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Existence and uniqueness of mild solution for stochastic partial integro-differential equations with delays

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Abstract

In this work, we study the existence results for stochastic partial integro-differential equations with delays. The mild solution of the problem is derived by using a different resolvent operator given in [18]. The existence of solution is proved by using contraction mapping principle.

Keywords

Resolvent operator, Mild solution, Stochastic partial integro-differential equations, Delays.

AMS Subject Classification

34A12, 35R60, 47H10, 60H15.

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Contents

1. Introduction

Stochastic differential equation have been used with great success in many application areas including biology, epidemiology, mechanics, economics and finance. In the past few decades many scholars have studied qualitative theory of stochastic partial differential equation (see [13, 16] and references therein). When we are concerned with mild solution of stochastic partial differential equation, the Lyapunov second method is not as appropriate as in the non-delays case. A difficulty is that mild solutions do not have stochastic differentials, so Ito's formula cannot be applied directly. Liu [14] solve this problem following Ichikawa [12] introducing approximating systems and then using a limiting argument. Recently Burton [6] has successfully used the fixed-point theory to investigate the stability of deterministic systems.

The non-local cauchy problem was first introduced by Byszewski and Lakshmikantham [5]. Since it is demonstrated that the non-local problem have better effects in applications then the classical ones, differential equations with non-local conditions have bean studied for more details on this topic. We refer extensively to [2, 4, 11, 17] and references therein. By using Leray schauder fixed point approach the existence of mild solutions for semi-linear stochastic delay evolution equations with non-local conditions have bean studied in [4]. In [3] Balasubramaniam et,al started the presence of mild and strong solution of semi-linear netural functional differential evolution equations with non-local conditions by using fractional power of operators and Krasnoselskii fixed point theorem. In [9] Hernandez observed that many authors used the resolvent operators in appropriately to study the existence of mild solution for fractional derivaties. To make the mild solutions to be more appropriate, Hernandez [10] introduced resolvent operator for integralequations defined in [18]. This resolvent operator for integral equation have bean used many authors to study the existence results for abstract differential equations [1, 10]. In [1] A.Anguraj et,al. started the presence of mild solution of fractional non-instantaneous impulsive integro-differential equations with non-local conditions by using fixed point theorem of condensing map and the resolvent operators defined in [18]. As far our knowledge, no one used the resolvent operators given in [18] to study the existence results for stochastic partial differential equation. In this article we introduce the resolvent operator of the above type to study the existence results for stochastic partial differential equation.

In this paper, we study the existence and uniqueness of the following stochastic partial integro-differential equation with delays of the form

$$\begin{aligned} du(t) &= \left[Au(t)+\int_0^t B(t-s)u(s)ds+F(t,u(t-\rho(t)))\right]dt \\ &+ G(t,u(t-\delta(t)))dW(t) \;\; for \;\; t\geq 0 \\ u(\theta) &= \phi(\theta) \;\; where \;\; \phi\in D^b_{\mathscr{F}_0}([-\tau,0];H), \;\; \tau>0 \quad (1.1) \end{aligned}$$

where A is the infinitesimal generator of a C_0- semigroup $(T(t))_{t\geq 0}$on H. For all $t\geq 0$, $B(t)$ is a continuous linear operator from $(Y,|.|_Y)$ into $(H,|.|_H)$. The mapping $F:R^+\times D([-\tau,0];H)\to H, G:R^+\times D([-\tau,0];H)\to L_2^0(K,H)$ are both Borel measurable,$\rho:R_+\to[0,\tau]$,$\delta:R_+\to[0,\tau]$ are continuous.

This paper is organized as follows. In section 2, we given some preliminaries, basic definitions and results, which will be used in the sequel. In section 3, the existence result for the considered system $(1)-(2)$ is proved.

2. Preliminaries

Let $\{\Omega,\mathscr{F},\mathbb{P}\}$ be a complete probability space equipped with some filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathscr{F}_0$ contains all P-null sets). Let $\mathbb{H},\mathbb{K}$ be two real separable Hilbert spaces. We denote by $\mathscr{L}(\mathbb{K},\mathbb{H})$ the set of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$,which is equipped with the usual operator norm $\|.\|$. In this paper, we always use the same symbol $\|.\|$ to denote norms of operators regardless of the spaces potentially involved when no confusion possibly arises. Let $\tau>0$ and $D:=D([-\tau,0];\mathbb{H})$ denote the family of all right-continuous functions with left-hand limits φ from $[-\tau,0]$ to $\mathbb{H}$. The space $D([-\tau,0];\mathbb{H})$ is assumed to be equipped with the norm $\|\varphi\|_D = sup_{-\tau\leq\theta\leq 0}\|\varphi(\theta)\|_{\mathbb{H}}$. We also use the space $\mathscr{D}^b_{F_0}([-\tau,0];H)$ denote the family of all almost surely bounded, $\mathscr{F}_0$- measurable, $D([-\tau,0];H)$- valued random variables. The space S endowed with the norm $\|u\|^p_s = sup_{t\geq 0}E\|u(t)\|^p_{\mathbb{H}}$ is a Banach space. Also $S=C([0,a];X)$ denotes the space of all continuous functions with the norm $\|.\|_{C([0,a];X)} = sup_{t\in[0,a]}\|x(t)\|_X$.

With the symbol $\{W(t),\, t\geq 0\}$, we denote a $\mathbb{K}$- valued $\{\mathscr{F}_t\}_{t\geq 0}$ Wiener process defined on the probability space $\{\Omega,\mathscr{F},\mathbb{P}\}$ with covariance operator Q,
i.e. $E\langle W(t),x\rangle_{\mathbb{K}}\langle W(t),y\rangle_{\mathbb{K}} = (t\wedge s)\langle Qx,y\rangle_{\mathbb{K}} \quad \forall x,y\in\mathbb{K}$,
where Q is a positive, self-adjoint, trace class operator on $\mathbb{K}$. In particular, we call such $\{W(t),t\geq 0\}$ a $\mathbb{K}$-valued $\mathbb{Q}$-Wiener process relative to$\{\mathscr{F}_t\}_{t\geq 0}$.

In order to define stochastic integrals with respect to the Q-Wiener process $\mathscr{W}(t)$, we introduce the subspace $K_0 = Q^{1/2}(\mathbb{K})$ of $\mathbb{K}$, which endowed with the inner product

$$\langle u,v\rangle_{K_0} = \left\langle Q^{-1/2}u, Q^{-1/2}v\right\rangle_{\mathbb{K}},$$

is a Hilbert space. Let $\mathscr{L}_2^0=\mathscr{L}_2(\mathbb{K}_0,\mathbb{H})$ denote the space of all Hilbert-Schmidt operators from K_0 into $\mathbb{H}$. It turns out to be a separable Hilbert space, equipped with the norm

$$\|\psi\|^2_{\mathscr{L}_2^0} = tr\left(\left(\psi Q^{1/2}\right)\left(\psi Q^{1/2}\right)^*\right),$$

for any $\psi\in\mathscr{L}_2^0$. Clearly, for any bounded opreators $\psi\in\mathscr{L}(\mathbb{K},\mathbb{H})$, this norm reducess to

$$\|\psi\|^2_{\mathscr{L}_2^0} = tr(\psi Q\psi^*).$$

For arbitrary given $T\geq 0$, let $J(t,w)$, $t\in[0,T]$, be an $\mathscr{F}_{\sqcup}$-adapted, $\mathscr{L}_2^0$-valued process, and we define the following norm for arbitrary $t\in[0,T]$,

$$|J|_t = \left\{E\int_0^t tr\left(\left(J(s,w)Q^{1/2}\right)\left(J(s,w)Q^{1/2}\right)^*\right)ds\right\}$$

Consider the following Stochastic Partial Integro - Differential Equation with delays

$$\begin{aligned} du(t) &= \left[Au(t)+\int_0^t B(t-s)u(s)ds+F(t,u(t-\rho(t)))\right]dt \\ &+ G(t,u(t-\delta(t)))dW(t) \;\; \text{for} \;\; t\geq 0, \end{aligned}$$

the above equation is equivalent to the following integral equation

$$\begin{aligned} u(t) &= \phi(0)+\int_0^t Au(t)ds+\int_0^t\int_0^s B(\tau-s)u(s)d\tau ds \\ &+\int_0^t F[s,u(s-\rho(s))]ds+\int_0^t G[s,u(s-\delta(s))]dW(s) \quad (2.1) \end{aligned}$$

This can be written the following form

$$u(t) = f(t)+\int_0^t s'(t-s)f(s)ds \qquad (2.2)$$

Where,

$$\begin{aligned} f(t) &= \phi(0)+\int_0^t F[s,u(s-\rho(s))]ds \\ &+ \int_0^t G[s,u(s-\delta(s))]dW(s) \end{aligned}$$

Let us assume that the integral equation (4) has an associated resolvent operator $\{S(t)\}_{t\geq 0}$ on H.

Definition 2.1. *[18] A family $(S(t))_{t\geq 0}\subset\mathscr{L}(X)$ of bounded linear operators in X is called resolvent for (4)(or solution operator for (4), if the following conditions are satisfied*

(S1) $S(t)$ is strongly continuous on $\mathbb{R}^+$ and $S(0)=I$,

(S2) $S(t)$ commutes with A, which means that $S(t)\mathscr{D}(A)\subset\mathscr{D}(A)$ and $AS(t)x=S(t)Ax$ for all $x\in\mathscr{D}(A)$ and $t\geq 0$;

(S3) The resolvent equation holds

$$S(t)x = x+\int_0^t As(s)xds \tag{2.3}$$

Definition 2.2. *[18] A resolvent $S(t)$ for (4) is called differentiable, if $S(\cdot)x \in W^{1,1}(\mathbb{R}^+;X)$ for each $x \in \mathscr{D}(A)$ and there is $\phi_A \in L^1_{loc}(\mathbb{R}^+)$ such that $\|S'(t)x\| \leq \phi_A(t)\|x\|_{\mathscr{D}(A)}$ a.e. on $\mathbb{R}^+$, for each $x \in \mathscr{D}(A)$, where the notation [D(A)] stands the domain of the operator A provided with the graph norm $\|x\|_{[D(A)]} = \|x\| + \|Ax\|$.*

Lemma 2.3. *[18] Suppose (4) admits a differentiable resolvent $S(t)$ and if $f \in C([0,a];\mathscr{D}(A))$ then*

$$u(t) = f(t)+\int_0^t S'(t-s)f(s)ds,\ t \in [0,a]$$

is a mild solution of (4).

In order to prove that the existence result of the stochastic partial integro-differential equation with delays, we need the following assumptions

H1: The mapping $F(t,.)$ satisfies the following Lipschitz conditions, for any $x,y \in \mathbb{H}$ and $t \geq 0$

$$\|F(t,x)-F(t,y)\|_{\mathbb{H}} \leq L_1 \|x-y\|_{\mathbb{H}} \quad \text{for all } t \geq 0, x,y \in H$$

where $L_1 > 0$,

H2: The mapping $G(t,.)$ satisfies the following Lipschitz conditions, for any $x,y \in \mathbb{H}$ and $t \geq 0$

$$\|G(t,x)-G(t,y)\|_{\mathscr{L}_2^0} \leq L_2 \|x-y\|_{\mathbb{H}} \quad \text{for all } t \geq 0, x,y \in H$$

where $L_2 > 0$,

3. Existence and Uniqueness Results

In this section, we provide the existence results for (1.1). This problem is equivalent to the following integral equation

$$\begin{aligned} u(t) &= \phi(0)+\int_0^t Au(t)ds+\int_0^t\int_0^s B(\tau-s)u(s)d\tau ds \\ &+ \int_0^t F[s,u(s-\rho(s))]ds+\int_0^t G[s,u(s-\delta(s))]dW(s) \end{aligned}$$

By Lemma(1) and the above representation, the mild solution of (1.1) can be defined as follows

Definition 3.1. *A stochastic process $\{u(t), t \in [0,T]\}$, $0 \leq T \leq \infty$, is called a mild solution of (1) if*
1.$u(t)$ is adapted to $\mathscr{F}_t$, $t \geq 0$;
2.$u(t) \in \mathbb{H}$ has cadlag paths on $t \in [0,T]$ almost surely, and for arbitrary $0 \leq t \leq T$,

$$\begin{aligned} (u)(t) &= \phi(0)+\int_0^t F[s,u(s-\rho(s))]ds \\ &+ \int_0^t G[s,u(s-\delta(s))]dW(s) \\ &+ \int_0^t s'(t-s)\phi(\theta)ds \\ &+ \int_0^t s'(t-s)\int_0^s F[\tau,u(\tau-\rho(\tau))]d\tau ds \\ &+ \int_0^t s'(t-s)\int_0^s G[\tau,u(\tau-\delta(\tau))]dw(\tau)ds \end{aligned}$$

Theorem Assume that $(H1)$ and $(H2)$are hold, then the problem(1.1) is a unique mild solution.
Proof. Define the operator $\mathscr{K}: S \to S$ by $\mathscr{K}(u)(t) = \phi(t)$ for $t \in [-\tau,0]$ and for $t \geq 0$,

$$\begin{aligned} (\mathscr{K}u)(t) &= \phi(0)+\int_0^t F[s,u(s-\rho(s))]ds \\ &+ \int_0^t G[s,u(s-\delta(s))]dW(s) \\ &+ \int_0^t s'(t-s)\phi(\theta)ds \\ &+ \int_0^t s'(t-s)\int_0^s F[\tau,u(\tau-\rho(\tau))]d\tau ds \\ &+ \int_0^t s'(t-s)\int_0^s G[\tau,u(\tau-\delta(\tau))]dw(\tau)ds \end{aligned}$$

First we verify thet $\mathscr{K}$ is p- th mean continuous on $[0,\infty)$. Let $u \in S, t_1 \geq 0$ and $|h|$ be sufficiently small, then

$$E\|(\mathscr{K}u)(t_1+h)-(\mathscr{K}u)(t_1)\|_H^P \leq 5^{P-1}\sum_{i=1}^5 E\|I_i(t_1+h)-I_i(t_1)\|_H^P$$

By using Holder inequality and the Burkholder - Davies - Gundy inequality we have

$$\begin{aligned} &E\|I_i(t_1+h)-I_i(t_1)\|_H^P \\ &\leq 2^{p-1}\left\|\int_0^{t_1}\left(s'(t_1+h-s)-s'(t_1-s)\right)\int_0^s G[\tau,u(\tau-\delta(\tau))]dw(\tau)ds\right\|_H^P \\ &\quad + 2^{p-1}E\left\|\int_0^{t_1+h} s'(t_1+h-s)\int_0^s G[\tau,u(\tau-\delta(\tau))]dw(\tau)ds\right\|_H^P \\ &\leq 2^{p-1}c_p\left(\int_0^{t_1}\left(E\left\|(s'(t_1+h-s)-s'(t_1-s))\int_0^s G[\tau,u(\tau-\delta(\tau))]\right\|_{L_2^0}^p\right)^{2/p} d\tau ds\right)^{p/2} \end{aligned}$$

$\to 0$ as $h \to 0$ Similarly, we can verify that

$$E\|I_i(t_1+h)-I_i(t_1)\|_H^2 \quad \to \quad 0, i=1,2,3,4 \text{as} h \to 0.$$

Where $c_p = (p(p-1)/2)^{p/2}$. Thus $\mathscr{K}$ is indeed continuous in pth mean on $[0,\infty)$. Next we show that $\mathscr{K}(S) \subset S$. It follows from (1) then we have

$$\begin{aligned}
& E\|(\mathscr{K}u)(t)\|_H^P \\
& \leq 6^{P-1}E\|\phi(\theta)\|_H^p + 6^{p-1}E\left\|\int_0^t F[s,u(s-\rho(s))]ds\right\|_H^P \\
& +6^{P-1}E\left\|\int_0^t G[s,u(s-\delta(s))]dw(s)\right\|_H^P \\
& +6^{P-1}E\left\|\int_0^t s'(t-s)\phi(\theta)ds\right\|_H^P \\
& +6^{p-1}E\left\|\int_0^t s'(t-s)\int_0^s F[\tau,u(\tau-\rho(\tau))]d\tau ds\right\|_H^p \\
& +6^{p-1}E\left\|\int_0^t s'(t-s)\int_0^s G[\tau,u(\tau-\delta(\tau))]dw(\tau)ds\right\|_H^p \\
& = \sum_{i=1}^6 J_i(t)
\end{aligned}$$

Now we estimate $J_i, i=1,2,...,6$ First we have

$$J_1(t) \quad \leq \quad \|\phi\|_D^P < \infty$$

Now by$(H1)$, we obtain

$$\begin{aligned}
J_2(t) \quad & \leq \quad E\left[\int_0^t \|F[s,u(s-\rho(s))]\|_H ds\right]^P \\
& \leq \quad L_1^P \|x\|_D^P T
\end{aligned}$$

From the well known lemma (Da Prato and Zabczyk [8]) and by $(H2)$ we have

$$\begin{aligned}
J_3(t) \quad & \leq \quad c_p\left[\int_0^t \left(E\|G[s,u(s-\delta(s))]\|_H^P\right)^{\frac{2}{p}} ds\right]^{\frac{p}{2}} \\
& \leq \quad c_p L_2^P \|u\|_D^P T
\end{aligned}$$

Similarly, by $(H2)$ we obtain

$$\begin{aligned}
J_4(t) \quad & \leq \quad E\left[\int_0^t \phi_A(t-s)\|\phi(0)\|_H^P ds\right] \\
& \leq \quad 6^{P-1}\|\phi(0)\|\,\|\phi_A\|_{L^1([0,t];\mathbb{R}^+)}
\end{aligned}$$

By $(H1),(H2)$ and the lemma (Da Prato and Zabczyk [8]) we have

$$\begin{aligned}
J_5(t) \quad & \leq \quad L_1^P\int_0^t \phi_A(t-s)\int_0^s E\|u(\tau-\rho(\tau))\|_H^P d\tau ds \\
& \leq \quad 6^{P-1}L_1^P\|u\|_D^P\int_0^t \phi_A(t-s)d\tau ds \\
& \leq \quad 6^{P-1}L_1^P\|u\|_D^P T\|\phi_A\|_{L^1([0,t];\mathbb{R}^+)}
\end{aligned}$$

$$\begin{aligned}
J_6(t) \quad & \leq \quad C_P E\left[\int_0^t \phi_A(t-s)\int_0^s \|G[\tau,u(\tau-\delta(\tau))]\|_H^P d\tau ds\right] \\
& \leq \quad 6^{P-1}C_P L_2^P\|u\|_D^P\int_0^t \phi_A(t-s)sds \\
& \leq \quad 6^{P-1}C_P L_2^P\|u\|_D^P T\|\phi_A\|_{L^1([0,t];\mathbb{R}^+)}
\end{aligned}$$

It follows from the above estimations we have $\|(\mathscr{K}u)(t)\| < \infty$. So we conclude that $\mathscr{K}(S) \subset (S)$. Next we need to show $\mathscr{K}$ is contraction mapping. Let $u,v \in S$, Then

$$\begin{aligned}
& E\sup_{t\in[0,T]}\|(\mathscr{K}U)(t)-(\mathscr{K}V)(t)\|_H^P \\
& \leq 4^{P-1}\sup_{t\in[0,T]}E\|U(t)-V(t)\|_H^P T \\
& \quad \times\left(L_1^P + C_P L_2^P + L_1^P\|\phi_A\|_{L^1([0,t];\mathbb{R}^+)} + C_P L_2^P\|\phi_A\|_{L^1([0,t];\mathbb{R}^+)}\right) \\
& \leq 4^{P-1}\sup_{t\in[0,T]}E\|U(t)-V(t)\|_H^P T \\
& \qquad \times\left[\left(L_1^P + C_P L_2^2\right)\left(1+\|\phi_A\|_{L^1([0,t];\mathbb{R}^+)}\right)\right]
\end{aligned}$$

If $T > 0$ is sufficiently small, then we can ensure that

$$\left[\left(L_1^P + C_P L_2^2\right)\left(1+\|\phi_A\|_{L^1([0,t];\mathbb{R}^+)}\right)\right] T < 1$$

We conclude that the operator $\mathscr{K}$ satisfies the contracting mapping principle, and hence there exists a unique mild solution for (1)-(2) on $T \in [0,T]$.

Conclusion

In this paper, we study the existence results for stochastic partial integro-differential equations with delays. The mild solution of the problem is derived by using a different resolvent operator and by using contraction mapping principle.

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