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Operators in 2-fuzzy $n-n$ inner product space

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Abstract

In this paper various 2-fuzzy operators are introduced in 2-fuzzy $n-n$ inner product space and the properties of 2-fuzzy self-adjoint, 2-fuzzy normal, 2-fuzzy unitary and 2-fuzzy projection operators are studied.

Keywords

2- fuzzy $n-n$ inner product space, 2-fuzzy self-adjoint operator, 2-fuzzy normal operator, 2-fuzzy unitary operator, 2-fuzzy projection operator.

AMS Subject Classification

03E72.

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Contents

1. Introduction

Gahler [4] introduced the theory of 2-norm on a linear space in 1964. In 1984 Katsaras [7] gave the notion of fuzzy norm on a linear space. Further, fuzzy normed spaces were defined in various ways by Cheng and Mordeson [2] and by Bag and Samanta [1]. R.M. Somasundaram and Thangaraj Beaula [9] introduced the notion of fuzzy 2-normed linear space, $\{F(X), N\}$. The concept of 2-inner product space was introduced by C.R. Diminnie, S. Gahler and A. White [5]. Parijat Sinha, Ghanshayam Lal and Divya Mishra introduced the concept of fuzzy 2-inner product space and the notion of $\alpha-2$-norm in [8]. The notions of fuzzy inner product space and of fuzzy normed linear space were established in [6]. Also, Vijayabalaji and Thillaigovindan [10] introduced the fuzzy n-inner product space as a generalization of the concept of n-inner product space given by Y.J. Cho, M. Matic and J. Pecaric in [3]. Thangaraj Beaula and Daniel Evans introduced the concept of 2-fuzzy $n-n$ inner product space in [11] as an extension of [10]. In this paper operators are introduced in 2-fuzzy $n-n$ inner product space and their properties are studied.

2. Preliminaries

Definition 2.1. *Let $n \in N$ and X be a real linear space of dimension greater or equal to n. Then a real valued function $\|.,\ldots,.\|$ on X^n is called a n-norm on X, if it satisfies the following four properties*

i) $\|x_1,\ldots,x_n\| = 0$ *if and only if* $x_1,\ldots,x_n$ *linearly dependent.*

ii) $\|x_1,\ldots,x_n\|$ *is invariant under any permutation*

iii) $\|x_1,\ldots,\alpha x_n\| = |\alpha|\|x_1,\ldots,x_n\|$, *for any* α *is a real number*

iv) $\|x_1,\ldots,x_{n-1},y+z\|$
$\leq \|x_1,\ldots,x_{n-1},y\| + \|x_1,\ldots,x_{n-1},z\|$

The pair $(X,\|.,\ldots,.\|)$ is called a n-normed linear space.

Definition 2.2. *Let X be a nonempty set, let $F(X)$ be the set of all fuzzy sets in X and let K be the field of real numbers. Then $F(X)$ becomes a linear space over the field K, where the addition and scalar multiplication are defined by $f+g = \{(x,\mu)+(y,\eta)\} = \{(x+y,\mu\wedge\eta) : (x,\mu)\in f$ and $(y,\eta)\in g\}$ and $kf = \{(kf,\mu) : (x,\mu)\in f,\}, \quad k\in K$.*

The linear space $F(X)$ is said to be a normed space, if, every $f \in F(X)$, is associated with a non-negative real number $\|f\|$ called the norm of f in such a way that

(i) $\|f\| = 0$, *if and only if* $f = 0$. *For* $\|f\| = 0$
$\Leftrightarrow \{\|(x,\mu)\|/(x,\mu) \in f\} = 0,$
$\Leftrightarrow x = 0, \mu \in (0,1] \Leftrightarrow f = 0.\|(x,\mu)\|$

(ii) $\|kf\| = |k|\|f\|, k \in K$. *For* $\|kf\| = \{\|k(x,\mu)\|/(x,\mu) \in f$ *and* $k \in K\} = \{|k|\|(x,\mu)\|/(x,\mu) \in f\} = |k|\|f\|$.

(iii) $\|f+g\| \leq \|f\| + \|g\|$ *for every* $f, g \in F(X)$. *For*

$$\begin{aligned}&\|f+g\| \\ &= \{\|(x,\mu)+(y,\eta)\| : x,y \in X, \mu,\eta \in (0,1]\} \\ &= \{\|(x+y),(\mu \wedge \eta\|/x,y \in X, \mu,\eta \in (0,1]\} \\ &\leq \{\|(x,\mu \wedge \eta)\| + \|(y,\mu \wedge \eta)\|/(x,\mu) \in f \text{ and} \\ &\qquad (y,\eta) \in g\} = \|f\| + \|g\|.\end{aligned}$$

Definition 2.3. *Let $F(X^n)$ be a linear space over a real field. A fuzzy subset N of $F(X^n)^n \times R$ is called 2-fuzzy $n-n$ norm if and only if*

(N1) *for all* $t \in R, t \leq 0, N(f_1,\ldots,f_n,t) = 0$

(N2) *for all* $t \in R, t > 0, N(f_1,\ldots,f_n,t) = 1$ *if and only if* $f_1,\ldots,f_n$ *are linearly dependent*

(N3) $N(f_1,\ldots,f_n,t) =$ *is invariant under any permutation of* $f_1,\ldots,f_n$

(N4) *for all* $t \in R, t > 0, N(f_1,\ldots,cf_n,t) = N(f_1,\ldots,f_n,\frac{t}{|c|})$

(N5) *for all* $s,t \in R, N(f_1,\ldots,f_n+f_n,s+t)$
$\geq \min\{N(f_1,\ldots,s), N(f_1,\ldots,f_n,t)\}$

(N6) $N(f_1,\ldots,f_n,t)$ *is a non-decreasing function of* $t \in R$ *and* $\lim_{t\to\infty} N(f_1,\ldots,f_n,t)$

The space $(F(X^n)^n, N)$ is called a 2-fuzzy $n-n$ normed linear space.

Definition 2.4. *Let $F(X^n)$, be a linear space over $\mathbb{C}$. Define a fuzzy subset η defined as a mapping from $[F(X^n)]^{n+1} \times \mathbb{C}$ to $[0,1]$ such that $(f_1,\ldots,f_n,f_{n+1}) \in [F(X^n)]^{n+1} \alpha \in \mathbb{C}$ satisfying the following conditions*

(I_1) *for* $g,h \in F(X), s,t \in s$

$$\begin{aligned}&\eta(f_1+g,h,f_2,\ldots,f_n,|t|+|s|) \\ &\geq \min\{\eta,(f_1,f_2,\ldots,f_n,|t|), \\ &\qquad \eta(g,h,f_2,\ldots,f_n,|s|)\}\end{aligned}$$

(I_2) *for* $s,t \in \mathbb{C}$

$$\begin{aligned}&\eta(f_1,g,f_2,\ldots,f_n,|st|) \\ &\geq \min\{\eta,(f_1,f_2,\ldots,f_n,|s|^2), \\ &\qquad \eta(g,g,f_2,\ldots,f_n,|t|^2)\}\end{aligned}$$

(I_3) *for* $t \in \mathbb{C}$

$$\begin{aligned}&\eta(f_1,g,f_2,\ldots,f_n,|t|) \\ &= \eta,(g,f_1,f_2,\ldots,f_n,|t|)\end{aligned}$$

(I_4) *for* $\alpha_1,\alpha_2 \in \mathbb{C},\ \alpha_1 \neq 0, \alpha_2 \neq 0$

$$\begin{aligned}&\eta(\alpha_1 f_1 \alpha_2 f_1,\ldots,f_n,|t|) \\ &\eta(f_1,f_2,\ldots,f_n,\frac{t}{|\alpha_1,\alpha_2|})\end{aligned}$$

(I_5) $\eta(f_1,f_1,f_2,\ldots,f_n,t),= 0 \quad t \in \mathbb{C}/R^+$
$\eta(f_1,f_1,f_2,\ldots,f_n,t) = 1\ \forall\, t > 0$ *if and only if* $f_1,\ldots,f_n$ *are linearly independent.*

(I_6) $\eta(f_1,gf_2,\ldots,f_n,t)$ *is invariant under any permutation of* $(f_2,\ldots,f_n)$

(I_7) $t > 0, \eta(f_1,f_2,\ldots,f_n,t) = \eta(f_2,f_2,f_1,f_3,\ldots,f_n,t)$

(I_8) $\eta(f_1,g,f_2,\ldots,f_n,t)$ *is a monotonic non-decreasing function of* $\mathbb{C}$ *and* $\lim_{t\to\infty} \eta(f_1,g,f_2,\ldots,f_n,t) = 1$.

Then η is said to be the 2-fuzzy $n-n$ inner product $F(X)^n$ and the pain $(F(X)^n,\eta)$ is called 2-fuzzy $n-n$ IPS.

Definition 2.5. *Let $(F(X^n),\eta)$ be a 2-fuzzy $n-n$ IPS satisfying the condition $\eta(f_1,f_1,f_2,\ldots,f_n,t^2) > 0$, when $t > 0$ implies that $f_1,f_2,\ldots f_n$ are linearly dependent. Then for all $\alpha \in (0,1)$, define $\|f_1,\ldots,f_n\|_\alpha = \inf\{t; \eta(f_1,f_1,f_2,\ldots,f_n,t^2) \geq \alpha\}$ a crisp norm on $F(X^n)$ called the $\alpha-n-n$ norm and the space is $(F(X^n), \|\cdot\|_\alpha)$ generated by η.*

3. 2-Fuzzy operators

Let T be a 2-fuzzy operator on 2-fuzzy $n-n$ inner product space $F(X^n)$. Then T gives rise to a 2-fuzzy operator T^* on $[F(X^n)]^*$ where T^* is defined by $(T^*H)f = H(Tf)$. Let $f \in F(X^n)$ and Hf its corresponding 2-fuzzy functional in $[F(X^n)]^*$ operate with T^* on Hf to obtain a 2-fuzzy functional $Hg = T^*Hf$ and return to its corresponding 2-fuzzy set g in $F(X^n)$. There are three mappings here as,

$$f \to H_f \to T^*H_f = H_g \to g$$

write $g = T^*f$ and call this new mapping T^* to map $F(X^n)$ into itself the adjoint of T. The same symbol is used for the adjoint of T as for its conjugate since these two mappings are the same if $F(X^n)$ and $[F(X^n)]^*$ are identified by means of the natural correspondence.

It can be observed that

$$(T^*H_f)h = H_f(Th) = < Th, f >_\alpha$$

and

$$(T^*H_f)h = H_g(h) < h, g >_\alpha = < h, T^*f >_\alpha$$

so that

$$< Th, f >_\alpha = < h, T^*f >_\alpha$$

Theorem 3.1.

$$< f+h,g >_\alpha = < f,g >_\alpha + < h,g >_\alpha$$

where $< f,g >_\alpha = \inf\{t; \eta(f,g,f_2,\dots,f_n,t) \geq \alpha\}$

Proof.

$$\begin{aligned}
&< f,g >_\alpha + < h,g >_\alpha \\
&= \inf\{t : \eta(f,g,f_2,\dots,f_n,t) \geq \alpha\} \\
&+ \quad \inf\{s : \eta(h,g,f_2,\dots,f_n,t \geq \alpha\} \\
&= \inf\{t+s : \eta(f,g,f_2,\dots,f_n,t) \geq, \alpha, \\
&\quad \eta(h,g,f_2,\dots,f_n,t) \geq t\} \\
&= \inf\{t+s : \min[\eta(f,g,f_2,\dots,f_n,t) \geq \alpha, \\
&\quad \eta(h,g,f_2,\dots,f_n,t) \geq \alpha]\} \\
&\geq \inf\{t+s : \eta(f+h,g,f_2,\dots,f_n,t+s) \geq \alpha\} \\
&= \inf\{r : \eta(f+h,g,f_2,\dots,f_n,r) \geq \alpha\} \\
&\qquad \text{where} \quad r = t+s \\
&= < f+h,g >_\alpha \qquad (3.1)
\end{aligned}$$

Conversely for any $\varepsilon > 0$

let,

$$\begin{aligned}
A &= \min\{(1-(1-\eta(f,g,f_2,\dots,f_n, < f,g >_\alpha - \frac{\varepsilon}{2}), \\
&\quad (1-\eta(h,g,f_2,\dots,f_n, < h,g >_\alpha - \frac{\varepsilon}{2}))\} \\
&= \min\{(1-\eta(-f,g,f_2,\dots,f_n, - < f,g >_\alpha + \frac{\varepsilon}{2})), \\
&\quad \eta(-h,g,f_2,\dots,f_n, - < h,g >_\alpha + \frac{\varepsilon}{2})\} \\
&\geq 1-\eta(-f,-h,g,f_2,\dots,f_n, - < f,g >_\alpha \\
&\quad - < h,g >_\alpha + \varepsilon) \\
&= \eta(f|h,g,f_2,\dots,f_n, < f,g >_\alpha | < h,g >_\alpha \varepsilon)
\end{aligned}$$

By the definition of infimum

$\eta(f,g,f_2,\dots,f_n, < f,g >_\alpha - \frac{\varepsilon}{2}) < \alpha$

Hence $1-\eta(f,g,f_2,\dots,f_n, < f,g >_\alpha - \frac{\varepsilon}{2}) < 1-\alpha$

Similarly $1-\eta(h,g,f_2,\dots,f_n, < h,g >_\alpha - \frac{\varepsilon}{2}) < 1-\alpha$

Then

$$\begin{aligned}
&\min\{(1-\eta(f,g,f_2,\dots,f_n, < f,g >_\alpha - \frac{\varepsilon}{2})), \\
&(1-\eta(h,g,f_2,\dots,f_n, < h,g >_\alpha - \frac{\varepsilon}{2}))\} > 1-\alpha \\
&(i.e) 1-A > 1-\alpha
\end{aligned}$$

Hence $A < \alpha$

which implies

$\eta(f+h,g,f_2,\dots,f_n, < f,g >_\alpha + < h,g >_\alpha - \varepsilon) \leq A < \varepsilon$

$(i.e) < f+h,g >_\alpha \geq < f,g >_\alpha + < h,g >_\alpha - \varepsilon$

Since ε is arbitrary,

$$< f+h,g >_\alpha \geq < f,g >_\alpha + < h,g >_\alpha \qquad (3.2)$$

From (3.1) and (3.2)

$$< f+h,g >_\alpha = < f,g >_\alpha + < h,g >_\alpha .$$

□

3.1 Linearity

T^* is linear.

Consider for any $f,g \in F(X^n)$ and for all $h \in F(X^n)$

$$\begin{aligned}
&< h, T^*(f+h) >_\alpha - < Th, f+g >_\alpha \\
&= \inf\{t_1+t_2 : \eta(Th, f+g, f_2,\dots,f_n, t_1+t_2) \geq \alpha\} \\
&= \inf\{t_1+t_2 : \min[\eta(Th,f,f_2,\dots,f_n,t_1), \\
&\quad \eta(Th,g,f_2,\dots,f_n,t_2) \geq \alpha\} \\
&= \inf\{t_1+t_2 : \eta(Th,f,f_2,\dots,f_n,t_1) \geq \alpha, \\
&\quad \eta(Th,g,f_2,\dots,f_n,t_2)\alpha\} \\
&= \inf\{t_1|t_2 : \eta(h,T^*f,f_2,\dots,f_n,t_1) \\
&\qquad \geq \alpha, \eta(h,T^*g,f_2,\dots,f_n,t_2) \geq \alpha\} \qquad (3.3)
\end{aligned}$$

Consider

$$\begin{aligned}
&< h, T^*f + T^*g >_\alpha \\
&= \inf\{t_1+t_2 : \eta(h, T^*f + T^*g, f_2,\dots,f_n, t_1+t_2) \geq \alpha\} \\
&\geq \inf\{t_1+t_2 : \eta(h,T^*f,f_2,\dots,f_n,t_1) \\
&\qquad \geq \alpha, \eta(h,T^*g,f_2,\dots,f_n,t_2) \geq \alpha\} \qquad (3.4)
\end{aligned}$$

From (3.3) and (3.4)

$$< h, T^*(f+g) >_\alpha = < h, T^*f + T^*g >_\alpha$$

hence

$T^*(f+g) = T^*f + T^*g$

$$\begin{aligned}
&< h, T^*(\beta f) >_\alpha \\
&= < Th, \beta f >_\alpha \\
&= \inf\{t : \eta(Th, \beta f, f_2,\dots,f_n,t) \geq \alpha\} \\
&= \inf\{t : \eta(Th, f, f_2,\dots,f_n, \frac{t}{|\beta|}) \geq \alpha\} \\
&= \inf\{t : \eta(h, T^*f, f_2,\dots,f_n, \frac{t}{|\beta|}) \geq \alpha\} \\
&= \inf\{t : \eta(h, \beta T^*f, f_2,\dots,f_n,t) \geq \alpha\} \\
&= < h, \beta T^*f >_\alpha
\end{aligned}$$

hence $T^*(\beta f) = \beta T^* f$ and so T^* is linear.

Consider

$$\begin{aligned}
&\|T^*,f,f_1,f_3,\dots,f_n\|_\alpha^2 \\
&= < T^*f, T^*f >_\alpha \\
&= < TT^*f, f >_\alpha \\
&= \|TT^*f,f,f_3,\dots,f_n\|_\alpha^2 \\
&\leq \|Tf,f,f_3,\dots,f_n\|_\alpha \|T^*f,f,f_3,\dots,f_n\|_\alpha
\end{aligned}$$

hence

$$\|T^*f,f,f_3,\dots,f_n\|_\alpha \leq \|Tf,f,f_3,\dots,f_n\|_\alpha$$

Theorem 3.2. *The* 2*-fuzzy adjoint operator* $T \to T^*$ *satisfies the following properties*

(i) $(T_1+T_2)^* = T_1^* + T_2^*$

(ii) $(\beta T)^* = \beta T^*$

(iii) $(T_1T_2)^* = T_2^*T_1^*$

(iv) $T^{**} = T$

(v) $\|T^*\| = T$

(vi) $\|T^*T\| = \|T\|^2$

Proof.

$$\begin{aligned}
&(i) < h,(T_1+T_2)^*g >_\alpha \\
&=< (T_1+T_2)h, g >_\alpha \\
&= \inf\{t_1+t_2 : \eta((T_1+T_2)h, g, f_2, \dots, f_n, t_1+t_2) \geq \alpha\} \\
&= \inf\{t_1+t_2 : \eta(T_1h+T_2h, g, f_2, \dots, f_n, t_1+t_2) \geq \alpha\} \\
&= \inf\{t_1+t_2 : \min[\eta(T_1h, g, f_2, \dots, f_n, t_1+t_2) \geq \alpha, \\
&\quad \eta(T_2h, g, f_2, \dots, f_n, t_1+t_2) \geq \alpha\}
\end{aligned}$$

The reverse inequality follows from Theorem 3.1

Therefore, $(T_1+T_2)^* = T_1^* + t_2^*$

$$\begin{aligned}
&(ii) < h,(\beta T)^*g >_\alpha \\
&=< (\beta T)h, g >_\alpha \\
&= \inf\{t : \eta(\beta Th, g, f_2, \dots, f_n, t) \geq \alpha\} \\
&= \inf\{t : \eta(Th, g, f_2, \dots, f_n, \frac{t}{|\beta|}) \geq \alpha\} \\
&= \inf\{t : \eta(h, T^*g, f_2, \dots, f_n, \frac{t}{|\beta|}) \geq \alpha\} \\
&= \inf\{t : \eta(h, T^*g, f_2, \dots, f_n, t) \geq \alpha\} \\
&=< h, \beta T^*g >_\alpha
\end{aligned}$$

$$(\beta T)^* = \beta T^*$$

$$\begin{aligned}
&(iii) < h,(T_1T_2)^*g >_\alpha \\
&=< (T_1T_2)h, g >_\alpha \\
&= \inf\{t : \eta((T_1T_2)h, g, f_2, \dots, f_n, t) \geq \alpha\} \\
&= \inf\{t : \eta(T_2h, T_1^*g, f_2, \dots, f_n, t) \geq \alpha\} \\
&= \inf\{t : \eta(h, T_2^*T_1^*g, f_2, \dots, f_n, t) \geq \alpha\} \\
&=< h, T_2^*T_1^*g >_\alpha
\end{aligned}$$

$$(T_1T_2)^* = T_2^*T_1^*$$

$$\begin{aligned}
&(iv) < h, T^{**}g >_\alpha \\
&=< h,(T^*)^*g >_\alpha \\
&= \inf\{t : \eta((h,T^*)^*, g, f_2, \dots, f_n, t) \geq \alpha\} \\
&= \inf\{t : \eta(T^*h, g, f_2, \dots, f_n, t) \geq \alpha\} \\
&= \inf\{t : \eta(h, Tg, f_2, \dots, f_n, t) \geq \alpha\} \\
&=< h, Tg >_\alpha
\end{aligned}$$

(v) Consider

$$\begin{aligned}
&\|T^*f, f_2, \dots, f_n\|_\alpha \\
&\leq \|Tf, f_2, \dots, f_n\|_\alpha
\end{aligned} \tag{3.5}$$

Applying (3.5) for T^*

$$\begin{aligned}
&\|(T^*)^*f, f_2, \dots, f_n\|_\alpha \\
&\leq \|T^*f, f_2, \dots, f_n\|_\alpha \\
&\|Tf, f_2, \dots, f_n\|_\alpha \\
&\leq \|T^*f, f_2, \dots, f_n\|_\alpha
\end{aligned} \tag{3.6}$$

From (3.5) and (3.6)

$$\|T^*f, f_2, \dots, f_n\|_\alpha \leq \|Tf, f_2, \dots, f_n\|_\alpha$$

$$\begin{aligned}
&(vi)\|T^*Tf, f_2, \dots, f_n\|_\alpha \\
&\leq \|T^*f, f_2, \dots, f_n\|_\alpha \|Tf, f_2, \dots, f_n\|_\alpha \\
&= \|Tf, f_2, \dots, f_n\|_\alpha \|Tf, f_2, \dots, f_n\|_\alpha \\
&= \|Tf, f_2, \dots, f_n\|_\alpha^2 \\
&\|T^*f, f_2, \dots, f_n\|_\alpha^2 \\
&=< Tf, Tf >_\alpha \\
&=< T^*Tf, f >_\alpha \\
&= \|T^*Tf, T^*Tf_2, \dots, f_n\|_\alpha \\
&= \inf\{t : \eta(T^*Tf, T^*Tf, f_2, \dots, f_n, t) > \alpha\} \\
&= \|T^*Tf, f_2, \dots, f_n\|_\alpha^2 \\
&\leq \|T^*Tf, f_2, \dots, f_n\|_\alpha.
\end{aligned}$$

From (3.5) and (3.6)

$$\|T^*Tf, f_2, \dots, f_n\|_\alpha = \|Tf, f_2, \dots, f_n\|_\alpha^2$$

□

3.2 2-Fuzzy self adjoint operator

$T \in \beta(\mathscr{F}(X^n)), T$ is said to be 2-fuzzy self adjoint when $T = T^*, 0^* = 0, I^* = I$

$$\begin{aligned}
< f, 0^*g >_\alpha &=< 0f, g >_\alpha \\
&= \inf\{t : \eta(0f, g, f_2, \dots, f_n, t) \geq \alpha\} \\
&= \inf\{t : \eta(0, g, f_2, \dots, f_n, t) \geq \alpha\} \\
&= 0.
\end{aligned}$$

Now to prove if A_1, A_2 are 2-fuzzy self adjoint then $(\beta_1A_1 + \beta_2A_2)^*$ is also 2-fuzzy self adjoint.

$$
\begin{aligned}
&< h,(\beta_1A_1+\beta_2A_2)*g>_\alpha\\
&=< h(\beta_1A_1+\beta_2A_2)g>_\alpha\\
&=\inf\{t:\eta((\beta_1A_1+\beta_2A_2)h,g,f_2,\dots,f_n,t)\geq\alpha\}\\
&=\inf\{t_1+t_2:\eta((\beta_1A_1+\beta_2A_2)h,g,f_2,\dots,f_n,t)\geq\alpha\}\\
&\geq\inf\{t_1+t_2:\min[\eta(\beta_1A_1h,g,f_2,\dots,f_n,t)\geq\alpha,\\
&\quad\eta(\beta_2A_2h,g,f_2,\dots,f_n,t)\geq\alpha]\}\\
&=\inf\{t_1+t_2:\min[\eta(\beta_1A_1^*h,g,f_2,\dots,f_n,t)\geq\alpha,\\
&\quad\eta(\beta_2A_2^*h,g,f_2,\dots,f_n,t)\geq\alpha]\}
\end{aligned}
$$

the reverse inequality follows from Theorem 3.1

Hence

$(\beta_1A_1+\beta_2A_2)^*=\beta_1A_1^*+\beta_2A_2^*+\beta_1A_1+\beta_2A_2$

Theorem 3.3. *If A_1,A_2 are 2-fuzzy self adjoint then their product A_1A_2 is also 2-fuzzy self adjoint if and only if $A_1A_2=A_2A_1$.*

Proof. Since we have $(A_1A_2)^*=A_2^*A_1^*$

Let $A_1A_2=A_2A_1$

$(A_1A_2)^*=A_2^*A_1^*=A_2A_1=A_1A_2$.

Hence the product is 2-fuzzy self adjoint

Conversely assume that the product is 2-fuzzy self adjoint

Consider $(A_1A_2)^*=A_2^*A_1^*=A_2A_1$.

Since $(A_1A_2)^*=A_1A_2$,

we have $A_2A_1=A_1A_2$. □

Theorem 3.4. *If T is a 2-fuzzy operator for which*

$$\eta(Tf,f,f_2,\dots,f_n,t)=0$$

for all f then $T=0$.

Proof. Consider

$$
\begin{aligned}
&\eta(T(\beta_1f+\beta_2g),\beta_1f+\beta_2g,f_2,\dots,f_n,t)\\
&\geq\min[\eta(Tf,f,f_2,\dots,f_n,\frac{t}{|\beta|}),\\
&\quad\eta(Tf,f,f_2,\dots,f_n,\frac{t}{|\beta_1\beta_2|})\\
&(\times)\eta(Tg,f,f_2,\dots,f_n,\frac{t}{|\beta_2\beta_1|}),\\
&\quad\eta(Tg,g,f_2,\dots,f_n,\frac{t}{|\beta_2\beta_1|})]\\
&\Rightarrow\eta(Tf,g,f_2,\dots,f_n,t)=0
\end{aligned}
$$

If $T=0$, then $\eta(0f,g,f_2,\dots,f_n,t)=0$

when $T\neq0$, put $g=Tf$

then

$$
\begin{aligned}
&\eta(Tf,Tf,f_2,\dots,f_n,t)=0\\
&\Rightarrow\|Tf,f_2,\dots,f_n\|_\alpha=0\\
&\Rightarrow Tf=0\\
&\Rightarrow T=0.
\end{aligned}
$$

□

3.3 2-Fuzzy normal operator

An operator N is said to be 2-fuzzy normal if it commutes with its adjoint i.e) $NN^*=N^*N$.

Theorem 3.5. *An operator T is 2-fuzzy normal if and only if*

$$\|T^*f,f_2,\dots,f_n\|_\alpha=\|Tf,f_2,\dots,f_n\|_\alpha$$

for all f.

Proof.

$$
\begin{aligned}
&\|T^*f,f_2,\dots,f_n\|_\alpha=\|Tf,f_2,\dots,f_n\|_\alpha\\
&\|T^*f,f_2,\dots,f_n\|_\alpha^2=\|T^*f,f_2,\dots,f_n\|_\alpha^2\\
&\Leftrightarrow\inf\{t:\eta(T^*f,T^*f,f_2,\dots,f_n,t)\geq\alpha\}\\
&=\inf\{t:\eta(Tf,Tf,f_2,\dots,f_n,t)\geq\alpha\}\\
&\Leftrightarrow\inf\{t:\eta(TT^*f,f,f_2,\dots,f_n,t)\geq\alpha\}\\
&=\inf\{t:\eta(T^*Tf,f,f_2,\dots,f_n,t)\geq\alpha\}\\
&\Leftrightarrow\inf\{t:\eta((TT^*-T^*T)f,f,f_2,\dots,f_n,t)\geq\alpha\}=0\\
&\Leftrightarrow\eta((TT^*-T^*T)f,f,f_2,\dots,f_n,t)\geq\alpha\}=0\\
&\Leftrightarrow TT^*-T^*T=0\\
&\Leftrightarrow TT^*=T^*T.
\end{aligned}
$$

□

Theorem 3.6. *If N is a 2-fuzzy normal operator and 2-fuzzy self adjoint on $\mathscr{F}(X^n)$ then $\|N^2f,f_2,\dots,f_n\|_\alpha=\|Nf,f_2,\dots,f_n\|_\alpha^2$.*

Proof. If N is a 2-fuzzy normal operator, then

$$\|Nf,f_2,\dots,f_n\|_\alpha=\|N^*f,f_2,\dots,f_n\|_\alpha \tag{3.7}$$

by replacing f by Nf (3.7) becomes

$$
\begin{aligned}
&\|NNf,f_2,\dots,f_n\|_\alpha=\|NN^*f,f_2,\dots,f_n\|_\alpha\\
&\Rightarrow\|N^2f,f_2,\dots,f_n\|_\alpha=\|NN^*f,f_2,\dots,f_n\|_\alpha\\
&\|N^2f,f_2,\dots,f_n\|_\alpha\\
&=\inf\{t:\eta(N^2f,N^2f,f_2,\dots,f_n)\geq\alpha\}\\
&=\inf\{t:\eta(NNf,NNf,f_2,\dots,f_n)\geq\alpha\}\\
&=\inf\{t:\eta(N^*Nf,N^*Nf,f_2,\dots,f_n)\geq\alpha\}\\
&=\|N^*Nf,f_2,\dots,f_n\|_\alpha
\end{aligned}
$$

By Theorem 3.2,

$$\|N^*Nf,f_2,\dots,f_n\|_\alpha=\|Nf,f_2,\dots,f_n\|_\alpha^2 \tag{3.8}$$

From (3.7) and (3.8)

$$\|N^2f,f_2,\dots,f_n\|_\alpha=\|Nf,f_2,\dots,f_n\|_\alpha^2$$

□

3.4 2-Fuzzy unitary operator

An operator T is said to be 2-fuzzy unitary if $T^*T = T^*T = I$.

Theorem 3.7. *If T is a 2-fuzzy operator on a 2-fuzzy n-n Hilbert space $\mathscr{F}(X^n)$, then the following conditions are equivalent to one another.*

(i) $T^*T = I$

(ii) $<Tf,Tg>_\alpha = <f,g>_\alpha$ *for all* $f,g \in \mathscr{F}(X^n)$

(iii) $\|Tf,f_2,\dots,f_n\|_\alpha = \|f,f_2,\dots,f_n\|_\alpha$

Proof. $(i) \Rightarrow (ii)$
Given $T^*T = I, <Tf,Tg>_\alpha$
$=<f,T^*Tg>_\alpha = <f,g>_\alpha$
$(ii) \Rightarrow (iii)$
Given $<Tf,Tg>_\alpha = <f,g>_\alpha$
taking $f = g$

$$<Tf,Tf>_\alpha = <f,f>_\alpha$$
$$\|Tf,f_2,\dots,f_n\|_\alpha^2 = \|f,f_2,\dots,f_n\|_\alpha^2$$
$$\|Tf,f_2,\dots,f_n\|_\alpha = \|f,f_2,\dots,f_n\|_\alpha$$

$(iii) \Rightarrow (i)$ Given $\|Tf,f_2,\dots,f_n\|_\alpha = \|f,f_2,\dots,f_n\|_\alpha$
Therefore $\|Tf,f_2,\dots,f_n\|_\alpha^2 = \|f,f_2,\dots,f_n\|_\alpha^2$

$$\Rightarrow <Tf,Tf>_\alpha = <f,f>_\alpha$$
$$\Rightarrow <Tf,T^{**}f>_\alpha = <f,f>_\alpha$$

hence

$$<T^*Tf,f>_\alpha = <Tf,T^{**}f>_\alpha = <f,f>_\alpha$$
$$\Rightarrow <(T^*T-I)f,f>_\alpha = 0$$
$$\Rightarrow T^*T - I = 0$$
$$\Rightarrow T^*T = I.$$

□

Theorem 3.8. *An operator T on a 2-fuzzy $n-n$ Hilbert space $\mathscr{F}(X^n)$ is unitary if and only if it is an isomorphism of $\mathscr{F}(X^n)$ onto itself.*

Proof. Let T be a 2-fuzzy unitary operator on $\mathscr{F}(X^n)$. Then from the definition of the unitary operator, it is invertible. Hence it is onto. Also $T^*T = I$.

But by Theorem 3.7

$\|Tf,f_2,\dots,f_n\|_\alpha = \|f,f_2,\dots,f_n\|_\alpha$ hence T is an isometric isomorphism of $\mathscr{F}(X^n)$ onto itself.

Conversely, let T be an isometric isomorphism of $\mathscr{F}(X^n)$ onto itself, then T is $1-1$ and onto and T^{-1} exists.

But $\|Tf,f_2,\dots,f_n\|_\alpha = \|f,f_2,\dots,f_n\|_\alpha$, by Theorem 3.7
$T^*T = I$

Hence

$$(T^*T)T^{-1} = T^{-1}$$
$$\Rightarrow T^*(TT^{-1}) = T^{-1}$$
$$\Rightarrow T^*I = T^{-1}$$
$$\Rightarrow T^* = T^{-1}$$

Pre multiply (3.7) by T
$TT^* = TT^{-1} = I \Rightarrow TT^* = I$
Post multiply (3.7) by T
$T^*T = T^{-1}T = I$
$\Rightarrow T$ is 2-fuzzy unitary.

□

4. 2-Fuzzy projection

A projection P on a 2-fuzzy $n-n$ Hilbert space $\mathscr{F}(X^n)$ is an operator P an $\mathscr{F}(X^n)$ such that $P^2 = P$ and $P^* = P$.

Theorem 4.1. *If P is a projection on a 2-fuzzy $n-n$ Hilbert space with range M and null space N, then $M \perp N$ if and only if P is self adjoint and $N = M^\perp$.*

Proof. Let P be a 2-fuzzy projection on $\mathscr{F}(X^n)$ with the range M and null space N.

Then $\mathscr{F}(X^n) = M \oplus N$.

Let $M|N$

Now to prove P is 2-fuzzy self adjoint. Each $h \in \mathscr{F}(X^n)$ can be written uniquely in the form $h = f + g$, where $f \in M$ and $g \in N$.

Here $Ph = f$ and since

$$M \perp N, <f,g>_\alpha = 0 \tag{4.1}$$

From (4.1)

$$\begin{aligned} <Ph,h>_\alpha &= <f,h>_\alpha \\ &= <f,f+g>_\alpha \\ &= <f,f>_\alpha + <f,g>_\alpha \\ &= <f,f>_\alpha \end{aligned}$$

Also

$$\begin{aligned} <P^*h,h>_\alpha &= <h,Ph>_\alpha \\ &= <h,f>_\alpha \\ &= <f+g,f>_\alpha \\ &= <f,f>_\alpha \\ &\Rightarrow <P^*h,h>_\alpha = <Ph,h>_\alpha \\ &\Rightarrow <(P^*-P)h,h>_\alpha = 0 \\ &\Rightarrow P^* = P. \end{aligned}$$

Therefore P is 2-fuzzy self adjoint.

Conversely assume P is 2-fuzzy self adjoint. Now to prove $M \perp N$.

Let $f \subset M, g \subset N$

Then $Pf = f, Pg = 0$

$$\begin{aligned} < f,g >_\alpha &=< Pf,g >_\alpha \\ &=< f,P^* g >_\alpha \\ &=< f,Pg >_\alpha \\ &< f,0 >_\alpha \\ &= 0 \\ &\Rightarrow M \perp N. \end{aligned}$$

To prove in P is a 2-fuzzy projection on H with range M and null space N, then $M \perp N$.

$N = M^\perp$

Let $f \in N$, then $f \in M^\perp \Rightarrow N \subset M^\perp$ if $N \neq M^\perp$, assume N is a proper closed subspace of $M^\perp$

then exists a non zero $h_0 \in M^\perp$ such that $h_0 \perp N$. But $h_0 \in M^\perp$ implies $h_0 \perp M$.

Therefore $h_0 \perp M$ and $h_0 \perp N$.

Since $\mathscr{F}(X^n) = M \oplus N, h_0$ but $\mathscr{F}(X^n)$

$\Rightarrow h_0 = 0$ leads to a contradiction

$\Rightarrow N = M^\perp$.

□

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