frac{\eta}{1+\mu x_{n-2}^{p}}(y_{n-1}-\bar{y}), \\
y_{n+1}-\bar{y} &= \frac{\mu x_{n-1}}{1+\eta y_{n-2}^{p}}-\frac{\mu \bar{x}}{1+\eta \bar{y}^{p}} \\
&= \frac{\mu}{1+\eta y_{n-2}^{p}}(x_{n-1}-\bar{x})-\frac{\eta \mu \bar{x}(y_{n-2}^{p}-\bar{y}^{p})}{\left(1+\eta y_{n-2}^{p}\right)\left(1+\eta \bar{y}^{p}\right)(y_{n-2}-\bar{y})}(y_{n-2}-\bar{y}),
\end{aligned}
$$

that is

$$
\begin{aligned}
x_{n+1}-\bar{x} &= -\frac{\eta \mu \bar{y}(x_{n-2}^{p}-\bar{x}^{p})}{\left(1+\mu x_{n-2}^{p}\right)\left(1+\mu \bar{x}^{p}\right)(x_{n-2}-\bar{x})}(x_{n-2}-\bar{x})+\frac{\eta}{1+\mu x_{n-2}^{p}}(y_{n-1}-\bar{y}), \\
y_{n+1}-\bar{y} &= \frac{\mu}{1+\eta y_{n-2}^{p}}(x_{n-1}-\bar{x})-\frac{\eta \mu \bar{x}(y_{n-2}^{p}-\bar{y}^{p})}{\left(1+\eta y_{n-2}^{p}\right)\left(1+\eta \bar{y}^{p}\right)(y_{n-2}-\bar{y})}(y_{n-2}-\bar{y}).
\end{aligned} \tag{2.5}
$$

Setting

$$\varepsilon_n^1 = x_n-\bar{x},\ \varepsilon_n^2 = y_n-\bar{y},$$

system (2.5) can also be expressed as

$$\varepsilon_{n+1}^1 = g_n\varepsilon_{n-2}^1+h_n\varepsilon_{n-1}^2,$$

$$\varepsilon_{n+1}^2 = i_n\varepsilon_{n-1}^1+j_n\varepsilon_{n-2}^2,$$

where

$$g_n=-\frac{\eta \mu \bar{y}(x_{n-2}^{p}-\bar{x}^{p})}{\left(1+\mu x_{n-2}^{p}\right)\left(1+\mu \bar{x}^{p}\right)(x_{n-2}-\bar{x})},\ h_n=\frac{\eta}{1+\mu x_{n-2}^{p}},$$

$$i_n=\frac{\mu}{1+\eta y_{n-2}^{p}},\ j_n=-\frac{\eta \mu \bar{x}(y_{n-2}^{p}-\bar{y}^{p})}{\left(1+\eta y_{n-2}^{p}\right)\left(1+\eta \bar{y}^{p}\right)(y_{n-2}-\bar{y})}.$$

Taking the limits of $g_n,\ h_n,\ i_n$ and j_n, we obtain

$$\lim_{n\to\infty} g_n=-\frac{\eta\mu p\bar{y}\bar{x}^{p-1}}{(1+\mu\bar{x}^p)^2},\ \lim_{n\to\infty} h_n=\frac{\eta}{1+\mu\bar{x}^p},$$

$$\lim_{n\to\infty} i_n=\frac{\mu}{1+\eta\bar{y}^p},\ \lim_{n\to\infty} j_n=-\frac{\eta\mu p\bar{x}\bar{y}^{p-1}}{(1+\eta\bar{y}^p)^2},$$

that is

$$g_n=-\frac{\eta\mu p\bar{y}\bar{x}^{p-1}}{(1+\mu\bar{x}^p)^2}+A_{n-2},\ h_n=\frac{\eta}{1+\mu\bar{x}^p}+B_{n-1},$$

$$i_n=\frac{\mu}{1+\eta\bar{y}^p}+C_{n-1},\ j_n=-\frac{\eta\mu p\bar{x}\bar{y}^{p-1}}{(1+\eta\bar{y}^p)^2}+D_{n-1},$$

where $A_{n-2},\ B_{n-1},\ C_{n-1},\ D_{n-1}\to 0$ as $n\to\infty$.

Now we have system of the form (2.3)

$$\varepsilon_{n+1}=\left[G+D(n)\right]\varepsilon_n,$$

where

$$G=\begin{pmatrix} 0 & 0 & -\frac{\eta\mu p\bar{y}\bar{x}^{p-1}}{(1+\mu\bar{x}^p)^2} & 0 & \frac{\eta}{1+\mu\bar{x}^p} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{\mu}{1+\eta\bar{y}^p} & 0 & 0 & 0 & -\frac{\eta\mu p\bar{x}\bar{y}^{p-1}}{(1+\eta\bar{y}^p)^2} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$D(n)=\begin{pmatrix} 0 & 0 & A_{n-2} & 0 & B_{n-1} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & C_{n-1} & 0 & 0 & 0 & D_{n-1} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

and

$$\|D(n)\|\to 0, n\to\infty.$$

The limiting system of error terms about $(\bar{x},\bar{y})$ is

$$\begin{pmatrix} \varepsilon_{n+1}^1 \\ \varepsilon_n^1 \\ \varepsilon_{n-1}^1 \\ \varepsilon_{n+1}^2 \\ \varepsilon_n^2 \\ \varepsilon_{n-1}^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{\eta\mu p\bar{y}\bar{x}^{p-1}}{(1+\mu\bar{x}^p)^2} & 0 & \frac{\eta}{1+\mu\bar{x}^p} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{\mu}{1+\eta\bar{y}^p} & 0 & 0 & 0 & -\frac{\eta\mu p\bar{x}\bar{y}^{p-1}}{(1+\eta\bar{y}^p)^2} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_n^1 \\ \varepsilon_{n-1}^1 \\ \varepsilon_{n-2}^1 \\ \varepsilon_n^2 \\ \varepsilon_{n-1}^2 \\ \varepsilon_{n-2}^2 \end{pmatrix}.$$

This is similar to linearized system of (1.3) about $(\bar{x},\bar{y})$. In particular, the limiting system of error term about O of system (1.3) is given by

$$\begin{pmatrix} \varepsilon_{n+1}^1 \\ \varepsilon_n^1 \\ \varepsilon_{n-1}^1 \\ \varepsilon_{n+1}^2 \\ \varepsilon_n^2 \\ \varepsilon_{n-1}^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \eta & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_n^1 \\ \varepsilon_{n-1}^1 \\ \varepsilon_{n-2}^1 \\ \varepsilon_n^2 \\ \varepsilon_{n-1}^2 \\ \varepsilon_{n-2}^2 \end{pmatrix}.$$

This is similar to the linearized system of (1.3) about O.

(ii). Let $\{(x_n,y_n)\}$ be any solution of system (1.4) such that $\lim_{n\to\infty} x_n=\bar{x}$, and $\lim_{n\to\infty} y_n=\bar{y}$. Then

$$\begin{aligned} x_{n+1}-\bar{x} &= \frac{\eta y_{n-1}}{1+\mu y_{n-2}^p} - \frac{\eta\bar{y}}{1+\mu\bar{y}^p} \\ &= \frac{\eta}{1+\mu y_{n-2}^p}(y_{n-1}-\bar{y}) - \frac{\eta\mu\bar{y}(y_{n-2}^p-\bar{y}^p)}{\left(1+\mu y_{n-2}^p\right)(1+\mu\bar{y}^p)(y_{n-2}-\bar{y})}(y_{n-2}-\bar{y}), \\ y_{n+1}-\bar{y} &= \frac{\mu x_{n-1}}{1+\eta x_{n-2}^p} - \frac{\mu\bar{x}}{1+\eta\bar{x}^p} \\ &= \frac{\mu}{1+\eta x_{n-2}^p}(x_{n-1}-\bar{x}) - \frac{\eta\mu\bar{x}(x_{n-2}^p-\bar{x}^p)}{\left(1+\eta x_{n-2}^p\right)(1+\eta\bar{x}^p)(x_{n-2}-\bar{x})}(x_{n-2}-\bar{x}), \end{aligned}$$

that is

$$\begin{aligned} x_{n+1}-\bar{x} &= \frac{\eta}{1+\mu y_{n-2}^p}(y_{n-1}-\bar{y}) - \frac{\eta\mu\bar{y}(y_{n-2}^p-\bar{y}^p)}{\left(1+\mu y_{n-2}^p\right)(1+\mu\bar{y}^p)(y_{n-2}-\bar{y})}(y_{n-2}-\bar{y}), \\ y_{n+1}-\bar{y} &= \frac{\mu}{1+\eta x_{n-2}^p}(x_{n-1}-\bar{x}) - \frac{\eta\mu\bar{x}(x_{n-2}^p-\bar{x}^p)}{\left(1+\eta x_{n-2}^p\right)(1+\eta\bar{x}^p)(x_{n-2}-\bar{x})}(x_{n-2}-\bar{x}). \end{aligned} \tag{2.6}$$

Setting

$$\varepsilon_n^1 = x_n-\bar{x},\ \varepsilon_n^2 = y_n-\bar{y},$$

system (2.6) can also be expressed as

$$\varepsilon_{n+1}^1 = k_n\varepsilon_{n-1}^2 + l_n\varepsilon_{n-2}^2,$$

$$\varepsilon_{n+1}^2 = m_n\varepsilon_{n-1}^1 + n_n\varepsilon_{n-2}^1,$$

where

$$k_n = \frac{\eta}{1+\mu y_{n-2}^p},\ l_n = -\frac{\eta\mu\bar{y}(y_{n-2}^p-\bar{y}^p)}{\left(1+\mu y_{n-2}^p\right)(1+\mu\bar{y}^p)(y_{n-2}-\bar{y})},$$

$$m_n = \frac{\mu}{1+\eta x_{n-2}^p},\ n_n = -\frac{\eta\mu\bar{x}(x_{n-2}^p-\bar{x}^p)}{\left(1+\eta x_{n-2}^p\right)(1+\eta\bar{x}^p)(x_{n-2}-\bar{x})}.$$

Taking the limits of $k_n,\ l_n,\ m_n$ and n_n, we obtain

$$\lim_{n\to\infty} k_n = \frac{\eta}{1+\mu\bar{y}^p},\ \lim_{n\to\infty} l_n = -\frac{\eta\mu p\bar{y}^p}{(1+\mu\bar{y}^p)^2},$$

$$\lim_{n\to\infty} m_n = \frac{\mu}{1+\eta\bar{x}^p},\ \lim_{n\to\infty} n_n = -\frac{\eta\mu p\bar{x}^p}{(1+\eta\bar{x}^p)^2},$$

that is

$$k_n = \frac{\eta}{1+\mu\bar{y}^p} + A_{n-1},\ l_n = -\frac{\eta\mu p\bar{y}^p}{(1+\mu\bar{y}^p)^2} + B_{n-2},$$

$$m_n = \frac{\mu}{1+\eta\bar{x}^p} + C_{n-1},\ n_n = -\frac{\eta\mu p\bar{x}^p}{(1+\eta\bar{x}^p)^2} + D_{n-2},$$

where $A_{n-1},\ B_{n-2},\ C_{n-1},\ D_{n-2} \to 0$ as $n \to \infty$.

Now we have system of the form (2.3)

$$\varepsilon_{n+1} = [G + D(n)]\,\varepsilon_n,$$

where

$$D(n) = \begin{pmatrix} 0 & 0 & 0 & 0 & A_{n-1} & B_{n-2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & C_{n-1} & D_{n-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

and

$$\|D(n)\| \to 0, n \to \infty.$$

The limiting system of error terms about $(\bar{x},\bar{y})$ can then be written as

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{\eta}{1+\mu\bar{y}^p} & -\frac{\eta\mu p\bar{y}^p}{(1+\mu\bar{y}^p)^2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{\mu}{1+\eta\bar{x}^p} & -\frac{\eta\mu p\bar{x}^p}{(1+\eta\bar{x}^p)^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \varepsilon^1_{n+1} \\ \varepsilon^1_n \\ \varepsilon^1_{n-1} \\ \varepsilon^2_{n+1} \\ \varepsilon^2_n \\ \varepsilon^2_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{\eta}{1+\mu\bar{y}^p} & -\frac{\eta\mu p\bar{y}^p}{(1+\mu\bar{y}^p)^2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{\mu}{1+\eta\bar{x}^p} & -\frac{\eta\mu p\bar{x}^p}{(1+\eta\bar{x}^p)^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon^1_n \\ \varepsilon^1_{n-1} \\ \varepsilon^1_{n-2} \\ \varepsilon^2_n \\ \varepsilon^2_{n-1} \\ \varepsilon^2_{n-2} \end{pmatrix}.$$

This is similar to linearized system of (1.4) about $(\bar{x},\bar{y})$. In particular, the limiting system of error term about O of system (1.4) is given by

$$\begin{pmatrix} \varepsilon^1_{n+1} \\ \varepsilon^1_n \\ \varepsilon^1_{n-1} \\ \varepsilon^2_{n+1} \\ \varepsilon^2_n \\ \varepsilon^2_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \eta & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon^1_n \\ \varepsilon^1_{n-1} \\ \varepsilon^1_{n-2} \\ \varepsilon^2_n \\ \varepsilon^2_{n-1} \\ \varepsilon^2_{n-2} \end{pmatrix}.$$

This is similar to the linearized system of (1.4) about O. □

3. Discussion and numerical simulations

In the present work global dynamics of $(1,2)-$ type systems of difference equations has been studied. Our investigations reveal that for all parameter values both the systems under discussion have a unique equilibrium at the origin. Linear stability analysis shows that for both systems, if $\eta < 1,\ \mu < 1$ then $O(0,0)$ is locally asymptotically stable but unstable if $\eta > 1$ or $\mu > 1$. The global asymptotic stability about $O(0,0)$ has also been proved. Finally prime period two solution and rate of convergence of a solution that converges to $O(0,0)$ of systems (1.3) and (1.4) are also demonstrated.

The following numerical data confirm the above theoretical results: $\eta = 0.899,\ \mu = 0.898,\ p = 7.9$ with $x_{-2} = 0.7,\ x_{-1} = 0.9,\ x_0 = 0.7,\ y_{-2} = 0.9,\ y_{-1} = 1.1,\ y_0 = 0.9$. System (1.3) can then be written as:

$$x_{n+1} = \frac{0.899 y_{n-1}}{1+0.898 x^{7.9}_{n-2}},\ y_{n+1} = \frac{0.898 x_{n-1}}{1+0.899 y^{7.9}_{n-2}},\ n = 0,1,\cdots. \tag{3.1}$$

The results of numerical simulations are expressed in Fig. 1. Fig. 1a and Fig. 1b shows the plots of x_n and y_n respectively. More precisely these figures simulate the stable points for system (3.1) whereas its attractor is shown in Fig. 1c. The graphs clearly show that if $\eta = 0.899 < 1$ and $\mu = 0.898 < 1$ then all orbits are attracted to $O(0,0)$. This confirms the statement of Theorem 2.3.

An another example consider the following data: $\eta = \mu = 0.9,\ p = 10$ with $x_{-2} = 1.3,\ x_{-1} = 1.1,\ x_0 = 0.7,\ y_{-2} = 0.7,\ y_{-1} = 0.9,\ y_0 = 0.1$. System (1.3) can then be written as:

$$x_{n+1} = \frac{0.9 y_{n-1}}{1+0.9 x^{10}_{n-2}},\ y_{n+1} = \frac{0.9 x_{n-1}}{1+0.9 y^{10}_{n-2}},\ n = 0,1,\cdots. \tag{3.2}$$

The plot of numerical simulation are presented in Fig. 2.

Figs. 2a and 2b are plots of x_n and y_n respectively. These plots clearly show that for $\eta = \mu = 0.9 < 1$ the origin is a stable point for the system (3.2). And Fig. 2c shows that $O(0,0)$ is attractor, *i.e.* all the orbits are eventually attracted towards the origin, $O(0,0)$. This again proves the correctness of the results obtained in Theorems 2.2 and 2.3.

Now we conclude an example with data where parameters have values grater than one. Consider the following data: $\eta = 2.2,\ \mu = 1.9,\ p = 10$ with $x_{-2} = 0.3,\ x_{-1} = 0.1,\ x_0 = 0.7,\ y_{-2} = 3.7,\ y_{-1} = 0.9,\ y_0 = 0.1$. System (1.3) can then be written as:

$$x_{n+1} = \frac{2.2y_{n-1}}{1+1.9x_{n-2}^{10}},\ y_{n+1} = \frac{1.9x_{n-1}}{1+2.2y_{n-2}^{10}},\ n = 0,1,\cdots. \quad (3.3)$$

Fig. 3a and 3b show plots of x_n and y_n respectively of system (3.3). The plot shows that if the values of parameters $\eta = 2.2 > 1,\ \mu = 1.9 > 1$ then $O(0,0)$ is unstable providing our theoretical discussion about system (1.3).

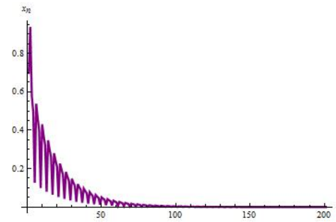
Following two examples are about system (1.4). By considering values of the parameters: if $\eta = 0.97,\ \mu = 0.96,\ p = 1112$ with $x_{-2} = 0.00003,\ x_{-1} = 0.88,\ x_0 = 0.777,\ y_{-2} = 0.88887,\ y_{-1} = 0.9,\ y_0 = 0.31$. System (1.4) can then be written as:

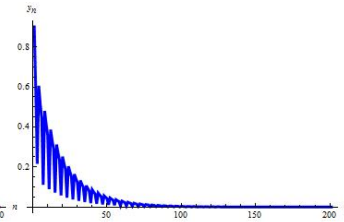
$$x_{n+1} = \frac{0.97y_{n-1}}{1+0.96y_{n-2}^{1112}},\ y_{n+1} = \frac{0.96x_{n-1}}{1+0.97x_{n-2}^{1112}},\ n = 0,1,\cdots. \quad (3.4)$$

Fig. 4 show results of numerical simulations of system (3.4). Figs. 4a and 4b are plots x_n and y_n respectively. The plot show that for a parameter values $\eta = 0.97 < 1$ and $\mu = 0.96 < 1$ the $O(0,0)$ is stable. Whereas Fig. 4c shows that $O(0,0)$ is attractor, *i.e.* all the orbits are eventually attracted towards the origin. Similarly, if $\eta = 0.9784,\ \mu = 0.9777,\ p = 11212,\ x_{-2} = 0.0009,\ x_{-1} = 0.8,\ x_0 = 0.7,\ y_{-2} = 0.7,\ y_{-1} = 0.8,\ y_0 = 0.1$. Then system (1.4) can be written as:

$$x_{n+1} = \frac{0.9784y_{n-1}}{1+0.9777y_{n-2}^{11212}},\ y_{n+1} = \frac{0.9777x_{n-1}}{1+0.9784x_{n-2}^{11212}},\ n = 0,1,\cdots. \quad (3.5)$$

The results of numerical simulation of system (3.5) are presented in Fig. 5. Plots of x_n and y_n are shown in Figs. 5a and 5b, respectively. The plots show that the choose values of parameter $\eta = 0.9784 < 1$ and $\mu = 0.9777 < 1$, the $O(0,0)$ is stable point of system (3.5). Fig. 5c, on the other hand, shows that $O(0,0)$ is attractor, *i.e.* all the orbits are eventually attracted to the origin.


(a)


(b)

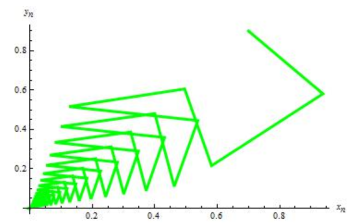
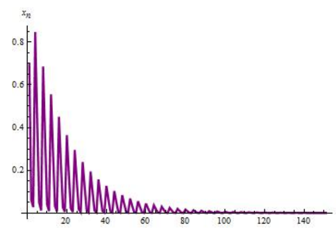
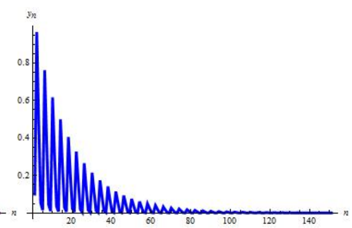

(c)

Figure 1. Stability of system (3.1)


(a)


(b)

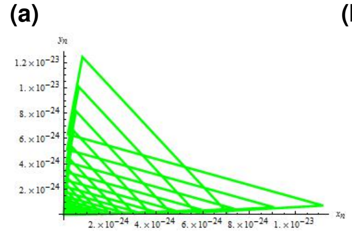
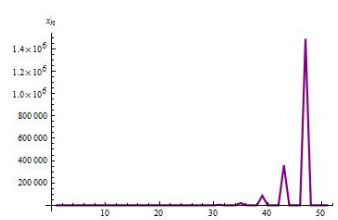

(c)

Figure 2. Stability of system (3.2)


(a)

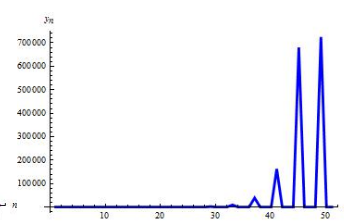
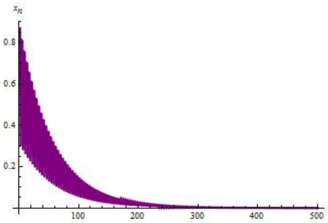
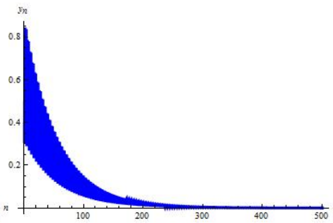
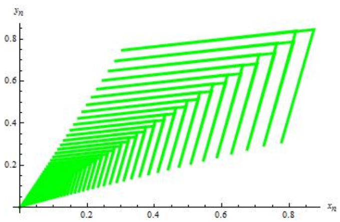

(b)

Figure 3. Stability of system (3.3)

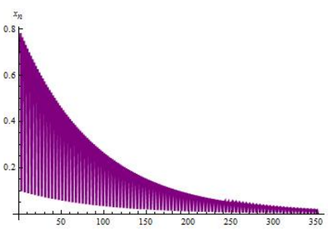
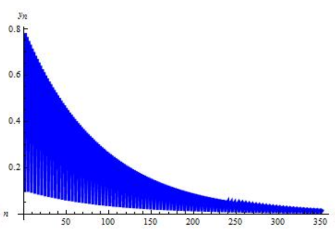



(a) **(b)** Plot of y_n for system (3.4)

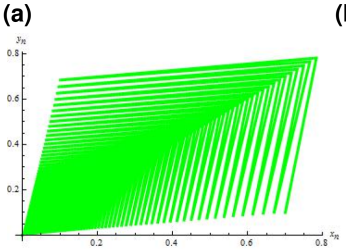


(c)

Figure 4. Stability of system (3.4)

(a) **(b)**



(c)

Figure 5. Stability of system (3.5)

Acknowledgment

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