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On fractional neutral integro-differential systems with state-dependent delay via Kuratowski measure of non-compactness in Banach spaces

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Abstract
Our aim in this work is to study the existence of solutions of fractional integro-differential equations with state-dependent delay with the strongly continuous α-order cosine family. The results are obtained by utilizing the Monch's fixed point theorem and the concept of measure of non-compactness.

Keywords
Fractional integro-differential equations, state-dependent delay, α-order cosine family, fixed point, measure of non-compactness.

AMS Subject Classification
34A09, 34G20, 34K30, 34K37.

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1. Introduction

Fractional differential equations have become an important object of study in recent years inspired by their various applications to problems arising in physics, mechanics and other fields (see [3, 7–9, 15]). The theory of differential equation of fractional order has recently received a lot of attention and now constitutes a significant branch of nonlinear analysis. Numerous research papers and monographs have appeared dedicated to fractional differential equations, for example see [6, 12–14, 19].

On the other hand, functional differential equations with state-dependent delay shows up frequently within applications as models of equations. Investigations of these classes of delay equations essentially differ from once of equations with constant or time dependent delay. Therefore the theory of differential equation with state-dependent delay has drawn the consideration of analysts in the recent years.

The cosine function theory is identified with abstract linear second order differential equations. For basic concepts and applications of this theory, we suggest the reader to refer Fattorini [10] and Travis and Webb [24].

In our study, we apply the method related with the technique of measure of non-compactness and the fixed point theorem of Monch type [20]. This technique was mostly initiated in the monograph of Bana and Goebel [4] and later developed and used in several papers; see for example, Bana and Sadarangani [5], Guo et al. [13], Lakshmikantham and Leela [17], and Szufla [23].

This paper is concerned with existence of mild solutions defined on a compact real interval for fractional differential equations with state-dependent delay of the form

$$
\begin{aligned}
&{}^{C}D_t^{\alpha}[x(t)+g(t,x_t)] = A[x(t)+g(t,x_t)] \\
&\quad + f\left(t, x(t-\rho(x(t)))\right), \quad t \in J = [0,b], && (1.1)\\
&x(t) = \phi(t), \quad t \in [-r,0], \quad x'(0) = y_0 \in E, && (1.2)
\end{aligned}
$$

where ${}^{C}D_t^{\alpha}$ is the Caputo's fractional derivatives of order $1<\alpha<2$. A is the infinitesimal generator of a strongly continuous α-order cosine family $\{C_\alpha(t)\}_{t\geq 0}$ on E. The functions $f, g : J \times C([-r,0],E) \to E$ are a continuous functions, and $\phi : [-r,0] \to E$ is a given continuous function with $\phi(0)=0$ and $(E,|.|)$ a real Banach space. ρ is a positive bounded continuous function on $C([-r,0],E)$. r is the maximal delay defined by

$$r = \sup_{x \in C} \rho(x).$$

Further, we also consider the following fractional integro-differential equations with state-dependent delay of the form

$$\begin{aligned} {}^C D_t^\alpha &\left[x(t) + g\left(t, x_t, \int_0^t k_1(t,s,x_s)\,ds\right)\right] \\ &= A\left[x(t) + \left(t, x_t, \int_0^t k_1(t,s,x_s)\,ds\right)\right] \\ &+ f\left(t, x(t-\rho(x(t))), \int_0^t k_2(t,s,x(s-\rho(x(s))))\,ds\right), \\ & \qquad t \in J, \qquad (1.3) \end{aligned}$$

$$x(t) = \phi(t), \quad t \in [-r, 0], \quad x'(0) = y_0 \in E, \qquad (1.4)$$

where ${}^C D_t^\alpha, A, \phi$ and y_0 are same as defined in (1.1) - (1.2). Further $k_i : J \times J \times C([-r,0],E) \to E$, (for $i = 1,2$) and $f, g : J \times C([-r,0],E) \times E \to E$ are continuous functions.

The rest of this paper is organized as follows. In second section, we recall some preliminaries about fractional calculus and the Kuratowski's measure of non-compactness and auxiliary results. In the third section, the results are based on Monch's fixed point theorem combined with the technique of measure of non-compactness.

2. Preliminaries

In this section, we recall some basic definitions, lemmas and notations which will be used throughout this paper.

let E be a Banach space. By $C(J,E)$ we denote the Banach space of continuous functions from J into E with norm

$$\|x\| = \sup\{|x| : t \in J\}.$$

$C([-r,0],E)$ is endowed with norm defined by

$$\|\psi\| = \sup\{|\psi| : \theta \in [-r,0]\}.$$

$B(E)$ denotes the Banach space of all bounded linear operators from E into E, with the norm

$$\|N\|_{B(E)} = \sup\{|N(x)| : |x| = 1\}.$$

$L^1(J,E)$ denotes the Banach space of measurable functions $x : J \to E$ which are Bochner integrable, normed by

$$\|x\|_{L^1} = \int_0^b |x(t)|\,dt.$$

$L^\infty(J,E)$ denotes the Banach space of measurable functions $x : J \to E$ which are bounded, equipped with the norm

$$\|x\|_{L^\infty} = \inf\{c > 0 : \|x(t)\| < c, \quad a.e \quad t \in J\}.$$

For a given set V of functions $v : [-r,b] \to E$, let us denote by

$$V(t) = \{v(t) : v \in V\}, \quad t \in [-r,b]$$

and

$$V(J) = \{v(t) : v \in V, \quad t \in [-r,b]\}.$$

Let I be the identity operator on E. If A is a linear operator on E, then $R(\lambda, A) = (\lambda I - A)^{-1}$ denotes the resolvent operator of A. We use the notation for $\eta > 0$,

$$k_\eta(t) = \frac{t^{\eta-1}}{\Gamma(\eta)}, \quad t > 0, \qquad (2.1)$$

where $\Gamma(\eta)$ is the Gamma function. If $\eta = 0$, we set $k_0(t) = \delta(t)$, the delta distribution.

Definition 2.1. *[6] The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by*

$$J_t^\alpha x(t) = \int_0^t k_\alpha(t-s)x(s)ds, \qquad (2.2)$$

where $x(t) \in L^1([0,b],E)$.

Definition 2.2. *[6] The Riemann-Liouville fractional derivative of order $\alpha \in (1,2]$ is defined by*

$$D_t^\alpha x(t) = \frac{d^2}{dt^2} J_t^{2-\alpha} x(t), \qquad (2.3)$$

where $x(t), D_t^\alpha x(t) \in L^1([0,b];E)$.

Definition 2.3. *[6] The Caputo fractional derivative of order $\alpha \in (1,2]$ is defined by*

$${}^C D_t^\alpha x(t) = D_t^\alpha (x(t) - x(0) - x'(0)t), \qquad (2.4)$$

$D_t^\alpha x(t) \in L^1([0,b];E) \cap C^1([0,b];E), D_t^\alpha x(t) \in L^1([0,b];E)$.

The Laplace transform for the Riemann-Liouville fractional integral is given by

$$L\{J_t^\alpha x(t)\} = \frac{1}{\lambda^\alpha} \widehat{x}(\lambda), \qquad (2.5)$$

where $\widehat{x}(\lambda)$ is the Laplace transform of x given by

$$\widehat{x}(\lambda) = \int_0^\infty e^{-\lambda t} x(t)dt, \quad \operatorname{Re}\lambda > \omega. \qquad (2.6)$$

The Laplace transform for the Caputo derivative is given by

$$L[{}^C D_t^\alpha x(t)] = \lambda^\alpha \widehat{x}(\lambda) - \lambda^{\alpha-1} x(0) - \lambda^{\alpha-2} x'(0). \quad (2.7)$$

Consider the following problem,

$${}^C D_t^\alpha x(t) = Ax(t), \quad x(0) = \phi(0), \quad x'(0) = 0. \qquad (2.8)$$

where $\alpha \in (1,2], A : D(A) \subset X \to X$ is a closed densely defined linear operator in E.

Definition 2.4. *[6] Let $\alpha \in (1,2]$. A family $\{C_\alpha(t)\}_{\geq 0} \subset B(E)$ is called a solution operator (or a strongly continuous α - order fractional cosine family) for (2.8) if the following conditions are satisfied:*

(i) $C_\alpha(t)$ *is strongly continuous for* $t \geq 0$ *and* $C_\alpha(0) = I$;

(ii) $C_\alpha(t)D(A) \subset D(A)$ *and* $AC_\alpha(t)\phi = C_\alpha(t)A\phi$ *for all* $\phi \in D(A), t \geq 0$;

(iii) $C_\alpha(t)$ *is a solution of* $x(t) = \phi + \int_0^t k_\alpha(t-s)Ax(s)ds$ *for all* $\phi \in D(A), t \geq 0$.

A is called the infinitesimal generator of $C_\alpha(t)$. The strongly continuous α-order fractional cosine family is also called α-order cosine family for short.

Definition 2.5. *[16] The fractional sine family* $S_\alpha : \mathbb{R}_+ \to B(E)$ *associated with* C_α *is defined by*

$$S_\alpha(t) = \int_0^t C_\alpha(s)ds, \quad t \geq 0. \tag{2.9}$$

Definition 2.6. *[16] The fractional Riemann - Liouville family* $P_\alpha : \mathbb{R}_+ \to B(E)$ *associated with* C_α *is defined by*

$$P_\alpha(t) = J_t^{\alpha-1}C_\alpha(t), \quad t \geq 0. \tag{2.10}$$

Definition 2.7. *[16] The* α*-order cosine family* $C_\alpha(t)$ *is called exponentially bounded if there are constants* $M \geq 1$ *and* $\omega \geq 0$ *such that*

$$\|C_\alpha(t)\| \leq Me^{\omega t}, \quad t \geq 0. \tag{2.11}$$

An operator A is said to belong to $C^\alpha(M,\omega)$, if the problem (2.8) has an α-order cosine family $C_\alpha(t)$ satisfying (2.11).

In the following, we will derive the appropriate definition of mild solutions of (1.1) - (1.2). Assume $A \in C^\alpha(M,\omega)$ and let $C_\alpha(t)$ be the corresponding α-order cosine family. Then, we have (see [15])

$$\lambda^{\alpha-1}R(\lambda^\alpha,A)\mu = \int_0^\infty e^{-\lambda t}C_\alpha(t)\mu dt, \operatorname{Re}\lambda > \omega, \mu \in E. \tag{2.12}$$

By (2.9), (2.12), we have

$$\lambda^{\alpha-2}R(\lambda^\alpha,A)\mu = \int_0^\infty e^{-\lambda t}S_\alpha(t)\mu dt, \operatorname{Re}\lambda > \omega, \mu \in E. \tag{2.13}$$

By (2.10), (2.12), we have

$$R(\lambda^\alpha,A)\mu = \int_0^\infty e^{-\lambda t}P_\alpha(t)\mu dt, \operatorname{Re}\lambda > \omega, \mu \in E. \tag{2.14}$$

Before we define the mild solution for the system (1.1) - (1.2), first we consider the following linear problem

$${}^CD_t^\alpha[x(t)+g(t)] = A[x(t)+g(t)] + f(t), \quad t \in [0,b], \tag{2.15}$$

$$x(t) = \phi(t), \quad t \in [-r,0]. \quad x'(0) = y_0 \in E. \tag{2.16}$$

Assume that the Laplace transform of $x(t), g(t), f(t)$, with respect to t exists. Taking the Laplace transform to (2.15) - (2.16), by (2.7), we obtain

$$\begin{aligned}&\lambda^\alpha[\widehat{x}(\lambda)+\widehat{g}(\lambda)] - \lambda^{\alpha-1}[x(0)+g(0)] - \lambda^{\alpha-2}[y_0+\eta]\\&= A[\widehat{x}(\lambda)+\widehat{g}(\lambda)] + \widehat{f}(\lambda),\end{aligned}$$

where $\widehat{x}(\lambda), \widehat{g}(\lambda), \widehat{f}(\lambda)$ denote the Laplace transform of $x(t), g(t), f(t)$ and $\frac{d}{dt}g(t)|_{t=0} = \eta$, where η is independent of x. Then

$$\begin{aligned}&\widehat{x}(\lambda)+\widehat{g}(\lambda) = \lambda^{\alpha-1}R(\lambda^\alpha,A)[\phi+g(0)]\\&+\lambda^{\alpha-2}R(\lambda^\alpha,A)[y_0+\eta] + R(\lambda^\alpha,A)\widehat{f}(\lambda).\end{aligned}$$

By (2.12) - (2.14) and the property of Laplace transforms,

$$\begin{aligned}x(t)+g(t) &= C_\alpha(t)[\phi+g(0)] + S_\alpha(t)[y_0+\eta]\\&\quad+\int_0^t P_\alpha(t-s)f(s)ds.\\x(t) &= C_\alpha(t)[\phi+g(0)] + S_\alpha(t)[y_0+\eta] - g(t)\\&\quad+\int_0^t P_\alpha(t-s)f(s)ds.\end{aligned} \tag{2.17}$$

Next, we show that the solution (2.17) satisfies the given problem (2.15) - (2.16).

Indeed, taking Caputo derivative on both sides of (2.17), we get,

$$\begin{aligned}&{}^CD_t^\alpha[x(t)+g(t)]\\&= {}^CD_t^\alpha\{C_\alpha(t)[\phi+g(0)]\} + {}^CD_t^\alpha\{S_\alpha(t)[y_0+\eta]\}\\&\quad+{}^CD_t^\alpha\left\{\int_0^t P_\alpha(t-s)f(s)ds\right\}\\&= AC_\alpha(t)[\phi+g(0)] + D_t^\alpha\{S_\alpha(t)[y_0+\eta]\}\\&\quad+D_t^\alpha\left\{\int_0^t P_\alpha(t-s)f(s)ds\right\}\\&= AC_\alpha(t)[\phi+g(0)] + D_t^\alpha\{t[y_0+\eta]\\&\quad+J_t^\alpha S_\alpha(t)A[y_0+\eta] - ty_0 - t\eta\}\\&\quad+\frac{d^2}{dt^2}J_t^{2-\alpha}[P_\alpha(t)*f(t)]\\&= AC_\alpha(t)[\phi+g(0)] + D_t^\alpha J_t^\alpha S_\alpha(t)A[y_0+\eta]\\&\quad+\frac{d^2}{dt^2}[g_{2-\alpha}*g_{\alpha-1}*C_\alpha(t)*f(t)]\\&= AC_\alpha(t)[\phi+g(0)] + S_\alpha(t)A[y_0+\eta]\\&\quad+\frac{d^2}{dt^2}[1*C_\alpha(t)*f(t)]\\&= AC_\alpha(t)[\phi+g(0)] + AS_\alpha(t)[y_0+\eta]\\&\quad+\frac{d}{dt}[C_\alpha(t)*f(t)]\end{aligned}$$

$$= AC_\alpha(t)[\phi + g(0)] + AS_\alpha(t)[y_0 + \eta] + \frac{d}{dt}\left[\int_0^t C_\alpha(t-s)f(s)ds\right]$$
$$= AC_\alpha(t)[\phi + g(0)] + AS_\alpha(t)[y_0 + \eta] + A\left[\int_0^t P_\alpha(t-s)f(s)ds\right] + f(t)$$
$$= A\Big\{C_\alpha(t)[\phi + g(0)] + S_\alpha(t)[y_0 + \eta] + \left[\int_0^t P_\alpha(t-s)f(s)ds\right]\Big\} + f(t).$$

That is

$$^C D_t^\alpha[x(t) + g(t)] = A[x(t) + g(t)] + f(t), \quad t \in J,$$
$$x(t) = \phi(t), \quad x'(0) = y_0.$$

Motivated by the above results, we define the mild solution for the given system (1.1) - (1.2).

Definition 2.8. *We say that a continuous function $x : [-r,b] \to E$ is a mild solution of problem (1.1) - (1.2) if $x_0 = \phi \in [-r,0]$, $x'(0) = y_0 \in E$,*

$$x(t) = C_\alpha(t)[\phi(0) + g(0,\phi(0))] + S_\alpha(t)[y_0 + \eta] - g(t,x_t) + \int_0^t P_\alpha(t-s) f(s, x(s-\rho(x(s))))\, ds, t \in J. \quad (2.18)$$

Definition 2.9. *A map $f : J \times C([-r,0],E) \to E$ is said to be Caratheodory if*

(i) $t \to f(t,u)$ is measurable for each $u \in C([-r,0],E)$;

(ii) $u \to f(t,u)$ is continuous for almost each $t \in J$.

Now, let us recall some fundamental facts of the notion of Kuratowski's measure of non-compactness.

Definition 2.10. *[4] Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski's measure of non-compactness is the $\beta : \Omega_E \to [0,\infty]$ defined by*

$$\beta(B) = \inf\{\varepsilon > 0 : B \subseteq \cup_{i=1}^n B_i \quad diam \quad (B_i) \le \varepsilon\};$$
$$here \quad B \in \Omega_E.$$

The Kuratowski's measure of non-compactness satisfies the following properties (for more details see [18])

(i) $\beta(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).

(ii) $\beta(B) = \beta\overline{B}$

(iii) $A \subset B \Rightarrow \beta(A) \le \beta(B)$

(iv) $\beta(A+B) \le \beta(A) + \beta(B)$

(v) $\beta(cB) = |c|\beta(B); c \in \mathbb{R}$

(vi) $\beta(convB) = \beta(B)$.

Theorem 2.11. *[2, 20] Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv}N(V) \quad or \quad V = N(V) \cup \{0\} \Rightarrow \beta(V) = 0$$

holds for every subset V of D, then N has a fixed point.

Lemma 2.12. *[23] Let D be a bounded, closed and convex subset of the Banach space $C(J,E)$, G a continuous function on $J \times J$ and f is a function from $J \times C([-r,0],E) \to E$ which satisfies the Caratheodory conditions and there exists $p \in L^1(J,\mathbb{R}_+)$ such that for each $t \in J$ and each bounded set $B \subset C([-r,0],E)$ we have*

$$\lim_{k\to 0^+} \beta(f(J_{t,k} \times B)) \le p(t)\beta(B); \quad J_{t,k} = [t-k,t] \cap J.$$

If V is an equi-continuous subset of D, then

$$\beta\left(\int_J G(s,t) f(s,x_s)ds; x \in V\right) \le \int_J \|G(t,s)\| p(s)\beta(V(s))ds.$$

3. Main Results

In this section, we present and prove the existence results for the problem (1.1) - (1.2) and (1.3) - (1.4) with the help of Monch's fixed point theorem.

In order to prove the existence result for the problem (1.1) - (1.2), we list the following hypotheses:

(H1) $A : D(A) \subset E \to E$ is the infinitesimal generator of a uniformly continuous cosine family $\{C_\alpha(t)\}_{t\ge 0}$. Let

$$M_c = \sup\{\|C_\alpha(t)\|_{C([-r,0],E)}; t \ge 0\}$$

and

$$M_s = \sup\{\|S_\alpha(t)\|_{C([-r,0],E)}; t \ge 0\}$$

(H2) $f : J \times C([-r,0],E) \to E$ is a Caratheodory.

(H3) There exist functions $p \in L^\infty(J,\mathbb{R}_+)$ such that

$$|f(t,u)| \le p(t)(\|u\|_C + 1), \text{ for a.e } t \in J \text{ and } u \in C([-r,0],E).$$

(H4) For almost each $t \in J$ and each bounded set $B \subset C([-r,0],E)$, we have

$$\lim_{k\to 0^+} \beta(f(J_{t,k} \times B)) \le p(t)\beta(B),$$
$$J_{t,k} = [t-k,t] \cap J.$$

(H5) The function $t \to g$ is a continuous on J and there exist constants $c_1, c_2 > 0$ such that

(a) $|g(t,u) - g(t,v)| \le c_1\|u - v\|_{C([-r,0],E)}$, for each $u, v \in C([-r,0],E)$.

(b) $|g(t,u)| \leq c_1\|u\|_{C([-r,0],E)} + c_2, \quad t \in J,$ $u \in C([-r,0],E)$.

(c) For each bounded set $B \subset C([-r,0],E)$, and $t \in J$, we have $\beta(g(t,B)) \leq c_1(t)\beta(B)$.

(H6) For each $t \in J$ and any bounded set $B \subset C([-r,0],E)$, the set $\{g(t,u) : u \in B\}$ is relatively compact in E.

Theorem 3.1. *Assume that the hypotheses (H1) - (H6) are satisfied. Then the problem (1.1) - (1.2) has at least one mild solution on $[-r,b]$, provided that*

$$c_1 + M\|p\|_{L^\infty} < 1. \tag{3.1}$$

Proof. Transform the problem (1.1) - (1.2) into a fixed point problem. Consider the operator $N : C([-r,0],E) \to C([-r,0],E)$ defined by

$$N(x)(t) = \begin{cases} \phi(t), \quad if \quad t \in [-r,0], \\ C_\alpha(t)[\phi(0) + g(0,\phi(0))] + S_\alpha(t)[y_0+\eta] - g(t,x_t) \\ + \int_0^t P_\alpha(t-s) f(s, x(s-\rho(x(s))))\, ds, \qquad t \in J. \end{cases} \tag{3.2}$$

Let $\nu > 0$ be such that

$$\nu \geq \frac{\Lambda + M\|p\|_{L^\infty}}{(1-[c_1+M\|p\|_{L^\infty}])}, \tag{3.3}$$

where $\Lambda = M_c[|\phi(0)| + |g(0,\phi(0))|] + M_s[|y_0| + |\eta|] + c_2$, and consider the set

$$D_\nu = \{x \in C([-r,0],E) : \|x\|_\infty \leq \nu\}.$$

Clearly the subset D_ν is closed, bounded and convex. We shall show that N satisfies the assumptions of Theorem 2.11.

Now, we prove that N is completely continuous. For our convenience, we break the proof into sequences of steps.

Step 1: We prove that N is continuous.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ as $n \to \infty$ in $C([-r,0],E)$, then for $t \in [0,b]$. Note that $-r \leq s - \rho(x(s)) \leq s$ for each $s \in J$, we have

$$\begin{aligned} &|N(x_n)(t) - N(x)(t)| \\ &\leq |g(t,x_{n_t}) - g(t,x_t)| \\ &\quad + \left| \int_0^t P_\alpha(t-s)[f(s,x_n(s-\rho(x_n(s)))) - f(s,x(s-\rho(x(s))))]ds \right|. \end{aligned}$$

Since f is a Caratheodory function for $t \in J$, and from the continuity of ρ, we have by the dominated convergence theorem of Lebesgue, the right member of the above inequality tends to zero as $n \to \infty$.

$$|N(x_n) - N(x)| \to 0 \text{ as } n \to \infty.$$

Thus the operator N is continuous.

Next, we will show that $N(D_\nu) \subset D_\nu$ is bounded. For each $x \in D_\nu$ by hypotheses (H1), (H3), (H5) we have for each $t \in [0,b]$,

$$\begin{aligned} &|N(x)(t)| \\ &\leq \Big| C_\alpha(t)[\phi(0)+g(0,\phi(0))] + S_\alpha(t)[y_0+\eta] - g(t,x_t) \\ &\quad + \int_0^t P_\alpha(t-s) f(s,x(s-\rho(x(s))))\, ds \Big| \\ &\leq M_c[|\phi(0)| + |g(0,\phi(0))|] + M_s[|y_0| + |\eta|] + c_1\|x(t)\| \\ &\quad + c_2 + M\int_0^t p(s)(\|x(s)\|+1)ds \\ &\leq M_c[|\phi(0)| + |g(0,\phi(0))|] + M_s[|y_0| + |\eta|] + c_1\nu \\ &\quad + c_2 + M\int_0^t p(s)(\nu+1)ds \\ &\leq \Lambda + c_1\nu + M(\nu+1)\|p\|_{L^\infty}, \end{aligned}$$

Then $N(D_\nu) \subset D_\nu$.

Now, we prove that $N(D_\nu)$ is equicontinuous. Let $\tau_1, \tau_2 \in J, \tau_2 > \tau_1$. Then if $\varepsilon > 0$ and $\varepsilon \leq \tau_1 \leq \tau_2$ we have for any $x \in D_\nu$;

$$\begin{aligned} &|N(x)(\tau_2) - N(x)(\tau_1)| \\ &\leq |\phi(0)+g(0,\phi(0))||C_\alpha(\tau_2) - C_\alpha(\tau_1)| \\ &\quad + |y_0+\eta||S_\alpha(\tau_2) - S_\alpha(\tau_1)| \\ &\quad + |g(\tau_2,x_{\tau_2}) - g(\tau_1,x_{\tau_1})| \\ &\quad + \Big| \int_0^{\tau_2} P_\alpha(\tau_2-s) f(s,x(s-\rho(x(s))))\, ds \\ &\quad - \int_0^{\tau_1} P_\alpha(\tau_1-s) f(s,x(s-\rho(x(s))))\, ds \Big| \\ &\leq |\phi(0)+g(0,\phi(0))||C_\alpha(\tau_2) - C_\alpha(\tau_1)| \\ &\quad + |y_0+\eta||S_\alpha(\tau_2) - S_\alpha(\tau_1)| \\ &\quad + |g(\tau_2,x_{\tau_2}) - g(\tau_1,x_{\tau_1})| \\ &\quad + \|p\|_{L^\infty}(\nu+1)\Bigg\{ \left| \int_0^{\tau_1-\varepsilon} \Big[P_\alpha(\tau_2-s) \right. \\ &\quad \left. - P_\alpha(\tau_1-s)\Big] ds \right| \\ &\quad + \left| \int_{\tau_1-\varepsilon}^{\tau_1} [P_\alpha(\tau_2-s) - P_\alpha(\tau_1-s)]\, ds \right| \\ &\quad + \left| \int_{\tau_1}^{\tau_2} P_\alpha(\tau_2-s) ds \right| \Bigg\}. \end{aligned}$$

As $\tau_1 \to \tau_2$ and ε is sufficiently small, the right hand side of the above inequality tends to zero, then $N(D_\nu)$ is continuous and completely continuous.

Now let V be a subset of D_ν such that $V \subset \overline{conv}(N(V) \cup \{0\})$.

V is bounded and equi-continuous and therefore the function $v \to v(t) = \beta(V(t))$ is continuous on $[-r,b]$. By hypothesis (H4) and Lemma 2.12 and the properties of the measure β, we have for each $t \in [-r,b]$

$$\begin{aligned}\|v(t)\| &\leq \beta(N(V)(t)\cup\{0\})\\ \|v\|_\infty &\leq \beta(N(V)(t))\\ &\leq c_1\beta(V(t))+M\int_0^t p(s)\beta(V(s))ds\\ &\leq c_1 v(t)+M\int_0^t p(s)v(s)ds\\ \|v\|_\infty &\leq c_1\|v\|_\infty+M\|p\|_{L^\infty}\|v\|_\infty.\end{aligned}$$

This means that

$$\|v\|_\infty[1-(c_1+M\|p\|_{L^\infty})]\leq 0.$$

By (3.1) it follows that $\|v\|_\infty=0$, that is $v(t)=0$ for each $t\in[-r,b]$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzela theorem, V is relatively compact in D_γ. Applying now Theorem 2.11, we conclude that N has a fixed point which is a mild solution for the problem (1.1) - (1.2). □

Our next existence result for the problem (1.3) - (1.4) is based on the Monch's fixed point theorem. Before, we present and prove the results for the problem, first we define the mild solution for the problem (1.3) - (1.4).

Definition 3.2. *We say that a continuous function $x:[-r,b]\to E$ is a mild solution of problem (1.3) - (1.4) if $x(0)=\phi\in[-r,0], x'(0)=y_0\in E$, we have*

$$\begin{aligned}x(t)&=C_\alpha(t)[\phi+g(0,\phi,0)]+S_\alpha(t)[y_0+\eta]\\ &-g\left(t,x_t,\int_0^t k_1(t,s,x_s)\,ds\right)\\ &+\int_0^t P_\alpha(t-s)f\Bigg(s,x(s-\rho(x(s))),\\ &\int_0^s k_2(s,\tau,x(\tau-\rho(x(\tau))))\,d\tau\Bigg)ds,\ t\in J.\end{aligned}\tag{3.4}$$

Next, to prove the existence result for the problem (1.3) - (1.4), we list the following additional hypotheses:

(H3*) There exist functions $p\in L^\infty(J,\mathbb{R}_+)$ such that

(a) $\|f(t,x,y)\|\leq p(t)\Omega[\|x\|_C+\|y\|_E],\quad t\in J$, $x\in C([-r,0],E)$ and $y\in E$.

(b) For almost each $t\in J$ and each bounded set $B\subset C([-r,0],E)$, $F\subset E$, we have

$$\lim_{k\to0^+}\beta(f(t,B(\theta),F))\leq p(t)\left[\sup_{-r\leq\theta\leq0}\beta B(\theta)+\beta(F)\right]\text{ for a.e } t\in J,$$

where $B(\theta)=\{u(\theta):u\in B\}$.

(H4*) The function $g:J\times C([-r,0],E)\times E\to E$ is a continuous and there exists a constant $c_1>0$ such that the function satisfies the following conditions:

(a) $\|g(t,x_1,x_2)-g(t,y_1,y_2)\|\leq c_1[\|x_1-y_1\|_C+\|x_2-y_2\|];\quad t\in J,\ x_1,y_1\in C$ and $x_2,y_2\in E$.

(b) There exist constants L_g and $\tilde{L}_g$ such that $\|g(t,x,y)\|\leq L_g[\|x\|_C+\|y\|]+\tilde{L}_g,\quad t\in J,\ x\in C,\ y\in E$.

(c) For each bounded set $B\in C([-r,0],E)$, $F\subset E$, there exists a positive function $\gamma\in L^1(J,\mathbb{R}_+)$, such that

$$\lim_{k\to0^+}\beta(g(t,B(\theta),F))\leq\gamma(t)\left[\sup_{-r\leq\theta\leq0}\beta B(\theta)+\beta(F)\right]\text{ for a.e } t\in J,$$

where $B(\theta)=\{u(\theta):u\in B\}$.

(H5*) The functions $k_i:J\times J\times C([-r,0],E)\to E, i=1,2$; are continuous maps and there exist positive constants $L_{k_i}>0$ such that

(i) $\left\|\int_0^t[k_i(t,s,z_1)-k_i(t,s,z_2)]ds\right\|\leq L_{k_i}\|z_1-z_2\|_C$ for each $z_1,z_2\in C$,

(ii) $\left\|\int_0^t k_i(t,s,z)ds\right\|\leq L_{k_i}[1+\|z\|_C]$ for each $z\in C$, $i=1,2$.

(iii) There exists $\mu\in L^1(J\times J,\mathbb{R}_+)$ such that

$$\|\beta(k_i(t,s,H))\|\leq\mu_i(t,s)\left[\sup_{-r\leq\theta\leq0}H(\theta)\right],$$

for a.e. $t,s\in J$, (for $i=1,2$)

where $H(\theta)=\{w(\theta):w\in H\}$ and β is the Hausdroff MNC.

Remark 3.3. *For our convenience, let us take $\mu_1^*=\sup_{(t,s)\in J}\int_0^t\mu_1(t,s)\,ds$ and $\mu_2^*=\sup_{(t,s)\in J}\int_0^t\mu_2(t,s)ds$.*

Theorem 3.4. *Assume that the conditions $(H1),(H2),(H3^*)-(H5^*)$ are satisfied. Then the problem (1.3) - (1.4) has at least one mild solution on $[-r,b]$, provided that*

$$[\|\gamma\|(1+\mu_1^*)+M\|p\|_{L^\infty}(1+\mu_2^*)]<1.\tag{3.5}$$

Proof. Transform the problem (1.3) - (1.4) into a fixed point problem. Consider the operator $N_1:C([-r,0],E)\to C([-r,0],E)$ defined by

$$N_1(x)(t)=\begin{cases}\phi(t),\quad if\quad t\in[-r,0],\\ C_\alpha(t)[\phi(0)+g(0,\phi(0),0)]+S_\alpha(t)[y_0+\eta]\\ -g\left(t,x_t,\int_0^t k_1(t,s,x_s)\,ds\right)\\ +\int_0^t P_\alpha(t-s)f\Bigg(s,x(s-\rho(x(s))),\\ \quad\int_0^s k_2(s,\tau,x(\tau-\rho(x(\tau))))\,d\tau\Bigg)ds,\quad t\in J.\end{cases}\tag{3.6}$$

Let $\nu > 0$ be such that

$$\nu \geq \frac{\Lambda_1 + L_g L_{k_1} - M\|p\|_{L^\infty}}{[1 - L_g(1+L_{k_1}) + M\|p\|_{L^\infty}(1+L_{k_2})]}, \tag{3.7}$$

where $\Lambda_1 = M_c[|\phi(0)| + |g(0,\phi(0),0)|] + M_s[|y_0+\eta|] + \tilde{L}_g$, and consider the set

$$D_\nu = \{x \in C([-r,0],E) : \|x\|_\infty \leq \nu\}.$$

Clearly the subset D_ν is closed, bounded and convex. We shall show that N_1 satisfies the assumptions of Theorem 2.11.

Now, we prove that N_1 is completely continuous. For our convenience, we break the proof into sequences of steps.

Step1: We prove that N_1 is continuous.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ as $n \to \infty$ in $C([-r,0],E)$, then for $t \in [0,b]$. Note that $-r \leq s - \rho(x(s)) \leq s$ for each $s \in J$, we have

$$\begin{aligned}
&|N_1(x_n)(t) - N_1(x)(t)| \\
&\leq \left| g\left(t, x_{n_t}, \int_0^t k_1(t,s,x_{n_s})\,ds\right) - g\left(t, x_t, \int_0^t k_1(t,s,x_s)\,ds\right) \right| \\
&\quad + \left| \int_0^t P_\alpha(t-s) f\left(s, x_n(s-\rho(x_n(s))), \int_0^s k_2(s,\tau,x_n(\tau-\rho(x_n(\tau))))\,d\tau\right) ds \right. \\
&\quad \left. - \int_0^t P_\alpha(t-s) f\left(s, x(s-\rho(x(s))), \int_0^s k_2(s,\tau,x(\tau-\rho(x(\tau))))\,d\tau\right) ds \right|.
\end{aligned}$$

Since f is a Caratheodory function for $t \in J$, and from the continuity of ρ, we have by the dominated convergence theorem of Lebesgue, the right member of the above inequality tends to zero as $n \to \infty$.

$$|N_1(x_n) - N_1(x)| \to 0 \text{ as } n \to \infty.$$

Thus N_1 is continuous.

Next, we will show that $N_1(D_\nu) \subset D_\nu$ is bounded. For each $x \in D_\nu$ by hypotheses (H3*) - (H5*) we have for each $t \in [0,b]$,

$$\begin{aligned}
&|N_1(x)(t)| \\
&\leq \left| C_\alpha(t)[\phi(0) + g(0,\phi(0),0)] + S_\alpha(t)[y_0+\eta] - g\left(t, x_t, \int_0^t k_1(t,s,x_s)\,ds\right) \right. \\
&\quad \left. + \int_0^t P_\alpha(t-s) f\left(s, x(s-\rho(x(s))), \int_0^s k_2(s,\tau,x(\tau-\rho(x(\tau))))\,d\tau\right) ds \right| \\
&\leq M_c[|\phi(0)| + |g(0,\phi(0),0)|] + M_s[|y_0| + |\eta|] + L_g\left[\|x(t)\|_C + \left\| \int_0^t k_1(t,s,x_s)ds \right\|\right] \\
&\quad + \tilde{L}_g + M\int_0^t p(s)\Omega\left[\|x(s)\|_C + L_{k_2}[1 + \|x(s)\|_C]\right] ds \\
&\leq M_c[|\phi(0)| + |g(0,\phi(0),0)|] + M_s[|y_0| + |\eta|] + L_g\left[\|x(t)\|_C + L_{k_1}[1 + \|x(t)\|_C]\right] \\
&\quad + \tilde{L}_g + M\|p\|_{L^\infty}\left[\|x\|_\infty + L_{k_2}[1 + |x\|_\infty]\right] \\
&\leq M_c[|\phi(0)| + |g(0,\phi(0),0)|] + M_s[|y_0| + |\eta|] + L_g[\nu + L_{k_1}(1+\nu)] + \tilde{L}_g \\
&\quad + M\|p\|_{L^\infty}[\nu + L_{k_2}(1+\nu)] \\
&\leq \Lambda_1 + L_g[\nu + L_{k_1}(1+\nu)] + M\|p\|_{L^\infty}[\nu + L_{k_2}(1+\nu)].
\end{aligned}$$

Then $N_1(D_\nu) \subset D_\nu$.

Now, we prove that $N_1(D_\nu)$ is equicontinuous. Let $\tau_1, \tau_2 \in J, \tau_2 > \tau_1$. Then if $\varepsilon > 0$ and $\varepsilon \leq \tau_1 \leq \tau_2$ we have for any $x \in D_\nu$;

$$\begin{aligned}
&|N_1(x)(\tau_2) - N_1(x)(\tau_1)| \\
&\leq |\phi(0) + g(0,\phi(0),0)||C_\alpha(\tau_2) - C_\alpha(\tau_1)| + |y_0+\eta||S_\alpha(\tau_2) - S_\alpha(\tau_1)| \\
&\quad + \left| g\left(\tau_2, x_{\tau_2}, \int_0^{\tau_2} k_1(\tau_2,s,x_s)\,ds\right) - g\left(\tau_1, x_{\tau_1}, \int_0^{\tau_1} k_1(\tau_1,s,x_s)\,ds\right) \right| \\
&\quad + \left| \int_0^{\tau_2} P_\alpha(\tau_2-s) f\left(s, x(s-\rho(x(s))), \int_0^s k_2(s,\tau,x(\tau-\rho(x(\tau))))\,d\tau\right) ds \right. \\
&\quad \left. - \int_0^{\tau_1} P_\alpha(\tau_1-s) f\left(s, x(s-\rho(x(s))), \int_0^s k_2(s,\tau,x(\tau-\rho(x(\tau))))\,d\tau\right) ds \right|
\end{aligned}$$

$$\begin{aligned}
&\leq |\phi(0)+g(0,\phi(0),0)||C_\alpha(\tau_2)-C_\alpha(\tau_1)| \\
&\quad + |y_0+\eta||S_\alpha(\tau_2)-S_\alpha(\tau_1)| \\
&\quad + \left| g\left(\tau_2, x_{\tau_2}, \int_0^{\tau_2} k_1(\tau_2,s,x_s)\,ds\right)\right. \\
&\quad \left. - g\left(\tau_1, x_{\tau_1}, \int_0^{\tau_1} k_1(\tau_1,s,x_s)\,ds\right)\right| \\
&\quad + \|p\|_{L^\infty}\Omega[\nu + L_{k_2}(1+\nu)]\left\{ \left| \int_0^{\tau_1-\varepsilon}\left[P_\alpha(\tau_2-s)\right.\right.\right. \\
&\quad \left.\left. - P_\alpha(\tau_1-s)\right] ds \right| \\
&\quad + \left| \int_{\tau_1-\varepsilon}^{\tau_1} [P_\alpha(\tau_2-s)-P_\alpha(\tau_1-s)]\,ds \right| \\
&\quad \left. + \left| \int_{\tau_1}^{\tau_2} P_\alpha(\tau_2-s)ds \right| \right\}.
\end{aligned}$$

As $\tau_1 \to \tau_2$ and ε is sufficiently small, the right hand side of the above inequality tends to zero, then $N_1(D_\nu)$ is continuous and completely continuous.

Now let V be a subset of D_ν such that $V \subset \overline{conv}(N_1(V) \cup \{0\})$.

V is bounded and equi-continuous and therefore the function $v \to v(t) = \beta(V(t))$ is continuous on $[-r,b]$. By hypotheses $(H3^*)-(H5^*)$ and Lemma 2.12 and the properties of the measure β, we have for each $t \in [-r,b]$

$$\begin{aligned}
\|v(t)\| &\leq \beta(N_1(V)(t) \cup \{0\}) \\
\|v\|_\infty &\leq \beta(N_1(V)(t)) \\
&\leq \beta\left[g\left(t, V(t), \int_0^t k_1(t,s,x_s)ds\right)\right] \\
&\quad + \beta\left[\int_0^t P_\alpha(t-s) f\left(s, V(s-\rho(V(s))),\right.\right. \\
&\quad \left.\left.\int_0^s k_2(s,\tau,x(\tau-\rho(x(\tau))))ds\right)\right] \\
&\leq \gamma(t)\left[\beta(V(t)) + \beta\left(\int_0^t k_1(t,s,x_s)ds\right)\right] \\
&\quad + Mp(t)\int_0^t p(s)\left[\beta(V(s))\right. \\
&\quad \left. + \beta\left(\int_0^s k_2(s,\tau,x(\tau-\rho(x(\tau))))d\tau\right)\right] ds \\
&\leq \gamma(t)\left[v(t) + \int_0^t \mu_1(t,s)\alpha(V(s))ds\right] \\
&\quad + Mp(t)\int_0^t p(s)\left[v(s) + \int_0^s \mu_2(s,\tau)V(\tau)d\tau\right] ds \\
&\leq \|\gamma\|\|v\|_\infty(1+\mu_1^*) + M\|p\|_{L^\infty}\|v\|_\infty(1+\mu_2^*) \\
\|v\|_\infty &\leq \|v\|_\infty[\|\gamma\|(1+\mu_1^*) + M\|p\|_{L^\infty}(1+\mu_2^*)]
\end{aligned}$$

This means that

$$\|v\|_\infty\{1 - [\|\gamma\|(1+\mu_1^*) + M\|p\|_{L^\infty}(1+\mu_2^*)]\} \leq 0.$$

By (3.5) it follows that $\|v\|_\infty = 0$, that is $v(t) = 0$ for each $t \in [-r,b]$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzela theorem, V is relatively compact in D_ν. Applying now Theorem 2.11, we conclude that N_1 has a fixed point which is a mild solution for the problem (1.3) - (1.4). □

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