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Existence results for differential evolution equations with nonlocal conditions in Banach space

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Abstract

Our aim in this paper is to study the existence and uniqueness of a mild solution to an initial value problem (IVP for short) for a class of nonlinear differential evolution equations with nonlocal initial conditions in a Banach space. We assume that the linear part is not necessarily densely defined and generates an evolution family. We give two results, the first one is based on a Krasnosel'skii fixed point Theorem, and in the second approach we make use Mönch fixed point Theorem combined with the measure of noncompactness and condensing.

Keywords

Nonlocal initial value problem, evolution family, measure of noncompactness, condensing map, nondensely defined operators, mild solution, Mönch fixed point Theorem.

AMS Subject Classification

34G20, 47D06, 47H10, 47H20..

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1. Introduction

Many authors have been attracted to problems on the existence and qualitative properties of solutions for abstract evolution equations.

Kato [16] studied the generation of an evolution operator associated with the linear evolution equation of "hyperbolic" type in a pair of Banach spaces (Y,X) such that Y is continuously and densely imbedded in X,

$$u'(t) = A(t)u(t) \quad \text{for } t \in [0,T] \tag{1.1}$$

$$u(0) = x_0. \tag{1.2}$$

Here $\{A(t), t \in [0,T]\}$ is a family of closed linear operators in X such that $Y \subset D(A(t))$ for $t \in [0,T]$. The concept of evolution operators is central for the theory of abstract linear evolution equations.

The nonlocal conditions can be applied in physics with better effect than the classical initial condition. For example, in [10], Deng used the nonlocal condition to describe the diffusion phenomenon of a small amount of gas in a transport tube. The nonlocal condition allows additional measurement which is more precise than the measurement just at $t=0$. Byszewski *et al.* [8, 9] established the existence, uniqueness and continuous dependence of a mild solution of a nonlocal Cauchy problem for a semilinear functional differential evolution equation

$$\begin{cases} u'(t) + Au(t) = f(t,u_t) & \text{for } t \in [0,T] \\ u(s) + g(u_{t_1}, \dots, u_{t_p})(s) = \phi(s) & s \in [-r,0], \end{cases}$$

where $0 < t_1 < \cdots < t_p \leq T$ $(p \in \mathbb{N})$, A is the infinitesimal generator of a C_0-semigroup of operators on a general Banach space.

Oka and Tanaka [19] proved that an evolution operator is generated by a family of closed linear operators whose common domain is not necessarily dense in the underlying Banach space, under the stability from the viewpoint of finite difference approximations. Tanaka [21] give some existence and uniqueness results for classical solutions to the semilinear initial value problem

$$\begin{cases} u'(t) = A(t)u(t) + B(t,u(t)) & \text{for } t \in [0,T] \\ u(0) = u_0. \end{cases}$$

Here $\{A(t): \ t \in [0,T]\}$ is a given family of closed linear operators in X satisfying all conditions which are usually referred to as the "hyperbolic" case except for the density of the common domain D of $A(t)$, and $B(t,u)$ is a nonlinear operator on $[0,T] \times X$.

In the paper [1], Benchohra *et al.* established sufficient conditions for the existence of mild and extremal mild solutions of first order impulsive functional evolution equations in a separable Banach space $(X.|\cdot|)$ of the form:

$$y'(t) - Ay(t) = F(t,y_t),\ a.e.\ t \in J = [0,T],\ t \neq t_k,\ k = 1,\ldots,m \tag{1.3}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k = 1,\ldots,m \tag{1.4}$$

$$y(t) = \phi(t),\ t \in [-r,0], \tag{1.5}$$

where $f: J \times D \to X$ is a given function,

$$D = \{\psi : [-r,0] \to X,$$

ψ is continuous everywhere except for a finite number of points s at which $\psi(s^-), \psi(s^+)$ exist and $\psi(s^-) = \psi(s)\}$, $\phi \in D$, $0 < r < \infty$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $I_k \in C(X,X)$, $k = 1,\ldots,m$, $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup $T(t)$, $t \geq 0$, and X a real separable Banach space with norm $|\cdot|$. In the case where the impulses are absent (i.e., $I_k = 0, k = 1,\ldots,m$) and F is a single or multivalued map and A is a densely defined linear operator generating a C_0-semigroup of bounded linear operators, the problem (1.3)-(1.5) has been investigated on compact intervals in, for instance, the monographs by Ahmed [3], Hu and Papageorgiou [13], Kamenskii *et al.* [14] and Wu [24] and the papers of Benchohra *et al.* [5–7].

Recently Kpoumiè *et al.* in [17] studied the following problem

$$\begin{cases} u'(t) = A(t)u(t) + \rho(t,u_t) & \text{for } t \geq 0, \\ u_0 = \phi \in \mathscr{B} \ \text{phase space}. \end{cases}$$

For further reading and details on partial differential equations, we refer the reader to the books of Freidman [12].

In this paper, we extend in some way the results obtained in [1, 16, 21]. We prove the existence a of mild solution to an initial value problem (IVP for short) for a nonlinear differential equation with nonlocal initial conditions by two approaches. More precisely we consider the IVP

$$x'(t) = A(t)x(t) + f(t,x(t)), \quad t \in [0,T] \tag{1.6}$$

$$x(0) + g(x) = x_0, \tag{1.7}$$

where $f : [0,T] \times X \to X$ and $g : C([0,T],X) \to X$ are functions that will be specified later, X is a real Banach space with the norm $\|\cdot\|$, and $\{A(t),\ t \geq 0\}$ is an evolution system of closed nondensely defined linear unbounded operators on the Banach space X with domain D

in Section (2) we recall some preliminary results on the evolution family and some definitions and properties on the measure of noncompactness and condensing operators, we recall also the Mönch fixed point Theorem which will be used throughout this paper to prove our result. In the third section, we will take two approaches for our main result; we prove first existence and uniqueness of a mild solution by using a Krasnosel'skii fixed point Theorem, and in the second part we make use the Mönch fixed point Theorem combined with the measure of noncompactness. Section (4) is devoted to an application to illustrate the main result of this work.

2. Preliminaries

We introduce in this section notations, definitions, fixed point Theorems and preliminary facts which are used throughout this paper.

In the remainder of this paper we denote by $J = [0,T]$, and $C(J,X)$ is the Banach space of continuous functions from J into X normed by

$$\|y\|_\infty = \sup\{\|y(t)\| : \ t \in J\}$$

2.1 Properties of an evolution system generator

In what follows, for the family $\{A(t),\ t \geq 0\}$ of closed non-densely defined linear unbounded operators on the Banach space X with domain D, we assume that the family satisfies the following assumptions:

(A_1) $D(A(t)) = D$ independent of t and is non-densely defined ($\overline{D} \neq X$).

(A_2) There exist $M \geq 1$, $\beta \geq 0$ with $(\beta,\infty) \subset \rho(A)$ and

$$\|\prod_{j=1}^{k} R(\lambda, A(t_j))\| \leq M(\lambda - \beta)^{-k} \tag{2.1}$$

$$\text{with } \ 0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq T. \tag{2.2}$$

(A_3) The mapping $t \mapsto A(t)x$ is continuously differentiable in X for all $x \in D$.

Theorem 2.1. *Assume that $\{A(t)\}_{t\geq 0}$ satisfies conditions $(A_1)-(A_3)$. Then the limit $\lim_{\lambda\to 0^+} U_\lambda(t,s)x = U(t,s)x$ exists for $x \in \overline{D}$ and $0 \leq s \leq t$, where the convergence is uniform on $\Delta := \{(t,s): \ 0 \leq s \leq t\}$. There exists an evolution system $\{U(t,s)\}_{(t,s)\in\Delta}$ satisfying the following properties*

(i) For $x \in \overline{D}$, $\lambda > 0$ and $0 \leq s \leq r \leq t$, one has

$$U_\lambda(t,t)x = x \quad \text{and} \quad U_\lambda(t,s)x = U_\lambda(t,r)U_\lambda(r,s)x.$$

(ii) $U(t,s) : \overline{D} \longrightarrow \overline{D}$ for $(t,s) \in \Delta$.

(iii) $U(t,t)x = x$ and $U(t,s)x = U(t,r)U(r,s)x$ for $x \in \overline{D}$ and $0 \leq s \leq r \leq t$.

(iv) The mapping $(t,s) \mapsto U(t,s)x$ is continuous on Δ for any $x \in \overline{D}$.

(v) $\|U(t,s)x\| \leq Me^{\beta(t-s)}\|x\|$ for $x \in \overline{D}$ and $(t,s) \in \Delta$.

In the following we give some results on existence of solutions for the nondensely nonautonomous partial differential equations (1.6)-(1.7). The following Theorem gives a generalized variation of constants formula for (1.6)-(1.7).

Theorem 2.2. *[21] Let $x_0 - g(x) \in \overline{D(A)}$, $f \in L^1(J,R)$. Then the limit*

$$x(t) := U(t,0)(x_0 - g(x)) + \lim_{\lambda\to 0^+}\int_0^t U_\lambda(t,s)f(x(s))ds$$

exists uniformly for $t \in J$ and x is continuous function on J

Lemma 2.3. *Assume $f \in L^1(J,R)$. If x is mild solution of the problem $(1.6)-(1.7)$, then*

$$\|x(t)\| \leq M\, e^{\beta T}(x_0 + \|g(x)\| + \|g(0)\|) + \int_0^t M\, e^{\beta\ (t-s)}\|f(s,x(s))\|ds.$$

Proof. Let, for $\lambda > 0$, $0 \leq s \leq t$ and $x \in \overline{D}$,

$$\begin{aligned} U_\lambda(t,s)x &:= \prod_{i=[\frac{s}{\lambda}]+1}^{[\frac{t}{\lambda}]} \left(\frac{1}{\lambda}R\left(\frac{1}{\lambda}, A(i\lambda)\right)\right)x \\ &= \prod_{i=[\frac{s}{\lambda}]+1}^{[\frac{t}{\lambda}]} (I - \lambda A(i\lambda))^{-1}x. \end{aligned}$$

Then,

$$\begin{aligned} \|U_\lambda(t,s)x\| &\leq M\left(\frac{1}{1-\lambda\beta}\right)^{[\frac{t}{\lambda}]-[\frac{s}{\lambda}]}\|x\| \\ &\leq M\left(\frac{1}{1-\lambda\beta}\right)^{\frac{t-s}{\lambda}+1}\|x\| \\ &\leq M\left(\frac{1}{1-\lambda\beta}\right)e^{-\beta(t-s)\frac{\ln(1-\lambda\beta)}{\lambda\beta}}\|x\|. \end{aligned}$$

Letting $\lambda \to 0$ one has

$$\|U_\lambda(t,s)x\| \leq M\, e^{\beta(t-s)}\|x\|.$$

Now it is clear that the statement of the Lemma is satisfied. □

More details on evolution systems and their properties from semigroup theory can be found in the books of Ahmed [3], Engel and Nagel [11] and Pazy [20].

2.2 Measure of noncompactness

We shall define the measure of noncompactness on $\mathscr{P}_b(X)$. Recall that a subset $A \subset X$ is relatively compact provided the closure $\overline{A}$ is compact.

Definition 2.4. *Let X be a Banach space and $\mathscr{P}_b(X)$ the family of all bounded subsets of X. Then the function: $\alpha : \mathscr{P}_b(X) \to R_+$ defined by*

$$\alpha(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ admits a finite cover by sets of diameter } \leq \varepsilon\}$$

is called the Kuratowski measure of noncompactness, (the α-MNC for short). The function $\chi : \mathscr{P}_b(X) \to R_+$ defined by:

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon - net\}.$$

is called the Hausdorff measure of noncompactness.

Definition 2.4 is very useful since α and χ have interesting properties, some of which are listed in the following.

Proposition 2.5. *[22] Let X be a Banach space and*

$$\gamma : \mathscr{P}_b(X) \to R_+$$

be either α or χ. Then:

(a) $\gamma(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact)

(b) $\gamma(B) = 0 = \gamma(\overline{B}) = 0$

(c) $A \subset B \Rightarrow \gamma(A) \leq \gamma(B)$

(d) $\gamma(A+B) \leq \gamma(A) + \gamma(B)$

(e) $\gamma(c.B) \leq |c|\gamma(B)$

(f) $\gamma(coB) = \gamma(B)$.

(e) The function $\gamma : \mathscr{P}_b(X) \to R_+$ is continuous with respect to the metric H_d on $\mathscr{P}_b(X)$.

Remark 2.6. *For every $A \in \mathscr{P}_b(X)$, we have $\chi(A) \leq \alpha(A) \leq 2\chi(A)$.*

The following result is a generalized Arzela-Ascoli Theorem using the Kuratowski measure of noncompactness.

Lemma 2.7. *[22] If $H \subset C(I,X)$ is bounded and equicontinuous, then $\alpha(H(t))$ is continuous on I, where*

$$\alpha_C(H) = \max_{t \in I} \alpha(H(t)) \, , \, \alpha\left(\int_I x(t)dt, x \in H\right) \leq \int_I \alpha(H(t))dt$$

and where $H(t) = \{x(t), x \in H\}$, $t \in I$, I is a compact interval of J.

Now, we present the abstract definition of MNC. For more details, we refer to [2, 4, 14, 22] and the references therein.

Definition 2.8. *Let $(\mathscr{A}, \geq)$ be a partially ordered set. A function $\beta : \mathscr{P}_b(X) \to \mathscr{A}$ is called a measure of noncompactness (MNC) in E if*

$$\beta(\overline{\text{co}}\Omega) = \beta(\Omega),$$

for every $\Omega \in \mathscr{P}_b(X)$.

Definition 2.9. *A measure of noncompactness β is called:*

(i) monotone if $\Omega_0, \Omega_1 \in \mathscr{P}_b(X)$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$.

(ii) nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in X$, $\Omega \in \mathscr{P}_b(X)$.

(iii) regular if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

The following property of the Hausdorff MNC can be easily verified. If $L : X \to X$ is a bounded linear operator, it is possible to define its χ-norm by

$$\|L\|^{(\chi)} := \chi(LB), \tag{2.3}$$

where $B \subset X$ is the unit ball. It is easy to see that

$$\|L\|^{(\chi)} \leq \|L\|.$$

2.3 Condensing maps

Definition 2.10. *A continuous map $F : X \to X$ is said to be condensing with respect to a MNC β (β-condensing) if for every bounded set $\Omega \subset X$ that is not relatively compact, we have*

$$\beta(F(\Omega)) \ngeq \beta(\Omega).$$

Definition 2.11. *The operator $G : L^1(J,X) \to C(J,X)$ defined by*

$$Gf(t) = \int_0^t T(t,s)f(s)ds \tag{2.4}$$

is called the generalized Cauchy operator, where $T(\cdot,\cdot)$ is the evolution operator generated by the family of operators $\{A(t) : t \in J\}$.

Lemma 2.12 ([14, Theorem 2])**.** *The generalized Cauchy operator G satisfies the properties*

(G1) There exists $\zeta \geq 0$ such that

$$\|Gf(t) - Gg(t)\| \leq \zeta \int_0^t \|f(s) - g(s)\| ds,$$

for every $f, g \in L^1(J,X)$, $t \in J$.

(G2) For any compact $K \subseteq X$ and sequence $\{f_n\}_{n\geq 1}$, $f_n \in L^1(J,X)$ such that for all $n \geq 1$, $f_n(t) \in K$, a. e. $t \in J$, the weak convergence $f_n \rightharpoonup f_0$ in $L^1(J,X)$ implies the convergence $Gf_n \to Gf_0$ in $C(J,X)$.

Lemma 2.13. *[14] Let $S : L^1(J,X) \to C(J,X)$ be an operator satisfying condition (G2) and the following Lipschitz condition (weaker than (G1)).*

(G1')

$$\|Sf - Sg\|_{C(J,X)} \leq \zeta \|f - g\|_{L^1(J,X)}.$$

Then for every semicompact set $\{f_n\}_{n=1}^{+\infty} \subset L^1(J,X)$ the set $\{Sf_n\}_{n=1}^{+\infty}$ is relatively compact in $C(J,X)$. Moreover, if $(f_n)_{n\geq 1}$ converges weakly to f_0 in $L^1(J,X)$ then $Sf_n \to Sf_0$ in $C(J,X)$.

Lemma 2.14. *[14] Let $S : L^1(J,X) \to C(J,X)$ be an operator satisfying conditions* (G1), (G2) *and let the set $\{f_n\}_{n=1}^{\infty}$ be integrably bounded with the property $\chi(\{f_n(t) : n \geq 1\}) \leq \eta(t)$, for a.e. $t \in J$, where $\eta(\cdot) \in L^1(J, \mathbb{R}^+)$ and χ is the Hausdorff MNC. Then*

$$\chi(\{Sf_n(t) : n \geq 1\}) \leq 2\zeta \int_0^t \eta(s)ds, \text{ for all } t \in J,$$

where $\zeta \geq 0$ is the constant in condition (G1).

Proposition 2.15. *[23] Let the space X be separable and the multifunction $\Phi : J \to \mathscr{P}(X)$ be integrable, integrably bounded and $\chi(\Phi(t)) \leq q(t)$ for a.a. $t \in J$ where $q(\cdot) \in L^1(J, \mathbb{R}^+)$. Then*

$$\chi\left(\int_0^\tau \Phi(s)ds\right) \leq \int_0^\tau q(s)ds, \quad \text{for all } \tau \in J.$$

In particular, if the multifunction $\Phi : J \to \mathscr{P}_k(X)$ is measurable and integrably bounded then the function $\chi(\Phi(\cdot))$ is integrable and

$$\chi\left(\int_0^\tau \Phi(s)ds\right) \leq \int_0^\tau \chi(\Phi(s))ds, \quad \text{for all } \tau \in J.$$

The following Theorem is due to Mönch.

Theorem 2.16. *[18] Let X be a Banach space, U an open subset of X and $0 \in U$. Suppose that $N : U \to X$ is a continuous map which satisfies Mönch's condition (that is, if $D \subseteq \overline{U}$ is countable and $D \subseteq \overline{co}(\{0\} \cup N(D))$, then $\overline{D}$ is compact) and assume that*

$$x \neq \lambda N(x), \quad \text{for } x \in \partial U \text{ and } \lambda \in (0,1)$$

holds. Then N has a fixed point in $\overline{U}$.

3. Main Results

Definition 3.1. *The function $x \in C(J,X)$ is a mild solution of* (1.6) − (1.7) *if it satisfies the following equation*

$$x(t) = U(t,0)[x_0 - g(x)] + \lim_{\lambda\to 0^+}\int_0^t U_\lambda(t,s)f(s,x(s))ds.$$

We introduce the following conditions:

(H_1) The evolution system $\{U(t,s)\}_{(t,s)\in\Delta}$ is compact, for $t > s > 0$.

(H_2) The function $f : J\times X \to X$ is Carathéodory .

(H_3) There exists a function $L > 0$ such that

$$\|f(t,x)\| \le L(1+\|x\|) \text{ for all } t \in J \text{ and all } x \in X.$$

(H_4) There is a constant $L_g > 0$ such that

$$\|g(u_2) - g(u_1)\| \le L_g\|u_2 - u_1\|, \text{ for all } u_1,u_2 \in X.$$

With

$$1 - Me^{\beta\ T}(L_g + L) > 0.$$

We make use Krasnosel'skii's fixed point Theorem to prove our first result.

Theorem 3.2. *Assume that* (H_1) − (H_4) *hold, and*

$$\eta = Me^{\beta T}L_g < 1.$$

Then the problem (1.6)-(1.7) *has at least one mild solution.*

Proof. Transform the problem (1.6)-(1.7) into a fixed point problem. Set

$$B_r = \{u \in C(J,X) : \ \|u\| \le r\}$$

where

$$r \ge \frac{Me^{\beta T}(\|x_0\| + \|g(0)\| + L)}{1 - Me^{\beta\ T}(L_g + L)}. \tag{3.1}$$

We define the operators P and Q on B_r as

$$(Px)(t) = U(t,0)\,(x_0 - g(x(t)))$$

$$(Qx)(t) = \lim_{\lambda\to 0^+}\int_0^t U_\lambda(t,s)f(s,x(s))ds.$$

For the sake of convenience, we divide the proof into several steps.

Step 1. For any $x \in B_r$, we prove that $Fx := Px + Qx \in B_r$.

$$\begin{aligned}
&\|(Px+Qx)(t)\| = \\
&\|U(t,0)g(x) + \lim_{\lambda\to 0^+}\int_0^t U_\lambda(t,s)f(s,x(s))ds\| \\
&\le \|U(t,0)g(x)\| + \lim_{\lambda\to 0^+}\int_0^t \|U_\lambda(t,s)f(s,x(s))\|ds \\
&\le Me^{\beta t}\|g(x)\| \\
&+ \lim_{\lambda\to 0^+}\int_0^t M\frac{1}{1-\lambda\beta}e^{-\beta(t-s)\frac{\ln(1-\lambda\beta)}{\lambda\beta}}L(1+\|x\|)ds \\
&\le M\ e^{\beta T}(L_g r + g(0)) + M\ Le^{\beta T}(1+r).
\end{aligned}$$

From (3.1), we infer that

$$\|(Px+Qx)\| \le r.$$

Hence

$$Fx = Px + Qx \in B_r.$$

Step 2. P is contraction on B_r.
For $u_1,u_2 \in B_r$ and for $t \in J$, we have

$$\begin{aligned}
\|(Pu_2)(t) - (Pu_1)(t)\| &= \|U(t,0)[g(u_2) - g(u_1)]\| \\
&\le Me^{\beta t}L_g\|u_2 - u_1\| \\
&\le Me^{\beta T}L_g\|u_2 - u_1\|.
\end{aligned}$$

From above, we obtain

$$\|Pu_2 - Pu_1\| \le \eta\|u_2 - u_1\|,$$

which implies that P is a contraction.

Step 3. We show that Q is continuous.
Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C(J,X)$. For each $t \in J$, we have

$$\begin{aligned}
\|(Qu_n)(t) - (Qu)(t)\| &= \Big\|\lim_{\lambda\to 0^+}\int_0^t U_\lambda(t,s)\,[f(s,u_n(s)) - \\
&\quad f(s,u(s))]\,ds\| \\
&\le M\ e^{\beta\ T}\,\|f(\cdot,u_n(\cdot)) - f(\cdot,u(\cdot))\|_{L^1}.
\end{aligned}$$

By (H_2) and (H_3), combined with Lebesgue dominated convergence Theorem, we have

$$\|Qu_n - Qu\| \to 0 \text{ as } n \to +\infty.$$

Thus Q is continuous.

Step 4. We prove Q is compact.
Define $\Gamma := QB_r$, that is, $\Gamma(t) := \{(Qu)(t);\ u \in B_r\}$ for $t \in J$.
We verify that $\Gamma(t)$ is relatively compact. We have
$\Gamma(0) = \{0;\ u \in B_r\} = 0$ which is independent on ε.
For $0 < \varepsilon < t \le T$ define

$$\Gamma_\varepsilon(t) := Q_\varepsilon B_r(t) = \{U(t,t-\varepsilon)(Qu)(t-\varepsilon);\ u \in B_r\},$$

then

$$\begin{aligned}
\Gamma_\varepsilon(t) &= (Q_\varepsilon u)(t) \\
&= U(t,t-\varepsilon)(Qu)(t-\varepsilon) \qquad t > \varepsilon \\
&= U(t,t-\varepsilon)\left(\lim_{\lambda\to 0^+}\int_0^{t-\varepsilon}U_\lambda(t-\varepsilon,s)f(s,u(s))ds\right) \\
&= \lim_{\lambda\to 0^+}\int_0^{t-\varepsilon}U_\lambda(t,s)f(s,u(s))ds.
\end{aligned}$$

Since $U(t,s)$ is a compact operator, the set

$$\Gamma_\varepsilon(t) := Q_\varepsilon B_r(t) = \{U(t,t-\varepsilon)(Qu)(t-\varepsilon);\ u \in B_r\}$$

is precompact in X for ε, $0<\varepsilon<t$. Moreover for every $u\in B_r$ we have

$$\begin{aligned}&\|(Qu)(t)-(Q_\varepsilon u)(t)\| = \\ &\left\|\lim_{\lambda\to0^+}\int_0^t U_\lambda(t,s)f(s,u(s))ds\right. \\ &\left.-\lim_{\lambda\to0^+}\int_0^{t-\varepsilon}U_\lambda(t,s)f(s,u(s))ds\right\| \\ &=\left\|\lim_{\lambda\to0^+}\int_{t-\varepsilon}^t U_\lambda(t,s)f(s,u(s))ds\right\| \\ &\leq \frac{M\,L(1+r)}{\beta}\left(e^{\beta\varepsilon}-1\right)\longrightarrow 0 \quad \text{as } \varepsilon\to0.\end{aligned}$$

Therefore, there are precompact sets arbitrary close to the set

$$\Gamma(t):=QB_r(t)=\{(Qu)(t);\ u\in B_r\}.$$

Hence the set $\Gamma(t)=\{Qu(t),\ u\in B_r\}$ is precompact in X and this yields the relatively compactness of $\Gamma(t)$.

On the other hand, Q is uniformly bounded on B_r since $\|Qu\|\leq r$.
We will prove now that Q maps bounded set into equicontinuous set of B_r.
Let $0\leq t_1<t_2\leq T$, $u\in B_r$, one has

$$\begin{aligned}&\|(Qu)(t_2)-(Qu)(t_1)\| = \\ &\left\|\lim_{\lambda\to0}\int_0^{t_2}U(t_2,s)f(s,u(s))ds\right. \\ &\left.-\lim_{\lambda\to0}\int_0^{t_1}U(t_1,s)f(s,u(s))ds\right\| \\ &\leq \left\|(U(t_2,t_1)-Id)\lim_{\lambda\to0}\int_0^{t_1}U(t_1,s)f(s,u(s))ds\right\| \\ &+\left\|\lim_{\lambda\to0}\int_{t_1}^{t_2}U(t_2,s)f(s,u(s))ds\right\|.\end{aligned}$$

Using assumption (H_1) and from the fact that the set

$$\left\{Q\,u(t_1)=\lim_{\lambda\to0}\int_0^{t_1}U(t_1,s)f(s,u(s))ds,\ u\in B_r\right\},$$

is relatively compact from previously , one can deduce

$$\lim_{t_1\to t_2 t_1<t_2}\ \sup_{u\in B_r}\|(U(t_2,t_1)-Id)\,Q\,u(t_1)\|=0.$$

In another hand,

$$\left\|\lim_{\lambda\to0}\int_{t_1}^{t_2}U(t_2,s)f(s,u(s))ds\right\| \quad\leq\quad M\,e^{\beta T}L(1+r)(t_2-t_1).$$

Finally

$$\lim_{t_1\to t_2 t_1<t_2}\ \sup_{u\in B_r}\|(Qu)(t_2)-(Qu)(t_1)\|=0.$$

Then, we get QB_r is equicontinuous.

As a consequence of the Arzela-Ascoli Theorem, we can conclude that

$$Q:B_r\to B_r,$$

is completely continuous. And then, Krasnoselskii's fixed point Theorem, infer us that $F=P+Q$ has a fixed-point which is a mild solution to problem (1.6)-(1.7). ☐

The following result is based on Mönch's fixed point Theorem combining with the measure of noncompactness.

Theorem 3.3. *Assume the following hypotheses hold:*

(H_5) *The system evolution $(t,s)\to U(t,s)$, is uniformly norm continuous for $(t,s)\in\Delta$.*

(H_6) *There exists a function $\zeta\in L^1(J,R^+)$ such that*

$$\|f(t,x)\|\leq\zeta(t)(1+\|x\|)\ \textit{for all } t\in J \textit{ and all } x\in X.$$

(H_7) *There exists a function $\delta\in L^1(J,R^+)$ such that for every nonempty, bounded set $\Omega\subset X$ we have*

$$\chi(f(t,\Omega))\leq\delta(t)\chi(\Omega)\ \textit{for each } t\in J \textit{ and all } x\in X.$$

with

$$1-Me^{\beta\,T}\,(\|\zeta\|_{L^1}+L_g)>0,$$

and χ is the Hausdorff measure of noncompactness in X.

(H_8) *There exists a constant $C_g>0$ such that*

$$\chi(g(\Omega))\leq C_g\chi(\Omega)\ \textit{for all subset } \Omega\subset C(J,R).$$

(H_9) *If $\Omega\subset C(J,R)$ is a bounded set, then*

$$mod_c U(.,.)g(\Omega)=0.$$

Then the nonlocal problem (1.6)-(1.7) has at least one mild solution on J.

Proof. Transform the problem (1.6)-(1.7) into a fixed point problem. Consider the operator

$$N:C(J,X)\to C(J,X)$$

defined by

$$N(x)(t)=U(t,0)[x_0-g(x(t))]+\lim_{\lambda\to0^+}\int_0^t U_\lambda(t,s)f(s,x(s))ds.$$

The fixed points of the operator N are solutions of the problem (1.6) – (1.7). We shall use Mönch's fixed point Theorem to prove that N has a fixed point.

Proof. We break the proof into a sequence of steps.

N is a continuous.

Arguing exactly by the same reasoning as in the previous result concerning the continuity of Q, one can easily prove the continuity of N.

N is a ν condensing operator.

We consider the measure of noncompactness defined in the following way. For every bounded subset $\Omega\subset$ ■

$$\nu(\Omega)=\max_{D\in\Delta(\Omega)}\{\gamma(D),\quad \mathrm{mod}_C(D)\}, \tag{3.2}$$

where $\Delta(\Omega)$ is the collection of all the denumerable subsets of Ω,

$$\gamma(D) = \sup_{t\in J} e^{-Lt}\chi(\{x(t) : x \in D\}), \tag{3.3}$$

where $\text{mod}_C(D)$ is the modulus of equicontinuity of the set of functions D given by the formula

$$\text{mod}_C(D) = \lim_{\delta\to 0}\sup_{x\in D}\max_{|t_1-t_2|\le\delta} \|x(t_1) - x(t_2)\|, \tag{3.4}$$

and $L > 0$ is a positive real number chosen so that

$$q := \sup_{t\in J}\left(2\, M^* \int_0^t \delta(s)e^{-L(t-s)}ds + M^*\, C_g\, e^{Lt}\right) < 1. \tag{3.5}$$

where $\sup_{(t,s)\in\Delta} \|U(t,s)\| \le M^*$.

From the Arzela-Ascoli Theorem, the measure ν gives a nonsingular and regular measure of noncompactness, (see [14]).

We shall prove now that N is ν condensing operator. Let $\{y_n\}_{n=1}^{+\infty}$ be the denumerable set which achieves that maximum $\nu(N(\Omega))$, i,e.,

$$\nu(N(\Omega)) = \max\{\gamma(\{y_n\}_{n=1}^{+\infty}), \quad \text{mod}_C(\{y_n\}_{n=1}^{+\infty})\}.$$

Then there exists a set $\{x_n\}_{n=1}^{+\infty} \subset \Omega$ such that $y_n = N(x_n)$, $n \ge 1$. Then

$$y_n(t) = U(t,0)[x_0 - g(x_n(t))] + \lim_{\lambda\to 0^+}\int_0^t U_\lambda(t,s)f(s,x_n(s))ds. \tag{3.6}$$

Suppose that

$$\nu(N(\Omega)) \ge \nu(\Omega). \tag{3.7}$$

We define the operators

$$\Upsilon : L^1(J,E) \to C(J,E),$$

by

$$\Upsilon(f)(t) = \lim_{\lambda\to 0^+}\int_0^t U_\lambda(t,s)f(s,x(s))ds,$$

and

$$\Upsilon^*(x)(t) = U(t,0)[x_0 - g(x)].$$

We have immediately

$$N(x) = \Upsilon N_f(x) + \Upsilon^*(x)$$

where ΥN_f is the Nemytskii operator corresponding to the nonlinearity f.

From the construction of Υ and Υ^* we have

$$y_n = \Upsilon^*(x_n) + \Upsilon(f_n) \tag{3.8}$$

where

$$f_n(t) = f(t,x_n(t)),$$

$$\Upsilon^*(x_n)(t) = U(t,0)[x_0 - g(x_n)],$$

$$\Upsilon(f_n)(t) = \lim_{\lambda\to 0^+}\int_0^t U_\lambda(t,s)f(s,x(s))ds.$$

We give an upper estimate for $\gamma(\{y_n\}_{n=1}^{+\infty})$.

For fixed $t \in J$, by using condition (H_7), for all $s \in J$, we have

$$\begin{aligned}\chi(\{f_n(s)\}_{n=1}^{+\infty}) &\le \chi(f(s,\{x_n(s)\}_{n=1}^{+\infty}))\\ &\le \delta(s)\chi(\{x_n(s)\}_{n=1}^{+\infty})\\ &\le \delta(s)e^{Ls}\sup_{t\in J} e^{-Lt}\chi(\{x_n(t)\}_{n=1}^{+\infty})\\ &= \delta(s)e^{Ls}\gamma(\{x_n\}_{n=1}^{+\infty}).\end{aligned} \tag{3.9}$$

By using condition (H_6), the set $\{f_n\}_{n=1}^{+\infty}$ is integrably bounded. In fact, for every $t \in J$, we have

$$\begin{aligned}\|f_n(t)\| &= \|f(t,x_n(t))\|\\ &\le \zeta(t)(1 + \|x_n(t)\|).\end{aligned}$$

By using hypothesis (H_5) and the same arguments as those in [14], Lemma 4.2.1, Theorem 4.2.2, Proposition 4.2.1, and Theorem 5.1.1, one can verify the following lemmas (2.12), (2.13), (2.14) for the operator Υ.

Thus by applying Lemma (2.12), (2.14) and (3.9), it follows that

$$\begin{aligned}\chi(\{\Upsilon f_n(s)\}_{n=1}^{+\infty}) &\le 2M^*\int_0^s \delta(t)e^{Lt}(\gamma(\{x_n\}_{n=1}^{+\infty}))dt\\ &= 2M^*\gamma(\{x_n\}_{n=1}^{+\infty})\int_0^s \delta(t)e^{Lt}dt.\end{aligned} \tag{3.10}$$

Noting that

$$\Upsilon^*(x_n)(t) = U(t,0)[x_0 - g(x_n)],$$

and using (H_8), we have

$$\begin{aligned}\chi(\{\Upsilon^* x_n(s)\}_{n=1}^{+\infty}) &= \chi(\{U(t,0)(x_0 - g(x_n)\}_{n=1}^{+\infty})\\ &\le \chi(U(t,0)\{g(x_n)\}_{n=1}^{+\infty})\\ &\le M^*\, C_g\chi(\{x_n\}_{n=1}^{+\infty})\\ &\le M^*\, e^{Lt}C_g\gamma(\{x_n\}_{n=1}^{+\infty}).\end{aligned} \tag{3.11}$$

Thus, we get from (3.7), (3.8), (3.10) and (3.11),

$$\begin{aligned}\gamma(\{x_n\}_{n=1}^{+\infty}) &\le \gamma(\{y_n\}_{n=1}^{+\infty})\\ &\le \gamma(\{\Upsilon f_n\}_{n=1}^{+\infty}) + \gamma(\{\Upsilon^* x_n\}_{n=1}^{+\infty})\\ &= \sup_{t\in J} 2M^*\gamma(\{x_n\}_{n=1}^{+\infty})\int_0^t \delta(s)e^{-L(t-s)}ds\\ &+ M^*\, C_g\, \gamma(\{x_n\}_{n=1}^{+\infty}\\ &\le q\gamma(\{x_n\}_{n=1}^{+\infty}).\end{aligned} \tag{3.12}$$

Therefore, we have that

$$\gamma(\{x_n\}_{n=1}^{+\infty}) \le q\gamma(\{x_n\}_{n=1}^{+\infty}). \tag{3.13}$$

From (3.5) combined with (3.7) and (3.13), we obtain that

$$\gamma(\{x_n\}_{n=1}^{+\infty}) = 0. \tag{3.14}$$

Then $\gamma(\{x_n(t)\}_{n=1}^{+\infty}) = 0$ for every $t \in J$, and thus

$$\gamma(\{y_n(t)\}_{n=1}^{+\infty}) = 0 \text{ for every } t \in J.$$

Consequently

$$\gamma(\{y_n\}_{n=1}^{+\infty}) = 0. \tag{3.15}$$

It remains now to prove that $\mod_C(\{y_n\}_{n=1}^{+\infty})) = 0$.

By using (3.9)-(3.14) and assumption (H_6) we can prove that set $\{f_n\}_{n=1}^{+\infty}$ is semicompact. Now, by applying Lemma (2.13), we can conclude that the set $\{\Upsilon f_n\}_{n=1}^{+\infty}$ is relatively compact in $C(J,E)$.

Next,

$$\mod_C(\{\Upsilon f_n\}_{n=1}^{+\infty}) = 0,$$

and from (H_9) we have

$$\mod_C(\{\Upsilon^*(x_n)\}_{n=1}^{+\infty}) = 0.$$

Taking (3.8) into account we deduce

$$\mod_C(\{(y_n)\}_{n=1}^{+\infty}) = 0,$$

and then

$$\nu(N(\Omega)) = (0,0).$$

From the meaning of (3.7), $\nu(\Omega) = (0,0)$, and from the regularity of ν we deduce that Ω is relatively compact. Hence N is ν condensing operator.

We shall now verify the Mönch condition. Let $D \subseteq \overline{U}$ be countable, bounded and $D \subseteq \overline{co}(\{0\} \cup N(D))$. Since ν is a monotone, nonsingular, regular MNC, one has

$$\nu(D) \leq \nu(\overline{co}(\{0\} \cup N(D))) \leq \nu(N(D)).$$

Therefore $\nu(D) = (0,0)$. Then D is a relatively compact set.

A priori bounds..

We will demonstrate that the solutions set is *a priori* bounded.

Indeed, let $x \in \lambda N_x$ and $\lambda \in (0,1)$. For every $t \in J$ we have

$$\begin{aligned}
\|x(t)\| &\leq \|U(t,0)\| \|x_0 - g(x)\| \\
&+ \lim_{\lambda \to 0^+} \int_0^t \|U_\lambda(t,s)\| \|f(s,x(s))\| ds \\
&\leq M\, e^{\beta\, T} (\|x_0\| + L_g \|x\| + \|g(0)\|) \\
&+ M\, e^{\beta\, T} \|\zeta\|_{L^1} (1 + \|x\|).
\end{aligned}$$

Hence

$$\|x\| \Big(1 - M e^{\beta\, T} (\|\zeta\|_{L^1} + L_g) \tag{3.16}$$

$$\leq M\, e^{\beta\, T} (\|x_0\| + \|g(0)\| + \|\zeta\|_{L^1})\Big). \tag{3.17}$$

Consequently

$$\|x\| \leq \frac{M\, e^{\beta\, T} (\|x_0\| + \|g(0)\| + \|\zeta\|_{L^1})}{\big(1 - M e^{\beta\, T} (\|\zeta\|_{L^1} + L_g)\big)} = C.$$

So, there exists N^* such that $\|x\| \neq N^*$. Set

$$U = \{x \in \Omega : \quad \|x\| < N^*\}.$$

From the choice of U there is no $x \in \partial U$ such that $x = \lambda N x$ for some $\lambda \in (0,1)$.

Thus, we get a fixed point of N in $\bar{U}$ due to the Mönch's Theorem. □

4. An example

As an application of our results we consider the following partial differential equation with nonlocal conditions of the form

$$\frac{\partial}{\partial t} z(x,t)) = a(x,t) \frac{\partial^2}{\partial x^2} z(x,t) + \varphi(t) \sin z(x,t), \quad x \in [0,\pi],\ t \in J := [0,T]. \tag{4.1}$$

$$z(x,0) = \sum_{i=1}^{m} c_i z(x,t_i)) + z_0, \quad x \in [0,\pi],\ t_i \in (0,T), \quad i = 1,\ldots,m,. \tag{4.2}$$

$$z(0,t) = z(\pi,t) = 0. \tag{4.3}$$

Where $a(.,.) : [0,\pi] \times J \to \mathbb{R}_*^+$ is continuous function and uniformly Hölder continuous in t, $z_0 \in R$, $c_i \in R$ for $i = 1 \ldots m$. Let $X = L^2([0,\pi])$ and the operator $A(t)$ defined by

$$A(t)w = a(x,t)w'',$$

with the domain

$$D(A) = \big\{w(.) \in X,\ \ w, w' \text{ absolutely continuous} \tag{4.4}$$

$$w'' \in X,\ w(0) = w(\pi) = 0\big\} \tag{4.5}$$

Let Δ be the Laplacien operator on $[0,\pi]$ with domain

$$D = \big\{w \in C^2([0,\pi],\mathbb{R}),\ \ w(0) = w(\pi) = 0\big\},$$

and

$$\Delta w = w''.$$

Then Δ satisfies the following conditions

$$(0,+\infty) \subset \rho(\Delta),\ \|R(\lambda,\Delta)\| \leq \frac{1}{\lambda},\ \lambda > 0,$$

we have $A(t) = a(x,t)\Delta$ with domain $D(A) = D$. For $\lambda > 0$

$$\begin{aligned} R(\lambda, A(t)) &= (\lambda I - A(t))^{-1} \\ &= (\lambda I - a(x,t)\Delta)^{-1} \\ &= \frac{1}{a(x,t)} R(\frac{\lambda}{a(x,t)}, \Delta), \end{aligned}$$

for every $\lambda > 0$, $\lambda \in \rho(A(t))$ and $\|R(\lambda, A(t))\| \leq \frac{1}{\lambda}$ then,

$$\left\| \prod_{i=1}^{n} R(\lambda, A(t_i)) \right\| \leq \frac{1}{\lambda^n}, \qquad t_1 \leq t_2 \leq \ldots \leq t_n.$$

Moreover

$$\overline{D} = \{w \in C([0,\pi], \mathbb{R}), \ \ w(0) = w(\pi) = 0\} \neq X.$$

The operator Δ has a discrete spectrum and the eigenvalues are "$-n^2$", $n \in \mathbb{N}$ with the corresponding normalized eigenvectors $v_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$. Thus for $w \in D(A)$, there holds

$$A(t)w = \sum_{n=1}^{\infty} (-n^2) < w, v_n > v_n,$$

where $< .,. >$ is the inner product in L^2 and the domain $D(A(t))$ coincides with that of the operator Δ.
We can verify that $A(t)$ generates an evolution operator $U(t,s)$ satisfying the assumptions (ii)-(v) in Theorem (2.1) and for each $w \in X$ it is given by

$$\begin{aligned} U(t,s)w &= \sum_{n=1}^{\infty} \exp(-n^2(t-s)) < w, v_n > v_n \\ &= \sum_{n=1}^{\infty} \frac{2}{\pi} \exp(-n^2(t-s)) \sin nx \int_0^{\pi} w(\zeta) \sin n\zeta \, d\zeta. \end{aligned}$$

From these expressions it follows that $\{U(t,s)\}$, $(0 \leq s \leq t \leq T)$ is uniformly bounded compact evolution system (H_1), then there exists a constant $M \geq 1$ and $\beta \geq 0$ such that

$$\|U(t,s)w\| \leq Me^{\beta T} \|w\|$$

To write system (4.1)-(4.3) in the form (1.6)-(1.7) we define $f : J \times X \to X$, $g : C(J,X) \to X$ defined by

$$f(t, x(t)) = \varphi(t) \sin z(x,t)$$

$$g(z(x,t)) = \sum_{i=1}^{m} c_i z(x, t_i),$$

note that f is Carathéodory function which yields condition (H_2),

$$\|g(u(t)) - g(v(t))\| \leq \sum_{i=1}^{m} |c_i| \|u - v\|, \text{Ł} = \sup_{t \in J} \|\varphi(t)\|,$$

$L_g = \sum_{i=1}^{m} |c_i|$, and choose c_i such that

$$\eta = Me^{\beta b} L_g < 1, \ 1 - Me^{\beta T}(L_g + L) > 1$$

, and

$$r \geq \frac{Me^{\beta T}(|z_0| + |g(0)| + L)}{1 - Me^{\beta T}(L_g + L)}.$$

An easy computation allow us to verify condition (H_3), and from the choose of $\{c_i\}_{i=1}^{n}$ it follows condition (H_4). Since the conditions $(H_1) - (H_4)$ of the Theorem (3.2) are satisfied, the problem (4.1)-(4.3) has at least one mild solution.

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