Tension spline technique for the solution of fourth-order parabolic partial differential equation

Talat Sultana[1]* and Pooja Khandelwal[2]

Abstract
In this paper, we propose a spline approach for the numerical solution of fourth order parabolic partial differential equation that governs the behavior of a vibrating beam. We have used nonpolynomial cubic tension spline in space and finite difference discretization in time. Class of methods and Stability analysis have been carried out. Finally, some numerical examples are presented to illustrate the efficiency and accuracy of the proposed method.

Keywords
Cubic tension spline; Parabolic partial differential equation; Stability analysis; Vibrating beam; Finite difference discretization.

AMS Subject Classification
65N06, 65N12.

[1] *Department of Mathematics, Lakshmibai College, University of Delhi, New Delhi-52, India.*
[2] *Department of Mathematics, M. L. V. Textile and Engineering College, Bhilwara, Rajasthan-311001, India.*
***Corresponding author**: [1] talat17m@gmail.com; [2] pooja2n@gmail.com

Contents

1. Introduction

Numerical methods for the solution of fourth-order parabolic partial differential equations have been carried out by many authors. Caglar and Caglar [3] proposed a fifth degree B-spline method, Conte [5], Crandall [6], Jain et al. [8] and Todd [16] have proposed both explicit and implicit methods successfully to solve fourth order parabolic partial differential equations. Collatz [4] and Evans and Yousif [7] used finite difference methods for such problems. The nonhomogeneous problem with constant coefficients has been studied by Aziz et al. [2] based on parametric quintic spline, Khan et al. [9] based on sextic spline and Khan and Sultana [10] based on parametric septic spline by using nodal points. Rashidinia and Aziz [13] developed three level methods based on parametric

quintic spline and Siddiqi and Arshed [15] developed quintic B-spline collocation method for the approximate solution of such problems.

The problem of undamped transverse vibrations of a flexible straight beam is considered in such a way that its support do not contribute to the strain energy of the system and is represented by the fourth-order parabolic partial differential equation of the form:

$$\frac{\partial^2 z}{\partial t^2} + \frac{\partial^4 z}{\partial x^4} = f(x,t), \ a \leq x \leq b, \ t > 0, \qquad (1.1)$$

subject to the initial conditions:

$$\left.\begin{array}{l} z(x,0) = g_0(x), \ a \leq x \leq b, \\[2mm] z_t(x,0) = g_1(x), \ a \leq x \leq b \end{array}\right\} \qquad (1.2)$$

and the boundary conditions are

$$\left.\begin{array}{l} z(a,t) = f_0(t), \ z(b,t) = f_1(t), \ t \geq 0, \\[2mm] z_{xx}(a,t) = q_0(t), \ z_{xx}(b,t) = q_1(t), \ t \geq 0, \end{array}\right\} \qquad (1.3)$$

where z is the transverse displacement of the beam, $g_0(x)$, $g_1(x)$, $f_0(t)$, $f_1(t)$, $q_0(t)$, $q_1(t)$ are continuous functions, t and x are time and distance variables respectively and $f(x,t)$ is dynamic driving force per unit mass [11].

In [12], Mittal and Jain discussed two methods. In Method-I, they decomposed equation (1.1) in a system of second order equations and have solved them by using cubic B-spline and in Method-II, they have solved equation (1.1) directly by using quintic B-spline method. Rashidinia and Mohammadi [14] developed three level implicit methods of $O(k^2 + h^4)$ and $O(k^4 + h^4)$ for the numerical solution of equation (1.1) with variable coefficients by using sextic spline. The analytic solution of homogeneous fourth-order parabolic partial differential equation based on Adomain decomposition method was given by Wazwaz [17].

The purpose of this paper is to present a new numerical method for obtaining smooth approximations for solving homogeneous and nonhomogeneous parabolic partial differential equations based on nonpolynomial cubic tension spline function approximation. Computational efforts of our method are less compare to other existing methods. Also, the numerical and graphical demonstration shows the practical usefulness of our method.

The paper is organised as follows: In section 2, we give a brief derivation of nonpolynomial cubic tension spline functions. In section 3, we have presented the formulation of our method with development of boundary equations. In section 4, truncation error is given. Stability analysis is discussed in section 5. Finally in section 6, four examples are considered to demonstrate the practical usefulness and superiority of our method.

2. Nonpolynomial cubic spline functions

We consider a set of grid points in the interval $[a,b]$ such that

$$x_i = a + ih, \ i = 0(1)n, \ h = \frac{(b-a)}{n}.$$

A nonpolynomial cubic spline function $S_i(x)$ of class $C^2[a,b]$ which interpolates $z(x)$ at mesh points x_i, $i = 0(1)n$ depends on a parameter τ, if we take $\tau \to 0$, then it reduces to ordinary cubic spline in $[a,b]$.

For each segment $[x_i, x_{i+1}]$, $i = 0,1,2\cdots n-1$, we consider the nonpolynomial cubic spline $S_i(x)$ of the form:

$$\begin{aligned} S_i(x) &= a_i(e^{\tau(x-x_i)} + e^{-\tau(x-x_i)}) + b_i(e^{\tau(x-x_i)} - e^{-\tau(x-x_i)}) \\ &\quad + c_i(x-x_i) + d_i, \ \ i = 0,1,\ldots,n, \end{aligned} \quad (2.1)$$

where a_i, b_i, c_i and d_i are unknown coefficients and τ is a free parameter which will be used to raise the accuracy of the method.

Let $z(x)$ be the exact solution and z_i be an approximation to $z(x_i)$, obtained by the segment $S_i(x)$ of the mixed splines function passing through the points (x_i, z_i) and (x_{i+1}, z_{i+1}). To determine the coefficients of equation (2.1) in terms of $z_i, z_{i+1}, M_i, M_{i+1}$, we first define:

$$\left. \begin{aligned} S_i(x_i) &= z_i, \ \ S_i(x_{i+1}) = z_{i+1}, \\ S_i''(x_i) &= M_i, \ \ S_i''(x_{i+1}) = M_{i+1}, \end{aligned} \right\} \quad (2.2)$$

From algebraic manipulation using equation (2.1) and equation (2.2), we obtain the following expressions:

$$\begin{aligned} a_i &= \frac{M_i}{2\tau^2}, \\ b_i &= \frac{2M_{i-1} - (e^\theta + e^{-\theta})M_i}{2\tau^2(e^\theta - e^{-\theta})}, \\ c_i &= \frac{M_i - M_{i+1} + \tau^2(z_{i+1} - z_i)}{\tau\theta}, \\ d_i &= z_i - \frac{M_i}{\tau^2}, \ \ \theta = \tau h, i = 0(1)n-1. \end{aligned}$$

Substituting these values in equation (2.1) and using the continuity condition of the first derivative at the point $x = x_i$, i.e. $S_j'(x_j) = S_{j+1}'(x_j)$, we obtain the following tridiagonal system for $i = 1,2,\cdots,n-1$:

$$z_{i-1} - 2z_i + z_{i+1} = h^2(\alpha \, M_{i-1} + 2\beta \, M_i + \alpha \, M_{i+1}), \quad (2.3)$$

where

$$\begin{aligned} \alpha &= \frac{1}{\theta^2}\left(1 - \frac{2\theta}{(e^\theta - e^{-\theta})}\right), \\ \beta &= \frac{1}{\theta^2}\left(\frac{\theta(e^\theta + e^{-\theta})}{(e^\theta - e^{-\theta})} - 1\right). \end{aligned}$$

If $\theta \to 0$, then $(\alpha, \beta) \to (\frac{1}{6}, \frac{1}{3})$, then the relation (2.3) reduces to a ordinary cubic spline relation:

$$(z_{i-1} - 2z_i + z_{i+1}) = \frac{h^2}{6}(M_{i-1} + 4 \, M_i + M_{i+1}). \quad (2.4)$$

Also for the differential equation in (1.1) at the knots x_i, we may have [1]

$$M_{i-1} - 2 \, M_i + M_{i+1} = h^2 z_i^{(4)} + O(h^6). \quad (2.5)$$

Using equation(2.3) and equation(2.5), we obtain the following fourth order scheme:

$$z_{i-2} - 4z_{i-1} + 6z_i - 4z_{i+1} + z_{i+2} =$$
$$h^4(\alpha \, F_{i-1} + (\beta + 4\alpha) \, F_i + \alpha \, F_{i+1}), \quad i = 2(1)(n-2), \quad (2.6)$$

where $S_i^{(4)}(x_i) = F_i$.

3. The method

Let the region $R = [a,b] \times [0,\infty)$ be discretized by a set of points $R_{h,k}$ which are the vertices of a grid points (x_i, t_m), where $x_i = ih$, $i = 0(1)n$, $nh = b - a$ and $t_m = mk$, $m = 0,1,2,3,...$ The quantities h and k are mesh sizes in the space and time directions respectively.

We replaced the time derivative by a finite difference approximation and space derivative by nonpolynomial cubic tension spline function for approximation of (1.1). We need

the following finite difference approximation for the time partial derivative of z :

$$\bar{z}_{tt_i}^m = k^{-2}\delta_t^2(1+\sigma\delta_t^2)^{-1}z_i^m, \qquad (3.1)$$

where σ is a parameter such that the finite difference approximation to the time derivative is $O(k^2)$ for arbitrary σ, z_i^m is the approximate solution of (1.1) at (x_i, t_m) and δ_t is the central difference operator with respect to t so that

$$\delta_t^2 z_i^m = z_i^{m+1} - 2z_i^m + z_i^{m-1}.$$

At the grid point (i, m) the differential equation may be discretized by

$$\bar{z}_{tt_i}^m + \bar{z}_{xxxx_i}^m = f_i^m, \qquad (3.2)$$

where $\bar{z}_{xxxx_i}^m$ is the fourth-order spline derivative at (x_i, t_m) denoted by $F_i^m = S_i^{(4)}(x_i, t_m)$ with respect to the space variable and $f_i^m = f(x_i, t_m)$. Using (3.1) and replacing fourth order spline derivative by F_i^m, in (3.2) we have

$$k^{-2}\delta_t^2(1+\sigma\delta_t^2)^{-1}z_i^m + F_i^m = f_i^m. \qquad (3.3)$$

Operating Λ_x on both sides of (3.3), we obtain

$$\delta_t^2[(6\alpha+\beta)+\alpha\delta_x^2+\sigma r^2\delta_x^4]z_i^m + r^2\delta_x^4 z_i^m$$
$$= k^2(1+\sigma\delta_t^2)[\alpha f_{i-1}^m + (\beta+4\alpha)f_i^m + \alpha f_{i+1}^m], \quad i=2(1)(n-1), \qquad (3.4)$$

where $r = \frac{k}{h^2}$, δ is a central difference operator and the operator Λ_x for any function L is defined by

$$\Lambda_x L_i = \alpha L_{i-1} + (\beta+4\alpha) L_i + \alpha L_{i+1}.$$

This scheme (3.4) is finite difference in time and spline scheme in space variable, which on simplification can be written as

$$[U_1(z_{i-2}^{m+1} + z_{i+2}^{m+1}) + U_2(z_{i-1}^{m+1} + z_{i+1}^{m+1}) + U_3 z_i^{m+1}]$$

$$+[V_1(z_{i-2}^m + z_{i+2}^m) + V_2(z_{i-1}^m + z_{i+1}^m) + V_3 z_i^m]$$

$$+[U_1(z_{i-2}^{m-1} + z_{i+2}^{m-1}) + U_2(z_{i-1}^{m-1} + z_{i+1}^{m-1}) + U_3 z_i^{m-1}]$$

$$= K_1[\alpha(f_{i-1}^{m+1} + f_{i+1}^{m+1}) + (\beta+4\alpha)f_i^{m+1}]$$

$$+K_2[\alpha(f_{i-1}^m + f_{i+1}^m) + (\beta+4\alpha)f_i^m]$$

$$+K_1[\alpha(f_{i-1}^{m-1} + f_{i+1}^{m-1}) + (\beta+4\alpha)f_i^{m-1}], \quad i=2(1)(n-2). \qquad (3.5)$$

where

$$U_1 = \sigma r^2, \ U_2 = \alpha - 4\sigma r^2, \ U_3 = (\beta+4\alpha)+6\sigma r^2,$$

$$V_1 = r^2 - 2\sigma r^2, \ V_2 = -2\alpha - 4r^2 + 8\sigma r^2,$$

$$V_3 = -2\beta - 8\alpha + 6r^2 - 12\sigma r^2, K_1 = \sigma k^2, K_2 = k^2(1-2\sigma).$$

The relation (3.5) gives $n-3$ linear algebraic equations in $n-1$ unknowns z_i, $i = 2(1)(n-2)$. We need two more equations, one at each end of the range of integration, for the direct computation of z_i, $i = 1(1)(n-1)$. For formula accuracy of

$O(k^2+h^2)$, we use the following equations for approximating the boundary equations:

(i) $\dfrac{52}{5}z_1^m - \dfrac{57}{5}z_2^m + \dfrac{28}{5}z_3^m - \dfrac{11}{10}z_4^m = \dfrac{7}{2}z_0^m - \dfrac{6}{5}h^2(z_0^m)'', \ i=1,$

(ii) $\qquad -\dfrac{11}{10}z_{n-4}^m + \dfrac{28}{5}z_{n-3}^m - \dfrac{57}{5}z_{n-2}^m + \dfrac{52}{5}z_{n-1}^m$

$$= \dfrac{7}{2}z_n^m - \dfrac{6}{5}h^2(z_n^m)'', \ i=n-1.$$

For high accuracy formula of $O(k^2+h^4)$, we use the following equations for approximating the boundary equations:

(i) $\dfrac{174}{7}z_1^m - \dfrac{585}{14}z_2^m + \dfrac{2540}{63}z_3^m - \dfrac{165}{7}z_4^m + \dfrac{54}{7}z_5^m - \dfrac{137}{126}z_6^m$

$$= \dfrac{58}{9}z_0^m - \dfrac{10}{7}h^2(z_0^m)'', \ i=1,$$

(ii)
$-\dfrac{137}{126}z_{n-6}^m + \dfrac{54}{7}z_{n-5}^m - \dfrac{165}{7}z_{n-4}^m + \dfrac{2540}{63}z_{n-3}^m - \dfrac{585}{14}z_{n-2}^m + \dfrac{174}{7}z_{n-1}^m$

$$= \dfrac{58}{9}z_n^m - \dfrac{10}{7}h^2(z_n^m)'', \ i=n-1.$$

4. Class of methods

Expanding (3.4) in Taylor series in terms of $z(x_i, t_m)$ and its derivatives, we obtain the following relations:

$$\delta_x^4 z(x_i,t_m) = \left[h^4 D_x^4 + \frac{120}{6!}h^6 D_x^6 + \frac{505}{8!}h^8 D_x^8 + \frac{1016}{10!}h^{10}D_x^{10}\right.$$
$$\left. + \frac{2040}{12!}h^{12}D_x^{12} + ...\right]z(x_i,t_m)$$

$$\delta_t^2 z(x_i,t_m) = \left[-r^2 h^4 D_x^4 + \frac{1}{12}r^4 h^8 D_x^8 - \frac{1}{360}r^6 h^{12}D_x^{12}\right.$$
$$\left. + \frac{1}{20160}r^8 h^{16}D_x^{16} + ...\right]z(x_i,t_m), \ (4.1)$$

where $(D_t^2 + D_x^4)z(x_i, t_m) = f(x_i, t_m)$. Using (3.4) and (4.1), we obtain the following truncation error:

$$T_i^m = \left[(6\alpha+\beta)\left(\frac{1}{12}-\sigma\right)k^4 D_t^4 + (6\alpha+\beta)\left(\frac{1}{360}-\frac{\sigma}{12}\right)k^6 D_t^6\right.$$

$$\alpha\left(\frac{1}{12}-\sigma\right)k^4 h^2 D_t^4 D_x^2 + \alpha\left(\frac{1}{360}-\frac{\sigma}{12}\right)k^6 h^2 D_t^6 D_x^2$$

$$+\alpha\left(\frac{1}{12}-\sigma\right)\frac{k^4 h^4}{12}D_t^4 D_x^4$$

$$+\frac{\alpha}{12}\left(\frac{1}{360}-\frac{\sigma}{12}\right)k^6 h^4 D_t^6 D_x^4 + \cdots$$

$$+\frac{\alpha}{360}\left(\frac{1}{12}-\sigma\right)k^4 h^6 D_t^4 D_x^6 + \frac{\alpha}{4320}\left(\frac{1}{30}-\sigma\right)k^6 h^6 D_t^6 D_x^6$$

$$+\frac{\alpha}{20160}\left(\frac{1}{12}-\sigma\right)k^4 h^8 D_t^4 D_x^8$$

$$+\frac{\alpha}{241920}\left(\frac{1}{30}-\sigma\right)k^6 h^8 D_t^6 D_x^8 + \cdots$$

$$+k^2(1-6\alpha-\beta)D_x^4 + \left(\frac{1}{6}-\alpha\right)k^2h^2D_x^6$$

$$+\left(\frac{1}{80}-\frac{\alpha}{12}\right)k^2h^4\frac{D_x^8}{80}+\cdots$$

$$+\sigma(1-6\alpha-\beta)k^4D_t^2D_x^4 + \sigma\left(\frac{1}{6}-\alpha\right)k^4h^2D_t^2D_x^6$$

$$+\sigma\left(\frac{1}{80}-\frac{\alpha}{12}\right)k^4h^4D_t^2D_x^8+\cdots$$

$$+\frac{\sigma}{12}(1-6\alpha-\beta)k^6D_t^4D_x^4 + \sigma\left(\frac{1}{72}-\frac{\alpha}{12}\right)k^6h^2D_t^4D_x^6$$

$$+\sigma\left(\frac{1}{960}-\frac{\alpha}{144}\right)k^6h^4D_t^4D_x^8+\cdots\Bigg]z_i^m$$

$$(4.2)$$

Here, we obtain the following class of methods:
Case 1: When $(1-6\alpha-\beta)=0$, we obtain various schemes of $O(k^2+h^2)$ for arbitrary values of α, β, and σ.
Case 2: when $\alpha=\frac{1}{6}$, $\beta=0$, we obtain a scheme of $O(k^2+h^4)$ for any σ.

5. Stability analysis and convergence

To investigate the stability analysis of the scheme (3.5), we use the Von Neumann method. We have assumed that the solution of (3.5) at the grid point (x_i, t_m) is of the form:

$$z_i^m = \eta^m e^{i\theta}, \quad (5.1)$$

where $i=\sqrt{-1}$, θ is real and η is complex in general. Substituting (5.1) in homogeneous part of (3.5), we obtain a characteristic equation

$$A\eta^2 + B\eta + C = 0, \quad (5.2)$$

where
$$A = C = U_1\cos 2\theta + U_2\cos\theta + 2U_3,$$
$$B = V_1\cos 2\theta + V_2\cos\theta + 2V_3.$$

Under the transformation $\eta=\frac{1+\xi}{1-\xi}$, equation (5.2) becomes

$$(A-B+C)\xi^2 + 2(A-C)\xi + (A+B+C) = 0. \quad (5.3)$$

The necessary and sufficient condition for $|\eta|\le 1$ is that $A-B+C>0$, $A-C>0$ and $A+B+C>0$.
The conditions $A-C>0$ and $A+B+C>0$ are always satisfied for all real values of θ.
From the condition $A-B+C>0$, we get that the scheme (3.5) is unconditionally stable if $\sigma\ge\frac{1}{4}$ and conditionally stable if $\sigma<\frac{1}{4}$ for all real values of α, β and θ.
We summarized the above results in the following theorem:

Theorem: The scheme (3.5) for solving (1.1) is unconditionally stable if $\sigma\ge\frac{1}{4}$ and conditionally stable if $\sigma<\frac{1}{4}$. By using the Lax theorem, we can conclude that the present method is converge as long as stability criterion is satisfied.

6. Computational results

We have applied the presented method on the fourth-order parabolic partial differential equation and have considered one homogeneous and three nonhomogeneous examples. The proposed method (3.5) is three level implicit method based on nonpolynomial cubic tension spline function.

Example 1. Consider a nonhomogeneous fourth-order parabolic partial differential equation [2, 7, 9, 10, 14]:

$$\frac{\partial^2 z}{\partial t^2} + \frac{\partial^4 z}{\partial x^4} = (\pi^4-1)\sin\pi x\cos t, \quad 0\le x\le 1,\ t\ge 0,$$

subject to the initial conditions:

$$z(x,0)=\sin\pi x,\ z_t(x,0)=0,\ 0\le x\le 1$$

and the boundary conditions

$$z(0,t)=z(1,t)=z_{xx}(0,t)=z_{xx}(1,t)=0,\ t\ge 0.$$

The analytical solution for this example is

$$z(x,t)=\sin\pi x\cos t.$$

The absolute errors for above example with $h=0.05$ at particular points $x=0.1,0.2,0.3,0.4,0.5$ and comparison with other existing methods are tabulated in table 1. Fig.1 illustrate the comparison of numerical solution and analytical solution for $n=20,r=2$ and time steps=10.

Example 2. Consider a homogeneous fourth-order parabolic partial differential equation [2, 6, 10, 16]:

$$\frac{\partial^2 z}{\partial t^2} + \frac{\partial^4 z}{\partial x^4} = 0,\ 0\le x\le 1,\ t\ge 0,$$

subject to the initial conditions:

$$z(x,0)=\frac{x}{12}(2x^2-x^3-1),\ z_t(x,0)=0,\ 0\le x\le 1$$

and the boundary conditions

$$z(0,t)=z(1,t)=z_{xx}(0,t)=z_{xx}(1,t)=0,\ t\ge 0.$$

The analytical solution for this example is

$$z(x,t)=\sum_{s=0}^{\infty}d_s\sin(2s+1)\pi x\cos(2s+1)^2\pi^2 t,$$

where

$$d_s=\frac{-8}{[(2s+1)^5\pi^5]}.$$

The absolute errors for above example and comparison with other existing methods are tabulated in table 2. Fig.2 illustrate the comparison of numerical solution and analytical solution for $n=10,r=4$ and time steps=50.

Example 3. Consider a nonhomogeneous fourth-order parabolic partial differential equation [10]:

$$\frac{\partial^2 z}{\partial t^2} + \frac{\partial^4 z}{\partial x^4} = [24 - x^2(1-x)^2]\cos t, \ \ 0 \le x \le 1,\ t > 0,$$

subject to the initial conditions:

$$z(x,0) = x^2(1-x)^2, \ \ z_t(x,0) = 0, \ \ 0 \le x \le 1$$

and the boundary conditions

$$z(0,t) = z(1,t) = 0,\ z_{xx}(0,t) = z_{xx}(1,t) = 2\cos t, \ \ t \ge 0.$$

The analytical solution for this example is

$$z(x,t) = x^2(1-x)^2 \cos t.$$

The maximum absolute errors for above example is tabulated in table 3 and comparison with other existing methods is tabulated in table 4. Fig.3 illustrate the comparison of numerical solution and analytical solution values for $n = 20, r = 2$ and time steps=10.

Example 4. Consider a nonhomogeneous fourth-order parabolic partial differential equation [15]:

$$\frac{\partial^2 z}{\partial t^2} + \frac{\partial^4 z}{\partial x^4} = (\pi^4 + 1)e^t \sin \pi x, \ \ 0 \le x \le 1,\ t > 0,$$

subject to the initial conditions:

$$z(x,0) = \sin \pi x, \ \ z_t(x,0) = \sin \pi x, \ \ 0 \le x \le 1$$

and the boundary conditions

$$z(0,t) = z(1,t) = 0,\ z_{xx}(0,t) = z_{xx}(1,t) = 0, \ \ t \ge 0.$$

The analytical solution for this example is

$$z(x,t) = e^t \sin \pi x.$$

The maximum absolute errors for this example and comparison with the existing method is tabulated in table 5. Fig.4 illustrate the comparison of numerical solution and analytical solution for $n = 10, r = \sqrt{\frac{1}{6}}$ and time steps=100.

Conclusion

Nonpolynomial cubic tension spline functions have been developed to obtain three level implicit methods for solving fourth-order parabolic partial differential equations. The developed methods are tested on four numerical examples using MATLAB and results are tabulated in tables 1-5. The performance of these methods have been examined by comparing solution of homogeneous and nonhomogeneous fourth-order parabolic partial differential equations with available results. We have also included results given by unconditionally stable methods. Tables show that our results are more accurate than the results obtained by previous existing methods. Figures show that the proposed numerical solution gives almost overlapping behavior with the corresponding analytical solution values.

Table 1. Observed maximum absolute errors for $h = 0.05$, Example 1.

Methods (α, β, σ)	r	Time steps	$x=0.1$	$x=0.2$	$x=0.3$	$x=0.4$	$x=0.5$
$O(k^2 + h^4)$							
$\left(\frac{1}{6},0,\frac{1}{4}\right)$	2	10	6.69(−7)	4.96(−8)	3.76(−8)	7.33(−9)	1.19(−9)
	0.5	16	1.66(−8)	1.99(−9)	5.55(−11)	1.34(−10)	1.28(−10)
$\left(\frac{1}{6},0,\frac{1}{12}\right)$	2	10	2.36(−7)	2.22(−8)	5.56(−9)	1.13(−9)	2.03(−9)
	0.5	16	1.58(−8)	5.91(−10)	1.22(−10)	1.23(−10)	1.29(−10)
$O(k^2 + h^2)$							
$\left(\frac{1}{2},-2,\frac{1}{4}\right)$	2	10	3.38(−5)	1.12(−5)	1.93(−5)	1.93(−5)	1.98(−5)
	0.5	16	7.28(−6)	1.30(−5)	1.71(−5)	1.96(−5)	2.04(−5)
$\left(\frac{1}{2},-2,\frac{1}{12}\right)$	2	10	7.78(−6)	1.51(−5)	1.62(−5)	1.91(−5)	2.02(−5)
	0.5	16	1.12(−5)	1.21(−5)	1.54(−5)	1.41(−5)	1.69(−5)
$\left(\frac{1}{3},-1,\frac{1}{4}\right)$	2	10	3.62(−5)	4.02(−6)	1.06(−5)	9.78(−6)	9.73(−6)
	0.5	16	2.01(−6)	6.54(−8)	5.46(−7)	5.95(−7)	6.22(−7)
$\left(\frac{1}{3},-1,\frac{1}{12}\right)$	2	10	1.20(−5)	8.18(−6)	8.26(−6)	9.48(−6)	1.01(−5)
	0.5	16	6.88(−6)	7.12(−6)	7.26(−6)	7.35(−6)	7.38(−6)
[14] $O(k^2+h^4), \sigma=\frac{1}{4}$,	2	10	3.09(−6)	4.04(−6)	1.65(−6)	2.44(−6)	2.73(−7)
	0.5	16	5.25(−7)	2.87(−7)	1.54(−7)	1.64(−7)	1.76(−7)
$O(k^4+h^4), \sigma=\frac{1}{12}$,	2	10	2.91(−6)	1.73(−6)	1.60(−6)	2.33(−6)	2.60(−7)
	0.5	16	4.47(−7)	2.66(−7)	1.39(−7)	1.55(−7)	1.57(−7)
[2]	2	10	1.80(−5)	2.00(−5)	1.40(−5)	8.30(−6)	5.70(−6)
	0.5	16	9.20(−6)	7.90(−6)	2.80(−6)	9.80(−7)	2.50(−6)
[7]	2	10	2.20(−4)	4.10(−4)	5.40(−4)	6.20(−4)	6.50(−4)
	0.5	16	2.50(−5)	4.70(−5)	6.60(−5)	7.80(−5)	8.20(−5)
[9]	2	10	1.87(−6)	2.13(−5)	1.49(−5)	8.60(−6)	5.96(−6)
	0.5	16	9.07(−6)	7.79(−6)	2.75(−6)	1.01(−6)	2.59(−6)
[10] $\sigma = \frac{1}{12}$	2	10	2.05(−6)	4.37(−7)	1.04(−7)	1.19(−8)	5.61(−8)
	0.5	16	4.04(−7)	7.81(−9)	8.34(−9)	4.56(−8)	5.10(−8)

Table 2. Observed maximum absolute errors for h=0.1, Example 2.

Methods (α, β, σ)	r	Time steps	$x=0.1$	$x=0.2$	$x=0.3$	$x=0.4$	$x=0.5$
$O(k^2 + h^4)$							
$\left(\frac{1}{6},0,\frac{1}{4}\right)$	4	50	3.74(−13)	6.88(−13)	2.74(−13)	4.51(−13)	2.29(−12)
	$\frac{1}{6}$	100	2.10(−13)	7.68(−13)	2.05(−12)	3.46(−12)	4.91(−12)
$\left(\frac{1}{6},0,\frac{1}{12}\right)$	4	50	1.82(−12)	1.43(−12)	2.28(−12)	5.44(−13)	6.57(−14)
	$\frac{1}{6}$	100	9.26(−14)	7.31(−13)	2.08(−12)	3.45(−12)	4.92(−12)
	$\frac{7}{60}$	100	2.01(−13)	5.79(−12)	1.42(−11)	2.38(−11)	3.39(−11)
	$\frac{1}{84}$	100	4.91(−14)	3.15(−13)	7.86(−13)	1.32(−12)	1.87(−12)
$O(k^2 + h^2)$							
$\left(\frac{1}{2},-2,\frac{1}{4}\right)$	4	50	3.42(−13)	2.17(−13)	1.76(−13)	2.83(−13)	1.98(−12)
	$\frac{1}{6}$	100	7.53(−13)	2.32(−12)	7.29(−12)	7.16(−12)	1.23(−11)
$\left(\frac{1}{2},-2,\frac{1}{12}\right)$	4	50	1.15(−12)	1.12(−12)	8.53(−13)	8.39(−13)	1.56(−12)
	$\frac{1}{9}$	100	2.20(−11)	8.84(−12)	2.55(−11)	1.02(−11)	1.93(−11)
	$\frac{7}{60}$	100	2.17(−11)	4.22(−12)	3.67(−11)	3.21(−11)	1.06(−11)
	$\frac{1}{84}$	100	1.74(−12)	5.92(−13)	2.53(−12)	2.22(−12)	1.16(−13)
$\left(\frac{1}{3},-1,\frac{1}{4}\right)$	4	50	3.65(−13)	1.31(−13)	3.13(−14)	6.57(−13)	2.38(−12)
	$\frac{1}{6}$	100	2.27(−13)	6.35(−13)	2.11(−12)	3.43(−12)	4.99(−12)
$\left(\frac{1}{3},-1,\frac{1}{12}\right)$	4	50	8.06(−13)	9.95(−13)	4.42(−13)	6.26(−13)	3.14(−13)
	$\frac{1}{9}$	100	3.81(−12)	5.99(−14)	7.09(−12)	3.37(−12)	1.02(−11)
	$\frac{7}{60}$	100	6.84(−12)	4.40(−12)	2.31(−11)	1.35(−11)	3.85(−11)
	$\frac{1}{84}$	100	3.34(−12)	9.68(−13)	3.23(−12)	4.80(−12)	2.31(−12)
[16], $\sigma = \frac{1}{4}$	4	50	3.19(−4)	6.19(−4)	8.81(−4)	1.87(−3)	1.15(−3)
	$\frac{1}{6}$	100	2.61(−4)	4.43(−4)	5.47(−4)	6.08(−4)	6.33(−4)
[2], $\sigma = \frac{1}{4}$	4	50	3.21(−4)	5.77(−4)	7.24(−4)	7.89(−4)	8.10(−4)
	$\frac{1}{6}$	100	3.81(−4)	3.33(−4)	7.74(−4)	7.81(−4)	7.66(−4)
[6], $\sigma = \frac{1}{12}$	4	50	4.32(−4)	8.34(−4)	1.18(−4)	1.42(−3)	1.52(−3)
	$\frac{1}{6}$	100	2.30(−4)	4.08(−4)	5.40(−4)	6.56(−4)	7.02(−4)
[2], $\sigma = \frac{1}{12}$	4	50	1.00(−5)	5.00(−5)	1.73(−4)	3.33(−4)	4.10(−4)
	$\frac{1}{9}$	100	3.52(−4)	6.30(−4)	7.77(−4)	7.72(−4)	7.38(−4)
	$\frac{7}{60}$	100	1.38(−4)	1.74(−4)	9.05(−5)	3.40(−4)	9.60(−4)
	$\frac{1}{84}$	100	3.53(−5)	6.22(−5)	7.11(−5)	6.11(−5)	5.53(−5)
[10], $\sigma = \frac{1}{12}$	4	50	2.66(−12)	5.48(−12)	1.12(−12)	6.07(−13)	7.42(−13)
	$\frac{1}{6}$	100	1.29(−12)	3.24(−12)	1.79(−12)	3.63(−12)	5.06(−12)
	$\frac{7}{60}$	100	1.79(−12)	4.02(−12)	1.38(−11)	2.40(−11)	3.11(−11)
	$\frac{1}{84}$	100	2.17(−13)	1.33(−13)	7.43(−13)	1.35(−12)	1.50(−12)

Table 3. Observed maximum absolute errors, Example 3.

Methods (α, β, σ)	r	h	10 time steps	20 time steps	30 time steps
$\left(\frac{1}{6},0,\frac{1}{4}\right)$	2	0.1	5.67(−3)	4.98(−3)	3.84(−3)
	0.5		1.45(−3)	1.43(−3)	1.41(−3)
	2	0.05	1.48(−3)	1.48(−3)	1.45(−3)
	0.5		3.63(−4)	3.62(−4)	3.62(−4)
$\left(\frac{1}{2},-2,\frac{1}{4}\right)$	2	0.1	6.70(−3)	5.88(−3)	4.50(−3)
	0.5		1.42(−3)	1.41(−3)	1.39(−3)
	2	0.05	1.73(−3)	1.72(−3)	1.70(−3)
	0.5		6.22(−4)	6.22(−4)	6.22(−4)
$\left(\frac{1}{3},-1,\frac{1}{4}\right)$	2	0.1	6.56(−3)	5.76(−3)	4.45(−3)
	0.5		2.14(−3)	2.12(−3)	2.10(−3)
	2	0.05	1.70(−3)	1.69(−3)	1.67(−3)
	0.5		5.36(−4)	5.35(−4)	5.35(−4)

Table 4. Observed maximum absolute errors for h=0.05, Example 3.

Methods (α, β, σ)	r	Time steps	$x=0.1$	$x=0.2$	$x=0.3$	$x=0.4$	$x=0.5$
$O(k^2+h^4)$							
$\left(\frac{1}{6},0,\frac{1}{4}\right)$	2	10	1.11(−3)	8.43(−5)	5.93(−5)	8.72(−6)	5.56(−6)
	0.5	16	2.76(−4)	3.18(−6)	2.65(−7)	1.91(−8)	2.43(−9)
$\left(\frac{1}{6},0,\frac{1}{12}\right)$	2	10	3.91(−4)	3.48(−5)	6.41(−6)	1.38(−6)	5.96(−8)
	0.5	16	2.60(−5)	1.11(−6)	2.85(−8)	4.28(−10)	4.02(−12)
$O(k^2+h^2)$							
$\left(\frac{1}{2},-2,\frac{1}{4}\right)$	2	10	1.29(−3)	2.25(−5)	9.70(−5)	2.98(−6)	1.42(−5)
	0.5	16	2.24(−4)	3.98(−4)	5.23(−4)	5.97(−4)	6.22(−4)
$\left(\frac{1}{3},-1,\frac{1}{4}\right)$	2	10	1.28(−3)	6.15(−5)	8.17(−5)	7.22(−6)	9.99(−6)
	0.5	16	5.90(−5)	9.90(−6)	1.18(−6)	1.19(−7)	2.07(−8)
$[10], \sigma=\frac{1}{4}$	2	10	1.05(−3)	7.46(−5)	5.42(−5)	7.48(−6)	4.98(−6)
	0.5	16	1.50(−4)	9.27(−5)	1.83(−5)	1.20(−4)	1.61(−4)

Table 5. Observed maximum absolute errors, Example 4.

| Methods (α, β, σ)	r	h	10 time steps	50 time steps	100 time steps
$O(k^2+h^4)$					
$\left(\frac{1}{6},0,\frac{1}{4}\right)$	$\sqrt{1/6}$	0.1	1.84(−5)	2.16(−5)	2.65(−5)
	$\sqrt{1/84}$		1.10(−5)	1.15(−5)	1.22(−5)
	$\sqrt{1/6}$	0.05	1.89(−7)	1.97(−7)	2.07(−7)
	$\sqrt{1/84}$		1.16(−7)	1.18(−7)	1.19(−7)
$\left(\frac{1}{6},0,\frac{1}{12}\right)$	$\sqrt{1/6}$	0.1	1.36(−5)	1.60(−5)	1.96(−5)
	$\sqrt{1/84}$		1.06(−5)	1.11(−5)	1.17(−5)
	$\sqrt{1/6}$	0.05	1.40(−7)	1.46(−7)	1.53(−7)
	$\sqrt{1/84}$		1.12(−7)	1.13(−7)	1.15(−7)
$O(k^2+h^2)$					
$\left(\frac{1}{2},-2,\frac{1}{4}\right)$	$\sqrt{1/6}$	0.1	6.35(−4)	7.47(−4)	9.17(−4)
	$\sqrt{1/84}$		1.22(−3)	1.27(−3)	1.35(−3)
	$\sqrt{1/6}$	0.05	2.88(−5)	3.00(−5)	3.15(−5)
	$\sqrt{1/84}$		1.69(−5)	1.71(−5)	1.73(−5)
$\left(\frac{1}{2},-2,\frac{1}{12}\right)$	$\sqrt{1/6}$	0.1	2.37(−2)	2.79(−2)	3.42(−2)
	$\sqrt{1/84}$		1.15(−3)	1.19(−3)	1.26(−3)
	$\sqrt{1/6}$	0.05	1.58(−5)	1.65(−5)	1.74(−5)
	$\sqrt{1/84}$		1.76(−5)	1.78(−5)	1.80(−5)
$\left(\frac{1}{3},-1,\frac{1}{4}\right)$	$\sqrt{1/6}$	0.1	3.93(−4)	4.62(−4)	5.67(−4)
	$\sqrt{1/84}$		2.71(−4)	2.83(−4)	2.98(−4)
	$\sqrt{1/6}$	0.05	1.35(−5)	1.41(−5)	1.48(−5)
	$\sqrt{1/84}$		3.44(−5)	3.48(−5)	3.52(−5)
$[15]$	$\sqrt{1/6}$	0.05	9.94(−4)	7.72(−4)	9.74(−5)
	$\sqrt{1/84}$		2.99(−4)	2.79(−4)	2.13(−4)
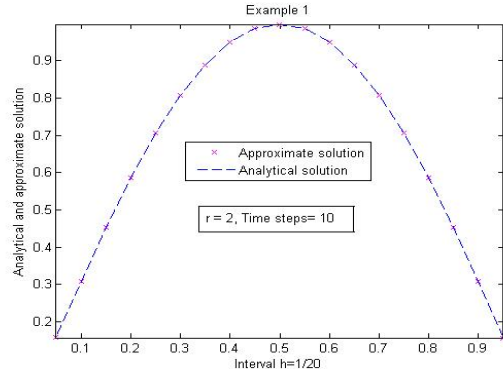
Figure 1. Comparison between analytical and approximate solution for example 1.
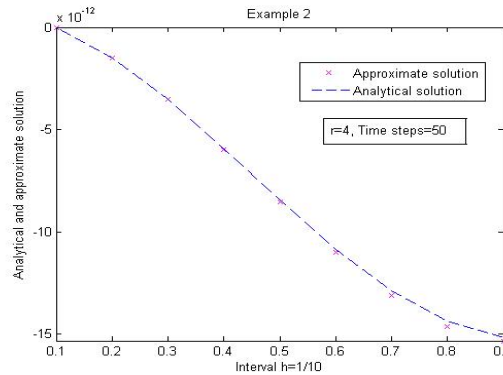

Figure 2. Comparison between analytical and approximate solution for example 2.
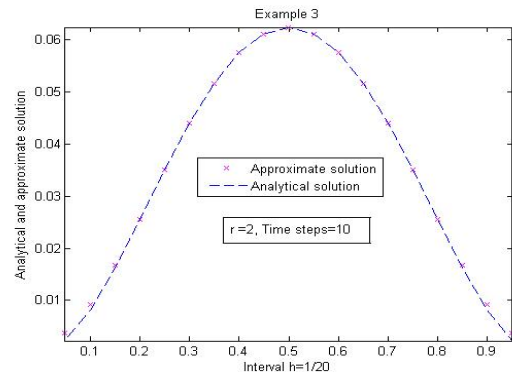

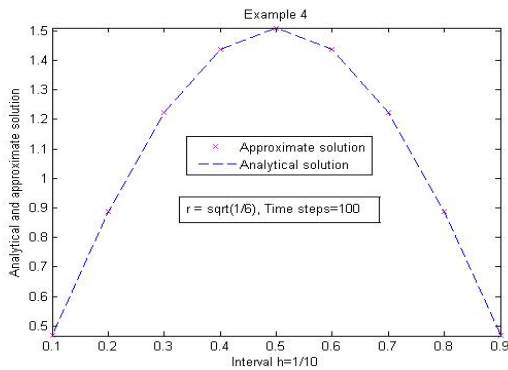**Figure 3.** Comparison between analytical and approximate solution for example 3.

Figure 4. Comparison between analytical and approximate solution for example 4.

Acknowledgment

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