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Modified least squares homotopy perturbation method for solving fractional partial differential equations

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Abstract
This paper introduces a new modification of least squares homotopy perturbation method (LSHPM) for solving linear and nonlinear fractional partial differential equations (FPDEs). The main advantage of the new modification is to approximate the solution for FPDEs in a full general set. Moreover, the convergence of the proposed modification is shown. Analytical and numerical solutions for the linear Navier-Stokes equation and the nonlinear gas dynamic equation are successfully obtained to confirm the accuracy and efficiency of the proposed modification.

Keywords
New modification of least squares homotopy perturbation method, Fractional partial differential equations, Time fractional linear Navier-Stokes equation, Time fractional nonlinear gas dynamic equation.

AMS Subject Classification
35R11, 93E24, 65H20.

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Contents

1. Introduction

Fractional partial differential equations (FPDEs) are becoming a useful tool due to their practical applications for describing the natural phenomena of science and engineering models, and they have played an important role in modeling that is so-called anomalous transport phenomena as well as in theory of complex systems, see [1–8]. Since the exact solutions to large fractional partial differential equations are rarely available, so approximate and numerical methods are applicable. Therefore, accurate methods for finding the solutions of FPDEs are yet under investigation.

Several analytical and numerical methods for solving FPDEs exist in the literature for example: Homotopy perturbation method (HPM) (Shaher Momani and Zaid Odibat [9]) where the authors applied HPM for nonlinear partial differential equations with fractional time derivative, Variational iteration method and Decomposition method (Zaid Odibat and Shaher Momani [10]) where these two methods applied to obtain the approximate solution of nonlinear fractional order partial differential equations, homotopy perturbation technique (Syed Tauseef Mohyud-Din [11]) where the idea of this technique was to utilize both the initial and boundary conditions in the recursive relation for obtaining approximate solution, the modified extended tanh-function method (Elsayed M.E. Zayed et al. [12]) where the method employed to solve fractional partial differential equations by turning them into nonlinear ordinary differential equations of integer orders. In [13] T. Bakkyaraj and R. Sahadevan applied homotopy analysis method to obtain the approximate analytical solution of two coupled time fractional nonlinear Schrödinger equations. Zigen Ouyang [14] obtained some conditions for the existence of the solutions of a class of nonlinear fractional order partial differential equations with delay. Rihuan et al. [15] proposed the fast direct method for solving the linear block lower triangular Toeplitz-like with tridiagonal blocks system which arises from the time-fractional partial differential equation. Komal Singla and R.K. Gupta [16] introduced an extension of the concept of nonlinear self-adjointness and Noether operators for calculating conserved vectors of the time fractional nonlinear systems of partial differential equations. Recently, Hayman Thabet et al. [17] proposed a new analytical technique for solving a system of nonlinear fractional partial differential equations in full general set.

The HPM which is a coupling of the traditional perturbation method and the homotopy in topology yields a very rapid convergence of the solution series in most cases. The HPM has a significant advantage in that it provides an approximate solution to a wide range of nonlinear problems in applied sciences, see [18–20] and some references cited therein. The LSHPM is a coupling of the least squares method and the standard HPM. In [21] Constantin Bota and Bogdan Caruntu recently applied the LSHPM to compute approximate analytical solutions for nonlinear differential equations.

The main aim of this paper is to demonstrate that full general FPDEs can be solved easily by using a new modification of LSHPM and that it gives good results in analytical and numerical experiments. The rest of the paper is organized in as follows: In Section 2, we present some basic definitions and theorems of fractional calculus theory which are needed in the sequel. In Section 3, we present basic idea of HPM with modification of nonlinear operator. In Section 4, we introduce a new modification of LSHPM for solving general FPDEs. Analytical and numerical solutions for linear Navier-Stokes equation and nonlinear gas dynamic equations in sense of Caputo fractional partial derivative are successfully obtained in Section 5.

2. Preliminaries.

There are various definitions and theorems of fractional calculus theory. This section presents some of theses definitions and theorems, which are needed in this paper and can be found in [22–26] and in some references cited therein.

Definition 2.1. Let $x,t,q \in \mathbb{R}$. Then, the Riemann-Liouville time fractional partial integral of order q for the function $u(x,t)$ is defined as follows:

$$\mathscr{I}_t^q u(x,t) = \frac{}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} u(x,\tau)d\tau, \ \ t>0. \quad (2.1)$$

Definition 2.2. Let $q \in \mathbb{R}$, $m-1<q<m \in \mathbb{N}$, the Riemann-Liouville time fractional partial derivative of order q for $u(x,t)$ is defined as follows:

$$D_t^q u(x,t) = \frac{\partial^m}{\partial t^m} \int_0^t \frac{(t-\tau)^{m-q-1}}{\Gamma(m-q)} u(x,\tau)d\tau, \ \ t>0. \quad (2.2)$$

Definition 2.3. Let $m-1<q<m \in \mathbb{N}, t \in \mathbb{R}$ and $t>0$, then

$$\begin{cases} \mathscr{D}_t^q u(x,t) = \displaystyle\int_0^t \frac{(t-\tau)^{m-q-1}}{\Gamma(m-q)} \frac{\partial^m u(x,\tau)}{\partial \tau^m} d\tau, \\ \mathscr{D}_t^q u(x,t) = \dfrac{\partial^m u(x,t)}{\partial t^m} = \mathscr{D}_t^m u(x,t), \ q=m \in \mathbb{N}, \end{cases} \quad (2.3)$$

is called the Caputo time fractional partial derivative of order q for $u(x,t)$.

Theorem 2.1. *Let $q_1,q_2 \in \mathbb{R}$, such that $n-1<q_1 \leq n$, $m-1<q_2 \leq m$, $n \neq m$ for $n,m \in \mathbb{N}$. Then, in general*

$$\begin{cases} \mathscr{D}_t^{q_1} \mathscr{D}_t^{q_2} u(x,t) = \mathscr{D}_t^{q_2} \mathscr{D}_t^{q_1} u(x,t) = \mathscr{D}_t^{q_1+q_2} u(x,t), \\ \mathscr{D}_t^{q_1} \mathscr{D}_t^{m} u(x,t) \neq \mathscr{D}_t^{m} \mathscr{D}_t^{q_1} u(x,t). \end{cases} \quad (2.4)$$

Theorem 2.2. *Let $q,t \in \mathbb{R}, m-1<q<m \in \mathbb{N}$ and $t>0$, then*

$$\begin{cases} \mathscr{I}_t^q \mathscr{D}_t^q u(x,t) = u(x,t) - \displaystyle\sum_{k=0}^{m-1} \frac{t^k}{k!} \frac{\partial^k u(x,0^+)}{\partial t^k}, \\ \mathscr{D}_t^q \mathscr{I}_t^q u(x,t) = u(x,t), \end{cases} \quad (2.5)$$

where $\mathscr{I}_t^q$ is the Riemann-Liouville fractional partial integral of order q.

Theorem 2.3. *Let $p,q,t \in \mathbb{R}, m-1<q \leq m$ and $m \in \mathbb{N}$, then*

$$\begin{cases} \mathscr{D}_t^q t^p = \dfrac{\Gamma(p+1)}{\Gamma(p-q+1)} t^{p-q} = \mathscr{D}_t^q t^p, \quad m-1<p \in \mathbb{R}, \\ \mathscr{D}_t^q t^p = 0, p \leq m-1, \ p \in \mathbb{N}. \end{cases} \quad (2.6)$$

3. Basic idea of HPM

This section discusses the basic idea of HPM for solving a nonlinear problem of the following form:

$$A(u) - f(r) = 0, \ \ B(u, \partial u/\partial n) = 0, \quad r \in \bar{\Omega}, \quad (3.1)$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function and $\bar{\Omega}$ is the boundary of the domain Ω.

The operator A in Eq. (3.1) can be divided into two parts, which are L and N where L is a linear and N is nonlinear operator. Therefore, Eq. (3.1) can be rewritten as follows

$$L(u) + N(u) - f(r) = 0. \quad (3.2)$$

However, to solve Eq. (3.1), we consider the homotopy $v(r,p): \Omega \times [0,1] \to \mathbb{R}$, which satisfies

$$\mathscr{H}(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad (3.3)$$

or

$$\mathscr{H}(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \quad (3.4)$$

where $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of Eq. (3.1), which satisfies the boundary conditions. According to HPM, we can first use the embedding parameter p as a small parameter, and assume that the solution of Eq. (3.4) can be written as a power series in p as follows:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \tag{3.5}$$

Letting $p \to 1$, the approximate solution of Eq. (3.1) is given by

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots \tag{3.6}$$

Theorem 3.1. *For $u(\bar{x},t) = \sum_{k=0}^{\infty} p^k u_k(\bar{x},t)$, the nonlinear operator $N(u(\bar{x},t))$ satisfies the following property:*

$$\begin{aligned} N(u(\bar{x},t)) &= N(\sum_{k=0}^{\infty} p^k u_k(\bar{x},t)) \\ &= \sum_{n=0}^{\infty} \Big[\frac{1}{n!}\frac{\partial^n}{\partial p^n}\big[N(\sum_{k=0}^{n} p^k u_k)\big]_{p=0}\Big] p^n. \end{aligned} \tag{3.7}$$

Proof. According to Maclaurin expansion of $N(\sum_{k=0}^{\infty} p^k u_k)$ with respect to p, we have

$$\begin{aligned} N(\sum_{k=0}^{\infty} p^k u_k) &= [N(\sum_{k=0}^{\infty} p^k u_k)]_{p=0} \\ &\quad + \big[\frac{\partial}{\partial p}[N(\sum_{k=0}^{\infty} p^k u_k)]_{p=0}\big] p \\ &\quad + \big[\frac{1}{2!}\frac{\partial^2}{\partial p^2}[N(\sum_{k=0}^{\infty} p^k u_k)]_{p=0}\big] p^2 + \cdots \\ &= \sum_{n=0}^{\infty} \big[\frac{1}{n!}\frac{\partial^n}{\partial p^n}[N(\sum_{k=0}^{\infty} p^k u_k)]_{p=0}\big] p^n \\ &= \sum_{n=0}^{\infty} \Big[\frac{1}{n!}\frac{\partial^n}{\partial p^n}\big[N(\sum_{k=0}^{n} p^k u_k \\ &\quad + \sum_{k=n+1}^{\infty} p^k u_k)\big]_{p=0}\Big] p^n \\ &= \sum_{n=0}^{\infty} \Big[\frac{1}{n!}\frac{\partial^n}{\partial p^n}\big[N(\sum_{k=0}^{n} p^k u_k)\big]_{p=0}\Big] p^n. \qquad \square \end{aligned}$$

Remark 3.1. Let the polynomials $H_n(u)$ be defined as follows:

$$H_n(u) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\Big[N(\sum_{k=0}^{n} p^k u_k)\Big]_{p=0}, \tag{3.8}$$

Then, from Theorem 3.1, the nonlinear operators $N_i u_\lambda$ can be expressed in terms of H_{in} as:

$$N_i u = N_i \sum_{k=0}^{\infty} p^k u_k = \sum_{n=0}^{\infty} H_{in} p^n, \ i = 1,2,\ldots n. \tag{3.9}$$

4. Modified LSHPM for solving FPDEs

This section introduces a new modification of LSHPM to solve full general FPDEs of the following form:

$$\begin{cases} \mathscr{D}_t^q u(\bar{x},t) + L(u(\bar{x},t)) + N(u(\bar{x},t)) = f(\bar{x},t), \\ \dfrac{\partial^{k_i} u(\bar{x},0)}{\partial t^k} = f_k(\bar{x}), \ \ k = 0,1,2,\ldots,n-1, \\ u(\bar{x},t)|_\Gamma = 0, \ \ (\bar{x},t) \in \Omega \subset \mathbb{R}^{n+1}, \end{cases} \tag{4.1}$$

for $n-1 < q < n \in \mathbb{N}$, and $\bar{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, t > 0$, where Γ is the boundary of Ω and $L[u(\bar{x},t)]$, $N[u(\bar{x},t)]$ are linear and nonlinear operators respectively of a function $u(\bar{x},t)$ and its partial derivatives which might include other fractional derivatives of order less than q, and $f(\bar{x},t)$ is a known analytic function and $\mathscr{D}_t^q$ is the Caputo partial derivative of fractional order q.

In processing to solve the system (4.1) by a new modification of LSHPM, we assume that the solution function $u(\bar{x},t)$ of the system (4.1) has the following analytic expansion:

$$u(\bar{x},t) = \sum_{m=0}^{\infty} u_m(\bar{x},t), \ i = 1,2,\ldots,n. \tag{4.2}$$

Next, we consider the following homotopy:

$$\begin{aligned} &(1-p)(\mathscr{D}_t^q \Phi(\bar{x},t,p) - f(\bar{x},t) + p(\mathscr{D}_t^q \Phi(\bar{x},t,p) \\ &\quad + L[\Phi(\bar{x},t,p)] + N[\Phi(\bar{x},t,p)] - f(\bar{x},t)) = 0, \end{aligned} \tag{4.3}$$

where $p \in [0,1]$ is an embedding parameter, $\Phi(\bar{x},t,p)$ is an unknown function which can be defined as

$$\Phi(\bar{x},t,p) = u_0(\bar{x},t) + \sum_{m\geq 1} p^m u_m(\bar{x},t). \tag{4.4}$$

When $p = 0$, $\Phi(\bar{x},t,0) = u_0(\bar{x},t)$ and when $p = 1$, $\Phi(\bar{x},t,1) = u(\bar{x},t)$. Thus, as p increases from 0 to 1, the solution $\Phi(\bar{x},t,p)$ varies from $u_0(\bar{x},t)$ to the solution $u(\bar{x},t)$, where $u_0(\bar{x},t)$ is obtained from the following system:

$$\begin{cases} \mathscr{D}_t^q u_0(\bar{x},t) - f(\bar{x},t) = 0, t > 0, \\ \dfrac{\partial^k u_0(\bar{x},0)}{\partial t^k} = f_k(\bar{x}), \ \ k = 0,1,2,\ldots,m-1, \\ u_0(\bar{x},t)|_\Gamma = 0, \ \ (\bar{x},t) \in \Omega. \end{cases} \tag{4.5}$$

When $p = 1$, $\Phi(\bar{x},t,1) = u(\bar{x},t)$, we obtain

$$\begin{cases} \mathscr{D}_t^q u_m(\bar{x},t) = -(L(u_{m-1}(\bar{x},t)) + N(u_0, u_1, \ldots, u_{m-1}), \\ \dfrac{\partial^k u_m(\bar{x},0)}{\partial t^k} = 0, \ \ k = 0,1,2,\ldots,n-1, \ m = 1,2,\ldots \\ u_m(\bar{x},t)|_\Gamma = 0, \ \ (\bar{x},t) \in \Omega., \ t > 0, \ m = 1,2,\ldots \end{cases} \tag{4.6}$$

By using Remark 3.1 and Theorem 3.1, the system (4.6) can be rewritten as:

$$\begin{cases} \mathscr{D}_t^q u_m(\bar{x},t) = -(L(u_{m-1}) + H_{m-1}(u_0, u_1, \ldots, u_{m-1})), \\ \dfrac{\partial^k u_m(\bar{x},0)}{\partial t^k} = 0, \ \ k = 0,1,2,\ldots,n-1, \ m = 1,2,\ldots \\ u_m(\bar{x},t)|_\Gamma = 0, \ \ (\bar{x},t) \in \Omega, \ t > 0, \ m = 1,2,\ldots. \end{cases} \tag{4.7}$$

Next, we consider the set $S_m (m = 0, 1, \ldots)$ containing the functions $\varphi_{m0}, \varphi_{m1}, \ldots, \varphi_{mn_m}$, chosen as linearly independent functions in a vector space of the continuous functions defined on the region Ω such that $S_{m-1} \subset S_m$ and $v_m = u_0 + u_1 + \cdots + u_m$ is a real linear combination of these functions.
We remark that such a combination is always possible for example we can choose $S_m = \{u_0, u_0, \ldots, u_m\}$. In this case, we may choose $\varphi_{m0} = u_0, \varphi_{m1} = u_1, \ldots, \varphi_{mn_m} = u_m$.

Definition 4.1. If $\tilde{u}(\bar{x},t)$ an approximate solution of equation (4.1) on the region Ω, we define

$$\mathscr{R}(\bar{x},t,\tilde{u}) = \mathscr{D}_t^q \tilde{u}(\bar{x},t) + L[\tilde{u}(\bar{x},t)] + N[\tilde{u}(\bar{x},t)] - f(\bar{x},t), \quad (4.8)$$

together with the initial and boundary conditions given by (4.1), where $\mathscr{R}(\bar{x},t,\tilde{u})$ is a remainder and it satisfies the following property:

$$|\mathscr{R}(\bar{x},t,\tilde{u})| < \varepsilon, \quad (4.9)$$

and we evaluate the error obtained by replacing the exact solution $u(\bar{x},t)$ with the approximate one $\tilde{u}(\bar{x},t)$ as the remainder.

Definition 4.2. We call a weak δ-approximate HP-solution of the problem (4.1) on the real region as an HP-function $\tilde{u}$ which satisfies the following relation:

$$\int_{A\in\Omega} \mathscr{R}^2(\bar{x},t,\tilde{u})dA \leq \delta, \quad (4.10)$$

together with the initial and boundary conditions from (4.1).

Next, we find a weak ε-approximate solution of the type

$$\tilde{u}(\bar{x},t) = \sum_{k=0}^{n} \tilde{u}_k(\bar{x},t) = \sum_{k=0}^{n_m} c_m^k \varphi_{mk}(\bar{x},t), \quad (4.11)$$

for $m \geq 0$, where c_m^k can be determined from the following formulas:

$$\begin{cases} \mathfrak{R}(\bar{x},t,c_m^k) = \mathscr{R}(\bar{x},t,\tilde{u}), \\ \quad J(c_m^k) = \displaystyle\int_{A\in\Omega} \mathfrak{R}^2(\bar{x},t,c_m^k)dA, \end{cases} \quad (4.12)$$

where by imposing the boundary conditions, we can determine $l \in \mathbb{N}, l \leq m$ such that $c_0^m, c_1^m, \ldots, c_l^m$ are computed as functions of $c_{l+1}^m, c_{l+2}^m, \ldots, c_n^m$. From equation (4.12), we compute the values of $c_{l+1}^{\sim m}, c_{l+2}^{\sim m}, \ldots, c_n^{\sim m}$ as the values which give the minimum of the functional J and the values of $c_0^{\sim m}, c_1^{\sim m}, \ldots, c_l^{\sim m}$ again as functions of $c_{l+1}^{\sim m}, c_{l+2}^{\sim m}, \ldots, c_n^{\sim m}$ by using the initial and boundary conditions. Then, the constants c_m^k can be computed. Consequently, a weak approximate solution given by (4.11) can be found. Thus the analytical solution of the problem (4.1) obtained by MLSHPM is given by

$$\underset{MLSHPM}{u(x,t)} = \lim_{n\to\infty} \sum_{k=0}^{n} \tilde{u}_k(\bar{x},t). \quad (4.13)$$

Theorem 4.1. *The HP-sequence of the problem (4.1) of the functions* $\{\mathbf{S}_m(\bar{x},t)\}_{m\in\mathscr{N}}$ *of the form* $\mathbf{S}_m(\bar{x},t) = \sum_{k=0}^{n_m} c_k^{\sim m} \varphi_{mk}$ *satisfies the following property:*

$$\lim_{m\to\infty} \int_{A\in\Omega} \mathscr{R}^2(\bar{x},t,\mathbf{S}_m(\bar{x},t))dA = 0. \quad (4.14)$$

Moreover, $\forall\varepsilon, \exists m_0 \in \mathbb{N}$ *such that* $\forall m \in \mathbb{N}, m > m_0$ *it follows that* $\mathbf{S}_m(\bar{x},t)$ *is a weak* ε*-approximate HP-solution of the problem (4.2).*

Proof. Define the sequence $\mathbf{S}_m$ of partial sums of the following series:

$$\begin{cases} \mathbf{S}_0 = \tilde{u}_0, \\ \mathbf{S}_1 = \tilde{u}_0 + \tilde{u}_1, \\ \mathbf{S}_2 = \tilde{u}_0 + \tilde{u}_1 + \tilde{u}_2, \\ \quad\vdots \\ \mathbf{S}_m = \tilde{u}_0 + \tilde{u}_1 + \tilde{u}_2 + \cdots + \tilde{u}_m. \end{cases}, \quad (4.15)$$

According to the components in (4.15), the sequence $\mathbf{S}_m$ can be obtained and the following inequality holds:

$$\begin{aligned} 0 &\leq \int_{A\in\Omega} \mathscr{R}^2(\bar{x},t,\mathbf{S}_m(\bar{x},t))dA \\ &\leq \int_{A\in\Omega} \mathscr{R}^2(\bar{x},t,v_m(\bar{x},t))dA. \end{aligned} \quad (4.16)$$

It follows that

$$\begin{aligned} 0 &\leq \lim_{m\to\infty} \int_{A\in\Omega} \mathscr{R}^2(\bar{x},t,\mathbf{S}_m(\bar{x},t))dA \\ &\leq \lim_{m\to\infty} \int_{A\in\Omega} \mathscr{R}^2(\bar{x},t,v_m(\bar{x},t))dA = 0. \end{aligned} \quad (4.17)$$

Thus we obtain

$$\lim_{m\to\infty} \int_{A\in\Omega} \mathscr{R}^2(\bar{x},t,\mathbf{S}_m(\bar{x},t))dA = 0. \quad (4.18)$$

From this limit we obtain that $\forall\varepsilon, \exists m_0 \in \mathbb{N}$ such that $\forall m \in \mathbb{N}, m > m_0$ it follows that $\mathbf{S}_m(\bar{x},t)$ is a weak ε-approximate HP-solution of the problem (4.2). ☐

Theorem 4.2. *The sequence* $\{\mathbf{S}_m(\bar{x},t)\}_{m=\in\mathscr{N}}$ *converges if* $\lim_{m\to\infty} \mathscr{R}(\bar{x},t,\mathbf{S}_m(\bar{x},t)) = 0.$

Proof. The proof comes immediately from Theorem 4.1. ☐

5. Applications

This section presents some well-known linear and nonlinear time fractional partial differential equations. These examples are chosen because of their closed form solutions are available or they have been solved previously by some other well-known methods.

Example 5.1. [27] Consider the following non-homogeneous linear Navier-Stokes initial value problem of time fractional order:

$$\begin{cases} \mathscr{D}_t^q u - r - u_{xx} - \frac{1}{x}u_x = 0,\ 0 < q < 1, \\ u(x,0) = 1 - x^2. \end{cases} \tag{5.1}$$

where r is a real constant. For $q = 1$, the exact solution of (5.1) is $u(x,t) = 1 - x^2 + (r-4)t$.

From systems (4.1) and (5.1), we have $L(u(x,t)) = -(u_{xx} + \frac{1}{x}u_x), N(u(x,t)) = 0$ and $f(x,t) = r$.
Next by using (4.5), we obtain

$$\mathscr{D}_t^q u_0(x,t) - r = 0,\ \ u_0(x,0) = 1 - x^2. \tag{5.2}$$

Solving the system (5.2), we obtain

$$u_0(x,t) = 1 - x^2 + \frac{r}{\Gamma(q+1)}t^q. \tag{5.3}$$

It follows that $S_0 = \{1, x^2, t^q\}$.
Consequently, the zero-order term approximate solution is given by

$$\tilde{u}_0(x,t) = c_0 + c_1 x^2 + c_2 t^q. \tag{5.4}$$

From the initial condition we have $1 - x^2 = \tilde{u}_0(x,0) = c_0 + c_1 x^2$, which implies that $c_0 = 1$ and $c_1 = -1$. Substituting the values of c_0 and c_1 in (5.4), we obtain

$$\tilde{u}_0(x,t) = 1 - x^2 + c_2 t^q. \tag{5.5}$$

By using equation (5.5) in Definition 4.1, we obtain

$$\mathscr{R}(\bar{x},t,\tilde{u}) = \Gamma(q+1)c_2 + 4 - r. \tag{5.6}$$

Next, with the help of system (4.12), we have

$$J(c_2) = \int_0^1 \int_0^1 (\Gamma(q+1)c_2 + 4 - r)^2 dxdt. \tag{5.7}$$

To compute the minimum of the functional $J(c_2)$, we determine the critical point of $J(c_2)$ as a real solution of $\frac{dJ}{dc_2} = 0$. It follows that $c_2 = \frac{r-4}{\Gamma(q+1)}$. Consequently, the zero-order term approximate solution of the problem (5.12) is

$$\tilde{u}_0(x,t) = 1 - x^2 + \frac{r-4}{\Gamma(q+1)}t^q. \tag{5.8}$$

From the system (4.7) for $m = 1$, we have

$$\begin{cases} \mathscr{D}_t^q u_1(\bar{x},t) = -(L(u_0(x,t)) + H_0(u_0(x,t))), \\ \frac{\partial u_1(x,0)}{\partial t} = 0. \end{cases} \tag{5.9}$$

By using Remark 3.1 and equation (5.3), the solution of system (5.9) is

$$u_1(x,t) = \frac{4}{\Gamma(q+1)}t^q. \tag{5.10}$$

It follows that $S_1 = \{t^q\}$. Therefore, the first-order term approximate solution $\tilde{u}_1(x,t) = c_0 t^q$ which implies by using Definition 4.1 and system (4.12) that $c_0 = 0$. Consequently, for $m = 1, 2\ldots$, we obtain $\tilde{u}_m(x,t) = 0$.
Hence by using equation (4.13), the exact analytical solution of (5.1) is

$$\underset{MLSHPM}{u(x,t)} = 1 - x^2 + \frac{r-4}{\Gamma(q+1)}t^q, \tag{5.11}$$

which is exactly the same result obtained in [27]. In case of $q = 1$, we get the same exact solution.

Example 5.2. [28] Consider the following non-homogeneous nonlinear time fractional gas dynamic equation:

$$\begin{cases} \mathscr{D}_t^q u + uu_x - u(1-u) = -e^{t-x},\ 0 < q < 1, \\ u(x,0) = 1 - e^{-x}. \end{cases} \tag{5.12}$$

For $q = 1$, the exact solution of the system (5.12) is $u(x,t) = 1 - e^{t-x}$.

By comparing the system (5.12) with the system (4.1) , we have $L(u) = u, N(u) = uu_x + u^2$ and $f(x,t) = -e^{t-x}$.
Now by using (4.5), we have

$$\begin{cases} \mathscr{D}_t^q u_0(x,t) + e^{t-x} = 0, \\ u_0(x,0) = 1 - e^{-x}. \end{cases} \tag{5.13}$$

Solving the system (5.13), we obtain

$$u_0(x,t) = 1 - e^{-x} - e^{-x}t^q E_{1,q+1}(t), \tag{5.14}$$

where $E_{1,q+1}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k+q+1)}$ is a Mittag-Leffler function. It follows that $S_0 = \{1, e^{-x}, e^{-x}t^q E_{1,q+1}\}$. Consequently, the zero-order term approximate solution is given by

$$\tilde{u}_0(x,t) = c_0 + c_1 e^{-x} + c_2 e^{-x} t^q E_{1,q+1}. \tag{5.15}$$

Next, we use the initial condition from the system (5.12) in Eq. (5.15), we obtain that $c_0 = 1, c_1 = -1$. By Substituting the values of c_0, c_1 in Eq. (5.15) and using Definition 4.1, we obtain

$$\mathscr{R}(\bar{x},t,\tilde{u}) = (c_2 + 1)e^{t-x}. \tag{5.16}$$

By using the system (4.12), we obtain that $c_2 = -1$. Therefore, the zero-order term approximate solution of the problem (5.12) is

$$\tilde{u}_0(x,t) = 1 - e^{-x} - e^{-x}t^q E_{1,q+1}. \tag{5.17}$$

From the system (4.7) for $m = 1$, we have

$$\begin{cases} \mathscr{D}_t^q u_1(\bar{x},t) = -(L(u_0(x,t)) + H_0(u_0)), \\ \dfrac{\partial u_1(x,0)}{\partial t} = 0. \end{cases} \tag{5.18}$$

By using Remark 3.1 and Eq. (5.14), the solution of the system (5.18) is $u_1(x,t) = 0$ which implies that $u_m(x,t) = 0$ for $m = 1,2,\ldots$. Therefore the approximate solution $\tilde{u}_m(x,t) = 0$ for $m = 1,2\ldots$

Hence by using Eq. (4.13), the exact analytical solution for the problem (5.12) is

$$\underset{MLSHPM}{u(x,t)} = 1 - e^{-x} - e^{-x} t^q E_{1,q+1}(t), \tag{5.19}$$

which is the same exact solution in case of $q = 1$.

6. Numerical experiments and discussion

Table 1 and Table 2 show numerical values of the solution obtained by MLSHPM and the exact solution u_{Ex} for Example 5.1 and Example 5.2 respectively among different values of x,t and q when $r = 1$. In Fig. 1 and Fig. 4, we plot the solution obtained by MLSHPM for Example 5.1 and Example 5.2 respectively when $q = 0.5$ and $r = 1$. In Fig. 2 and Fig. 5, we plot the solution obtained by MLSHPM for Example 5.1 and Example 5.2 respectively when $q = 0.75$ and $r = 1$. In Fig. 3 and Fig. 6, we plot the solution obtained by MLSHPM for Example 5.1 and Example 5.2 respectively when $q = r = 1$. The graphs are plotted in the region Ω where $\Omega = \{(x,t) : -40 \leq x \leq 40,\ 0.2 \leq t \leq 40, x,t \in \mathbb{R}\}$.

Table 1
Numerical values of the solution obtained by MLSHPM and the exact solution for Example 5.1 when $r = 1$.

x	t	$q = 0.5$	$q = 0.75$	$q = 1$	
		$u(x,t)$ MLSHPM	$u(x,t)$ MLSHPM	$u(x,t)$ MLSHPM	$u(x,t)$ Eact
0.25	0.20	-0.57638	-0.03872	0.33750	0.33750
	0.40	-1.20345	-0.70430	-0.26250	-0.26250
	0.60	-1.68460	-1.28781	-0.86250	-0.86250
0.75	0.20	-1.07638	-0.53872	-0.16250	-0.16250
	0.40	-1.70345	-1.20430	-0.76250	-0.76250
	0.60	-2.18462	-1.78781	-1.36250	-1.36250

Table 2
Numerical values of the approximate and exact solutions to Example 5.2 for different values of x,t and q .

x	t	$q = 0.5$	$q = 0.75$	$q = 1$	
		$u(x,t)$ MLSHPM	$u(x,t)$ MLSHPM	$u(x,t)$ MLSHPM	$u(x,t)$ Eact
0.25	0.20	-0.22865	-0.06342	0.04877	0.04877
	0.40	-0.50949	-0.31825	-0.16183	-0.16183
	0.60	-0.81001	-0.60566	-0.41907	-0.41907
0.75	0.20	0.25479	0.35501	0.42305	0.42305
	0.40	0.08445	0.20044	0.29531	0.29531
	0.60	-0.09782	0.02612	0.13929	0.13929

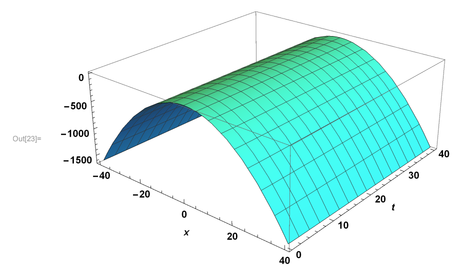

Fig. 1. The graph of the solution for Example 5.1 when $q = 0.5$ and $r = 1$

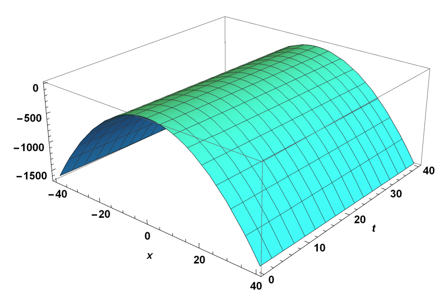

Fig. 2. The graph of the solution for Example 5.1 when $q = 0.75$ and $r = 1$

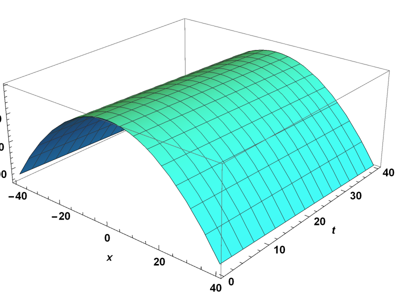

Fig. 3. The graph of the solution for Example 5.1 when $q = 1$ and $r = 1$

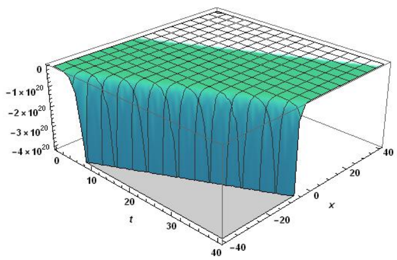

Fig. 4. The graph of the solution for Example 5.2 when $q = 0.5$

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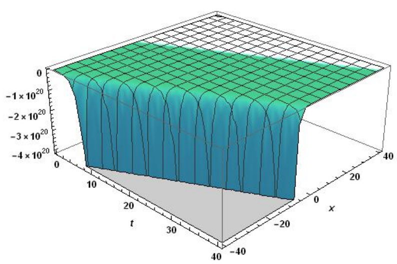

Fig. 5. The graph of the solution for Example 5.2 when $q = 0.75$

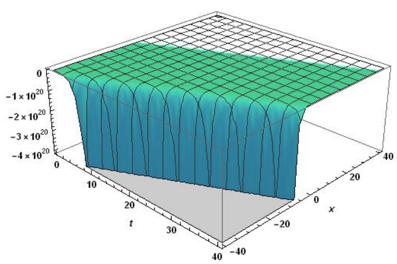

Fig. 6. The graph of the solution for Example 5.2 when $q = 1$

7. Conclusions

In this paper, a new modification of LSHPM for solving FPDEs in full general set was introduced. The solutions obtained by this modification were in excellent agreement with those obtained via previous works and also they were in very good conformity with the exact solution to confirm the effectiveness and accuracy of the proposed modification. We used Mathematica software to obtain the numerical solutions and plotting the graphs.

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