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Oscillation theorems for second order neutral difference equations with “Maxima”

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Abstract
In this paper we establish some sufficient conditions for the oscillation of all solutions of the equation

$$\Delta(r_n\Delta(x_n+p_nx_{n-k}))+q_n\max_{[n-\ell,n]}x_s^{\alpha}=0, n\in N_0$$

which improve and extend the known results. Examples are provided to illustrate the main results.

Keywords
Second order, oscillation, neutral difference equation with “maxima”.

AMS Subject Classification
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Contents

1. Introduction

This paper deals with the second order neutral difference equation of the form

$$\Delta(r_n\Delta(x_n+p_nx_{n-k}))+q_n\max_{[n-\ell,n]}x_s^{\alpha}=0, n\in N_0, \quad (1.1)$$

subject to the following conditions:

(C_1) $\{r_n\}$ is a positive sequence with $\sum_{n=n_0}^{\infty}\frac{1}{r_n}=\infty$;

(C_2) $\{p_n\}$ is a nonnegtive real sequence with $0\le p_n\le p<\infty$;

(C_3) $\{q_n\}$ is a positive real sequence;

(C_4) k and ℓ are positive integers;

(C_5) α is a ratio of odd positive integer.

Let $\theta=\max\{k,\ell\}$. By a solution of equation (1.1) we mean a real sequence $\{x_n\}$ defined for all $n\ge n_0-\theta$ and satisfying equation (1.1) for all $n\ge n_0$. A solution $\{x_n\}$ is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

From the review of literature it is well known that there is a lot of results available on the oscillatory and asymptotic behavior of solutions of neutral difference equations without maxima, see [1, 2, 9, 10], and the references cited therein. But very few results available in the literature dealing with the oscillatory and asymptotic behavior of solutions of neutral difference equations with “maxima”, see [3, 4, 7, 8] and the references cited therein. Therefore, in this paper, we investigate the oscillatory and asymptotic behavior of all solutions of equation (1.1).

In Section 2, we establish sufficient conditions for the oscillation of all solutions of equation (1.1) and in Section 3, we present some examples to illustrate the main results.

2. Main Results

To prove our main results we need the following lemmas.

Lemma 2.1. *If $A\ge 0, B\ge 0$ and $0<\alpha\le 1$, then*

$$A^{\alpha}+B^{\alpha}\ge(A+B)^{\alpha}. \quad (2.1)$$

Lemma 2.2. *If $A \geq 0, B \geq 0$ and $\alpha > 1$, then*

$$A^{\alpha} + B^{\alpha} \geq \frac{1}{2^{\alpha-1}}(A+B)^{\alpha}. \tag{2.2}$$

For the proof of Lemmas 2.1 and 2.2, see [6].

Lemma 2.3. *If $0 < \alpha < 1$, ℓ is a positive integer and $\{q_n\}$ is a positive real sequence with $\sum_{n=n_0}^{\infty} q_n = \infty$, then every solution of equation*

$$\Delta x_n + q_n x_{n-\ell}^{\alpha} = 0, \tag{2.3}$$

is oscillatory.

Lemma 2.4. *If $\alpha = 1$ and*

$$\liminf_{n\to\infty} \sum_{s=n-\ell}^{n-1} q_s > \left(\frac{\ell}{\ell+1}\right)^{\ell+1}, \tag{2.4}$$

then every solution of equation (2.3) is oscillatory.

Lemma 2.5. *Let $\alpha > 1$. If there exists a $\lambda > \frac{1}{\ell}\log\alpha$ such that*

$$\liminf_{n\to\infty} [q_n exp(-e^{\lambda n})] > 0, \tag{2.5}$$

then every solution of equation (2.3) is oscillatory.

For the proof of Lemmas 2.3 and 2.5, see [9], and Lemma 2.4, see [5].

Lemma 2.6. *The sequence $\{x_n\}$ is an eventually negative solution of equation (1.1) if and only if $\{-x_n\}$ is an eventually positive solution of equation*

$$\Delta(r_n\Delta(x_n + p_n x_{n-k})) + q_n \max_{[n-\ell,n]} x_s^{\alpha} = 0, n \in N_0.$$

The assertion of Lemma 2.6 can be verified easily.

Lemma 2.7. *If $\{x_n\}$ is a positive solution of (1.1), then $z_n = x_n + p_n x_{n-k}$ satisfies*

$$z_n > 0, \ \ r_n\Delta z_n > 0, \ \ \Delta(r_n\Delta z_n) < 0 \tag{2.6}$$

eventually.

Proof. Assume that $\{x_n\}$ is a positive solution of equation (1.1). Then $z_n = x_n + p_n x_{n-k} > 0$ for all $n \geq n_1 \geq n_0$. From the equation (1.1), we have

$$\Delta(r_n\Delta z_n) = -q_n \max_{[n-\ell,n]} x_s^{\alpha} < 0.$$

Consequently, $r_n\Delta z_n$ is nonincreasing and thus either $r_n\Delta z_n > 0$ or $r_n\Delta z_n \leq 0$. If $r_n\Delta z_n \leq 0$ then for $n \geq n_1$, we have

$$r_n\Delta z_n \leq r_{n_1}\Delta z_{n_1} < 0.$$

Dividing the last inequality by r_{n_1} and then summing the resulting inequality from n_1 to $n-1$, we obtain

$$z_n < z_{n_1} + r_{n_1}\Delta z_{n_1} \sum_{s=n_1}^{n-1} \frac{1}{r_s} \to -\infty \ as \ n \to \infty,$$

which is a contradiction for the positivity of z_n. This completes the proof. □

Before stating the next theorem, let us define

$$Q_n = \min\{q_n, q_{n-k}\} \ \ for \ n \in N_0, \tag{2.7}$$

and

$$Q_n^* = \begin{cases} Q_n\left(\sum_{s=n_1}^{n-\ell-1} \frac{1}{r_s}\right)^{\alpha}, & if \ 0 < \alpha \leq 1; \\ Q_n 2^{1-\alpha}\left(\sum_{s=n_1}^{n-\ell-1} \frac{1}{r_s}\right)^{\alpha}, & if \ \alpha \geq 1. \end{cases} \tag{2.8}$$

Theorem 2.8. *Assume that the first order neutral difference inequality*

$$\Delta(y_n + p^{\alpha} y_{n-k}) + Q_n^* \max_{[n-\ell,n]} y_s^{\alpha} \leq 0, \tag{2.9}$$

where $\{Q_n^*\}$ is as defined in (2.8), has no positive solution, then every solution of equation (1.1) is oscillatory.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $x_n > 0$ and $x_{n-k} > 0$ for all $n \geq n_1 \geq n_0 + \theta$. Then $z_n > 0$ and from the equation (1.1), we obtain

$$\Delta(r_n\Delta z_n) + q_n \max_{[n-\ell,n]} x_s^{\alpha} = 0, \tag{2.10}$$

and

$$p^{\alpha}\Delta(r_{n-k}\Delta z_{n-k}) + p^{\alpha} q_{n-k} \max_{[n-k-\ell,n-k]} x_s^{\alpha} = 0. \tag{2.11}$$

Combining (2.10) and (2.11), we get

$$\Delta(r_n\Delta z_n + p^{\alpha} r_{n-k}\Delta z_{n-k}) + Q_n \max_{[n-\ell,n]} [x_s^{\alpha} + p^{\alpha} x_{s-k}^{\alpha}] \leq 0. \tag{2.12}$$

Apply Lemma 2.1 when $0 < \alpha \leq 1$ and use Lemma 2.2 if $\alpha \geq 1$ in (2.12), we obtain

$$\Delta(r_n\Delta z_n + p^{\alpha} r_{n-k}\Delta z_{n-k}) + Q_n \max_{[n-\ell,n]} z_s^{\alpha} \leq 0, \tag{2.13}$$

and

$$\Delta(r_n\Delta z_n + p^{\alpha} r_{n-k}\Delta z_{n-k}) + Q_n 2^{1-\alpha} \max_{[n-\ell,n]} z_s^{\alpha} \leq 0, \tag{2.14}$$

respectively. Since $y_n = r_n\Delta z_n > 0$ is decreasing, we have

$$z_n \geq y_n \sum_{s=n_1}^{n-1} \frac{1}{r_s}. \tag{2.15}$$

Substituting (2.15) in (2.13) and (2.14), we get that $\{y_n\}$ is a positive solution of the inequality

$$\Delta(y_n + p^{\alpha} y_{n-k}) + Q_n^* \max_{[n-\ell,n]} y_s^{\alpha} \leq 0,$$

which is a contradiction. The proof is now complete. □

Theorem 2.9. *If the first order difference inequality*

$$\Delta w_n + \frac{1}{(1+p^\alpha)^\alpha} Q_n^* \max_{[n-\ell,n]} w_s^\alpha \leq 0, \tag{2.16}$$

where $\{Q_n^*\}$ is as defined in (2.8), has no positive solution, then every solution of equation (1.1) is oscillatory.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Then it follows from Lemma 2.6 and the proof of the Theorem 2.8 that $y_n = r_n \Delta z_n > 0$ is decreasing. Define

$$w_n = y_n + p^\alpha y_{n-k} \leq (1+p^\alpha) y_{n-k}. \tag{2.17}$$

Substituting (2.17) in (2.9), we get that $\{w_n\}$ is a positive solution of the inequality (2.16). This contradiction completes the proof. □

Corollary 2.10. *Let $\ell > k$ and $0 < \alpha < 1$ in equation (1.1). If*

$$\sum_{n=n_0}^{\infty} Q_n^* = \infty, \tag{2.18}$$

then every solution of equation (1.1) is oscillatory.

Proof. The proof follows by applying Lemma 2.3 in Theorem 2.2 and the details are left to the reader. □

Corollary 2.11. *Let $\ell > k$ and $\alpha = 1$ in equation (1.1). If*

$$\liminf_{n\to\infty} \sum_{s=n-\ell+k}^{n-1} Q_s^* > (1+p)\Big(\frac{\ell-k}{\ell-k+1}\Big)^{\ell-k+1}, \tag{2.19}$$

then every solution of equation (1.1) is oscillatory.

Proof. The proof follows by applying Lemma 2.4 in Theorem 2.2 and the details are left to the reader. □

Corollary 2.12. *Let $\ell > k$ and $\alpha > 1$ in equation (1.1). If there exists a $\lambda > 0$ such that $\lambda > \frac{1}{\ell}\log\alpha$ and*

$$\liminf_{n\to\infty} [Q_n^* exp(-e^{\lambda n})] > 0, \tag{2.20}$$

then every solution of equation (1.1) is oscillatory.

Proof. The proof follows by applying Lemma 2.5 in Theorem 2.2 and the details are omitted. □

3. Examples

In this section, we present some examples to illustrate the main results.

Example 3.1. *Consider the neutral difference equation*

$$\Delta\Big(\frac{1}{n}\Delta(x_n + 2x_{n-1})\Big) + \frac{1}{n^{5/3}} \max_{[n-2,n]} x_s^{1/3} = 0, \quad n \geq 1. \tag{3.1}$$

Here $r_n = \frac{1}{n}$, $p_n = 2$, $q_n = \frac{1}{n^{5/3}}$, $k = 1$, $\ell = 2$, and $\alpha = \frac{1}{3}$. It is easy to see that all conditions of Corollary 2.10 are satisfied. Hence every solution of equation (3.1) is oscillatory.

Example 3.2. *Consider the neutral difference equation*

$$\Delta\Big(\frac{1}{n}\Delta(x_n + 3x_{n-2})\Big) + \frac{e^{e^n}}{n^6} \max_{[n-4,n]} x_s^3 = 0, \quad n \geq 1. \tag{3.2}$$

Here $r_n = \frac{1}{n}$, $p_n = 3$, $q_n = \frac{e^{e^n}}{n^6}$, $k = 2$, $\ell = 4$, and $\alpha = 3$. Choose $\lambda = 1$, then it is easy to see that all conditions of Corollary 2.12 are satisfied. Hence every solution of equation (3.2) is oscillatory.

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