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Several results for high dimensional singular fractional systems involving n^2 Caputo derivatives

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Abstract
In this paper, we introduce a high dimensional systems of singular fractional nonlinear differential equations involving n^2 Caputo derivatives. Using Schauder fixed point theorem and the contraction mapping principle, we investigate new existence and uniqueness results. Furthermore, we define and study the Ulam-Hyers stability and the generalized Ulam-Hyers stability for such systems. The application of the main results is illustrated by some examples.

Keywords
Caputo derivative, fixed point, singular fractional differential equation, existence, uniqueness, Ulam-Hyers stability, generalized Ulam-Hyers stability.

AMS Subject Classification
30C45, 39B72, 39B82.

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Contents

1. Introduction and Preliminaries

Due to its demonstrated applications in engineering sciences and applied mathematics, the fractional calculs has gained considerable popularity and importance. For instance see, [12, 16, 19]. There are recently some results obtained for the existence and uniqueness by dealing with fractional differential equations, see [1, 7–10, 13, 21]. Also other researchs papers treated the existence and uniqueness of solutions for singular fractional differential equation. For more details, [2–6, 17, 20, 23, 24]. On the other hand, recent Ulam-Hyers stabilities results have been obtained in [11, 14, 15, 18, 22–25].

Now, we present some inspiring results for our work: In [3], R.P. Agarwal, D. O'Regan and S. Staněk examined the existence of solutions to the singular fractional boundary value problem:

$$\begin{cases} {}^cD^{\alpha}u(t)+f\left(t,u(t),u'(t),{}^cD^{\mu}u(t)\right)=0,\\ u'(0)=u(1)=0,\end{cases}$$

where $1<\alpha<2$, $0<\mu<1$, ${}^cD^{\alpha}$ stands for the Caputo fractional derivative, f is L^q-Carathéodory function on $[0,1]\times(0,+\infty)\times(-\infty,0)\times(-\infty,0)$, $q>\frac{1}{\alpha-1}$, and $f(t,x,y,z)$ may be singular at the value 0 of its space variables x,y,z.

In [5], Z. Bai and W. Sun studied the existence and multiplicity of solutions for the following singular fractional problem:

$$\begin{cases} D^{\alpha}_{0+}u(t)+f\left(t,u(t),D^{\nu}_{0+}u(t),D^{\mu}_{0+}u(t)\right)=0,\\ u(0)=u'(0)=u''(0)=u''(1)=0,\end{cases}$$

where $3<\alpha\leq4$, $0<\nu\leq1$, $1<\mu\leq2$, D^{α}_{0+}, D^{ν}_{0+} and D^{μ}_{0+} are the standard Riemann-Liouville fractional derivatives, f is a Carathédory function on $[0,1]\times D$; $\left(D\subset\mathbb{R}^3\right)$ and $f(t,x,y,z)$ is singular at the value 0 of its arguments x,y,z.

In [15], R.W. Ibrahim investigated the existence and uniqueness of solutions and different types of Ulam-Hyers stability for the following cauchy differential equation of fractional order:

$$\begin{cases} D^{\alpha}_{z}u(z)=f(z,u(z)),\\ au(0)+bu(1)=cu(\xi),\ z,\xi\in U,\ a+b\neq c,\end{cases}$$

where $u : U \to B$ is an analytic function for all z in the unit disk $U := \{z : |z| < 1\}$ and $f : U \times B \to B$ is an analytic function in $z \in U$. B is the space of all analytic and bounded functions in the unit disk.

In [7], Z. Dahmani and A. Taïeb discussed the existence and uniqueness of solutions to the following nonlinear coupled system:

$$\begin{cases} D^{\alpha_1} x_1(t) = \\ \sum_{i=1}^{m} f_i^1 \begin{pmatrix} t, x_1(t), x_2(t), ..., x_n(t), \\ D^{\beta_1} x_1(t), D^{\beta_2} x_2(t), \\ ..., D^{\beta_n} x_n(t) \end{pmatrix}, \\ D^{\alpha_2} x_2(t) = \\ \sum_{i=1}^{m} f_i^2 \begin{pmatrix} t, x_1(t), x_2(t), ..., x_n(t), \\ D^{\beta_1} x_1(t), D^{\beta_2} x_2(t), \\ ..., D^{\beta_n} x_n(t) \end{pmatrix}, \\ \vdots \\ D^{\alpha_n} x_n(t) = \\ \sum_{i=1}^{m} f_i^n \begin{pmatrix} t, x_1(t), x_2(t), ..., x_n(t), \\ D^{\beta_1} x_1(t), D^{\beta_2} x_2(t), \\ ..., D^{\beta_n} x_n(t) \end{pmatrix}, \\ x_k(0) = x_k'(0) = \\ ... = x_k^{(n-2)}(0) = x_k^{(n-1)}(1) = 0, \\ k = 1, 2, ..., n,\ t \in [0,1]. \end{cases}$$

Inspired by all the cited works, this paper is devoted to build the existence, uniqueness and Ulam-Hyers stability and the generalized Ulam-Hyers stability of solutions for the following problem of singular fractional nonlinear equations:

$$\begin{cases} D^{\alpha_1} u_1(t) = f_1 \begin{pmatrix} t, u_1(t), ..., u_n(t), \\ D^{\alpha_1^1} u_1(t), ..., D^{\alpha_1^{n-1}} u_1(t), \\ D^{\alpha_2^1} u_2(t), ..., D^{\alpha_2^{n-1}} u_2(t), ..., \\ D^{\alpha_n^1} u_n(t), ..., D^{\alpha_n^{n-1}} u_n(t) \end{pmatrix}, \\ 0 < t \le 1, \\ \vdots \\ D^{\alpha_n} u_n(t) = f_n \begin{pmatrix} t, u_1(t), ..., u_n(t), \\ D^{\alpha_1^1} u_1(t), ..., D^{\alpha_1^{n-1}} u_1(t), \\ D^{\alpha_2^1} u_2(t), ..., D^{\alpha_2^{n-1}} u_2(t), ..., \\ D^{\alpha_n^1} u_n(t), ..., D^{\alpha_n^{n-1}} u_n(t) \end{pmatrix}, \\ 0 < t \le 1, \\ n-1 < \alpha_k < n,\ k = 1, 2, ..., n, \\ i-1 < \alpha_k^i < i,\ i = 1, 2, ..., n-1, \\ u_k^{(j)}(0) = \omega_j^k,\ D^{\nu_k} u_k(1) + J^{\eta_k} u_k(1) = 0, \\ n-2 < \nu_k < n-1,\ \eta_k > 0,\ n \in N^* \setminus \{1\}, \end{cases} \tag{1.1}$$

where $f_k : (0,1] \times \mathbb{R}^{n^2} \to \mathbb{R}$ are continuous functions, singular at $t = 0$, and $\lim_{t \to 0^+} f_k(t) = \infty$. D^{α_k} and $D^{\alpha_k^i}$, $i = 1, 2, ..., n-1$, $k = 1, 2, ..., n$, stand for the Caputo fractional derivative defined by:

$$\begin{aligned} D^{\gamma} x(t) &= \frac{1}{\Gamma(m-\gamma)} \int_0^t (t-s)^{m-\gamma-1} x^{(m)}(s)\, ds \\ &= J^{m-\gamma} x^{(m)}(t); \\ m-1 &< \gamma < m,\ m \in \mathbb{N}^*. \end{aligned}$$

The Riemann-Liouville fractional integral J^{ϑ} of order $\vartheta \ge 0$ for a continuous function f on $[0, +\infty)$ is defined by:

$$J^{\vartheta} f(t) = \begin{cases} \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} f(s)\, ds, & \vartheta > 0, \\ f(t), & \vartheta = 0, \end{cases}$$

where $t \ge 0$, and $\Gamma(\vartheta) := \int_0^\infty e^{-x} x^{\vartheta-1} dx$.

We need to the following properties and the fundamental Lemma from the fractional calculus theory [16]:

(i) : For $\alpha, \beta > 0$; $n-1 < \alpha < n$, we have $D^{\alpha} t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1}$, $\beta > n$, and $D^{\alpha} t^j = 0$, $j = 0, 1, ..., n-1$.

(ii) : $D^p J^q f(t) = J^{q-p} f(t)$, where $q > p > 0$ and $f \in L^1([a,b])$.

(iii) : Let $n \in \mathbb{N}^*$, $n-1 < \alpha < n$, and $D^{\alpha} u(t) = 0$. Then, $u(t) = \sum_{j=0}^{n-1} c_j t^j$, and $J^{\alpha} D^{\alpha} u(t) = u(t) + \sum_{j=0}^{n-1} c_j t^j$, $(c_j)_{j=0,1,...,n-1} \in \mathbb{R}$.

Lemma 1.1. *(Shauder fixed point theorem) Let (E, d) be a complete metric space, let X be a closed convex subset of E, and let $A : E \to E$ be a mapping such that the set $Y := \{Ax : x \in X\}$ is relatively compact in E. Then, A has at last one fixed point.*

Let us now impart the integral solution of system (1) by the following auxiliary result:

Lemma 1.2. *For $n \in \mathbb{N}^* \setminus \{1\}$, $n-1 < \alpha_k < n$, $k = 1, 2, ..., n$, and $U_k \in C([0,1], \mathbb{R})$. The following system*

$$\begin{cases} D^{\alpha_k} u_k(t) = U_k(t),\ 0 < t < 1, \\ u_k^{(j)}(0) = \omega_j^k,\ D^{\nu_k} u_k(1) + J^{\eta_k} u_k(1) = 0, \\ n-2 < \nu_k < n-1,\ \eta_k > 0, \end{cases} \tag{1.2}$$

has a unique solution $(u_1, u_2, ..., u_n)(t)$:

$$\begin{aligned} & u_k(t) = \int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} U_k(s)\, ds + \sum_{j=0}^{n-2} \frac{\omega_j^k}{j!} t^j \\ & - \sum_{j=0}^{n-2} \frac{\omega_j^k \Gamma(n-\nu_k) \Gamma(n+\eta_k) t^{n-1}}{\Gamma(j+1+\eta_k)(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))} \\ & - \frac{\Gamma(n-\nu_k)\Gamma(n+\eta_k) t^{n-1}}{(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))} \\ & \times \int_0^1 \left(\frac{(1-s)^{\alpha_k-\nu_k-1}}{\Gamma(\alpha_k-\nu_k)} + \frac{(1-s)^{\alpha_k+\eta_k-1}}{\Gamma(\alpha_k+\eta_k)} \right) U_k(s)\, ds. \end{aligned} \tag{1.3}$$

Proof. Using the property (iii), we can write the problem (1.2) to an equivalent integral equations:

$$\begin{cases} u_1(t) = \int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} U_1(s)\, ds - \sum_{j=0}^{n-1} c_j^1 t^j, \\ \vdots \\ u_n(t) = \int_0^t \frac{(t-s)^{\alpha_n-1}}{\Gamma(\alpha_n)} U_n(s)\, ds - \sum_{j=0}^{n-1} c_j^n t^j, \end{cases} \tag{1.4}$$

where

$$\begin{pmatrix} c_0^1 & c_1^1 & \cdots & c_{n-1}^1 \\ c_0^2 & c_1^2 & \cdots & c_{n-1}^2 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ c_0^n & c_1^n & \cdots & c_{n-1}^n \end{pmatrix} \in M_n(\mathbb{R}).$$

Then, we get

$$\begin{cases} u_k^{(j)}(0) = -j!c_j^k, \ j = 0,1,...,n-2, \\ D^{\nu_k} u_k(1) = \int_0^1 \frac{(1-s)^{\alpha_k-\nu_k-1}}{\Gamma(\alpha_k-\nu_k)} U_k(s)\,ds \\ -\frac{\Gamma(n)}{\Gamma(n-\nu_k)} c_{n-1}^k, \\ J^{\eta_k} u_k(1) = \int_0^1 \frac{(1-s)^{\alpha_k+\eta_k-1}}{\Gamma(\alpha_k+\eta_k)} U_k(s)\,ds \\ -\sum_{j=0}^{n-2} \frac{\Gamma(j+1)}{\Gamma(j+1+\eta_k)} c_j^k - \frac{\Gamma(n)}{\Gamma(n+\eta_k)} c_{n-1}^k. \end{cases}$$

Using the conditions: $u_k^{(j)}(0) = \omega_j^k$, $D^{\nu_k} u_k(1) + J^{\eta_k} u_k(1) = 0$, we obtain

$$c_j^k = \begin{cases} -\frac{\omega_j^k}{j!}, \quad j = 0,1,...,n-2, \\ \frac{\Gamma(n-\nu_k)\Gamma(n+\eta_k)}{(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))} \\ \times \int_0^1 \left(\frac{(1-s)^{\alpha_k-\nu_k-1}}{\Gamma(\alpha_k-\nu_k)} + \frac{(1-s)^{\alpha_k+\eta_k-1}}{\Gamma(\alpha_k+\eta_k)} \right) U_k(s)\,ds \\ + \sum_{j=0}^{n-2} \frac{\omega_j^k \Gamma(n-\nu_k)\Gamma(n+\eta_k)}{\Gamma(j+1+\eta_k)(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}, \ j = n-1. \end{cases} \tag{1.5}$$

Substituting (1.5) in (1.4), we receive (1.3). This completes the proof. □

Let us introduce the Banach space

$$B := \left\{ \begin{array}{l} (u_1,u_2,...,u_n) : u_k \in C([0,1],\mathbb{R}), \\ D^{\alpha_k^i} u_k \in C([0,1],\mathbb{R}), \\ k = 1,2,...,n, \ i = 1,2,...,n-1 \end{array} \right\},$$

where $n \in \mathbb{N}^* \setminus \{1\}$, endowed with the norm:

$$\|(u_1,u_2,...,u_n)\|_B = \max_{\substack{1 \le k \le n \\ 1 \le i \le n-1}} \left(\|u_k\|_\infty, \left\| D^{\alpha_k^i} u_k \right\|_\infty \right);$$

$$\|u_k\|_\infty = \sup_{t \in J} |u_k(t)|, \left\| D^{\alpha_k^i} u \right\|_\infty = \sup_{t \in J} \left| D^{\alpha_k^i} u_k(t) \right|.$$

2. Existence and Uniqueness

In this section, we establish sufficient conditions for the existence and the uniqueness of solutions for the problem (1.1). Then, some examples are presented to illustrate tlhe application of the main results.

Let define the nonlinear operator $T : B \to B$ by

$$T(u_1,u_2,...,u_n)(t) :=$$

$$(T_1(u_1,u_2,...,u_n)(t),...,T_n(u_1,u_2,...,u_n)(t)), t \in [0,1],$$

such that for $k = 1,2,...,n$,

$$T_k(u_1,u_2,...,u_n)(t) :=$$

$$\int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} f_k \begin{pmatrix} s, u_1(s), \\ ..., u_n(s), \\ D^{\alpha_1^1} u_1(s), \\ ..., D^{\alpha_1^{n-1}} u_1(s), \\ ..., D^{\alpha_n^1} u_n(s), \\ ..., D^{\alpha_n^{n-1}} u_n(s) \end{pmatrix} ds$$
$$+ \sum_{j=0}^{n-2} \frac{\omega_j^k}{j!} t^j - \sum_{j=0}^{n-2} \frac{\omega_j^k \Gamma(n-\nu_k)\Gamma(n+\eta_k) t^{n-1}}{\Gamma(j+1+\eta_k)(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}$$

$$-\frac{\Gamma(n-\nu_k)\Gamma(n+\eta_k) t^{n-1}}{(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}$$

$$\times \int_0^1 \begin{pmatrix} \frac{(1-s)^{\alpha_k-\nu_k-1}}{\Gamma(\alpha_k-\nu_k)} \\ + \frac{(1-s)^{\alpha_k+\eta_k-1}}{\Gamma(\alpha_k+\eta_k)} \end{pmatrix} f_k \begin{pmatrix} s, u_1(s), \\ ..., u_n(s), \\ D^{\alpha_1^1} u_1(s), \\ ..., D^{\alpha_1^{n-1}} u_1(s), \\ ..., D^{\alpha_n^1} u_n(s), \\ ..., D^{\alpha_n^{n-1}} u_n(s) \end{pmatrix} ds.$$

Lemma 2.1. *Let $n-1 < \alpha_k < n, n \in \mathbb{N}^* \setminus \{1\}, F_k : (0,1] \to \mathbb{R}$ are continuous and $\lim_{t \to 0^+} F_k(t) = \infty$. Assume that $0 < \delta_k < 1$, and $t^{\delta_k} F_k(t)$ are continuous on $[0,1]$. Then,*

$$u_k(t) = \int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} F_k(s)\,ds + \sum_{j=0}^{n-2} \frac{\omega_j^k}{j!} t^j$$
$$-\sum_{j=0}^{n-2} \frac{\omega_j^k \Gamma(n-\nu_k)\Gamma(n+\eta_k) t^{n-1}}{\Gamma(j+1+\eta_k)(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}$$

$$-\frac{\Gamma(n-\nu_k)\Gamma(n+\eta_k) t^{n-1}}{(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}$$
$$\times \int_0^1 \left(\frac{(1-s)^{\alpha_k-\nu_k-1}}{\Gamma(\alpha_k-\nu_k)} + \frac{(1-s)^{\alpha_k+\eta_k-1}}{\Gamma(\alpha_k+\eta_k)} \right) F_k(s)\,ds,$$

are continuous on $[0,1]$.

Proof. From the continuity of $t^{\delta_k} F_k(t)$ and

$$u_k(t) = \int_0^t \frac{(t-s)^{\alpha_k-1} s^{-\delta_k}}{\Gamma(\alpha_k)} s^{\delta_k} F_k(s)\,ds + \sum_{j=0}^{n-2} \frac{\omega_j^k}{j!} t^j$$
$$-\sum_{j=0}^{n-2} \frac{\omega_j^k \Gamma(n-\nu_k)\Gamma(n+\eta_k) t^{n-1}}{\Gamma(j+1+\eta_k)(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}$$
$$-\frac{\Gamma(n-\nu_k)\Gamma(n+\eta_k) t^{n-1}}{(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}$$
$$\times \int_0^1 \left(\frac{(1-s)^{\alpha_k-\nu_k-1}}{\Gamma(\alpha_k-\nu_k)} + \frac{(1-s)^{\alpha_k+\eta_k-1}}{\Gamma(\alpha_k+\eta_k)} \right) s^{-\delta_k} s^{\delta_k} F_k(s)\,ds.$$

we can see that $u_k(0) = \omega_0^k$. The proof is given into three cases.

Case 1: For $t_0 = 0$ and $\forall t \in (0,1]$, it follows the continuity of $t^{\delta_k} F_k(t)$, that there exist nonnegative constants G_k:

$$\left| t^{\delta_k} F_k(t) \right| \le G_k,\ k = 1,2,...,n.$$

Then,

$$\begin{aligned}
&|u_k(t) - u_k(0)| \\
\le\ & \frac{G_k}{\Gamma(\alpha_k)} \int_0^t (t-s)^{\alpha_k - 1} s^{-\delta_k} ds + \sum_{j=1}^{n-2} \frac{\left|\omega_j^k\right|}{j!} t^j \\
& + \sum_{j=0}^{n-2} \frac{\left|\omega_j^k\right| \Gamma(n-\nu_k)\Gamma(n+\eta_k) t^{n-1}}{\left[\begin{array}{c}\Gamma(j+1+\eta_k)(n-1)! \\ \times(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))\end{array}\right]} \\
& + \frac{G_k \Gamma(n-\nu_k)\Gamma(n+\eta_k) t^{n-1}}{(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))} \\
& \times \int_0^1 \left(\begin{array}{c} \frac{(1-s)^{\alpha_k-\nu_k-1} s^{-\delta_k}}{\Gamma(\alpha_k-\nu_k)} \\ + \frac{(1-s)^{\alpha_k+\eta_k-1} s^{-\delta_k}}{\Gamma(\alpha_k+\eta_k)} \end{array} \right) ds \\
\le\ & \frac{G_k \beta(\alpha_k, 1-\delta_k) t^{\alpha_k-\delta_k}}{\Gamma(\alpha_k)} + \sum_{j=1}^{n-2} \frac{\left|\omega_j^k\right| t^j}{j!} \\
& + \sum_{j=0}^{n-2} \frac{\left|\omega_j^k\right| \Gamma(n-\nu_k)\Gamma(n+\eta_k) t^{n-1}}{\left[\begin{array}{c}\Gamma(j+1+\eta_k)(n-1)! \\ \times(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))\end{array}\right]} \\
& + \frac{G_k \Gamma(n-\nu_k)\Gamma(n+\eta_k) t^{n-1}}{(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))} \\
& \times \left(\begin{array}{c} \frac{\beta(\alpha_k-\nu_k, 1-\delta_k)}{\Gamma(\alpha_k-\nu_k)} \\ + \frac{\beta(\alpha_k+\eta_k, 1-\delta_k)}{\Gamma(\alpha_k+\eta_k)} \end{array} \right) \\
\le\ & \left(\begin{array}{c} \frac{G_k \Gamma(1-\delta_k) t^{\alpha_k-\delta_k}}{\Gamma(\alpha_k+1-\delta_k)} + \sum_{j=1}^{n-2} \frac{\left|\omega_j^k\right| t^j}{j!} \\ + \sum_{j=0}^{n-2} \frac{\left|\omega_j^k\right| \Gamma(n-\nu_k)\Gamma(n+\eta_k) t^{n-1}}{\Gamma(j+1+\eta_k)(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))} \\ + \frac{G_k \Gamma(n-\nu_k)\Gamma(n+\eta_k) t^{n-1}}{(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))} \\ \times \left(\frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k-\nu_k+1-\delta_k)} + \frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k+\eta_k+1-\delta_k)} \right) \end{array} \right) \\
\to\ & 0, \text{ as } t \to 0.
\end{aligned}$$

Case 2: For $t_0 \in (0,1)$ and $\forall t \in (t_0, 1]$,

$$\begin{aligned}
&|u_k(t) - u_k(t_0)| \\
\le\ & \left| \begin{array}{l} \int_0^t \frac{(t-s)^{\alpha_k-1} s^{-\delta_k}}{\Gamma(\alpha_k)} s^{\delta_k} F_k(s) ds \\ - \int_0^{t_0} \frac{(t_0-s)^{\alpha_k-1} s^{-\delta_k}}{\Gamma(\alpha_k)} s^{\delta_k} F_k(s) ds \end{array} \right| \\
& + \sum_{j=1}^{n-2} \frac{\left|\omega_j^k\right|}{j!} \left(t^j - t_0^j\right) \\
& + \sum_{j=0}^{n-2} \frac{\left|\omega_j^k\right| \Gamma(n-\nu_k)\Gamma(n+\eta_k) \left(t^{n-1} - t_0^{n-1}\right)}{\Gamma(j+1+\eta_k)(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))} \\
& + \frac{G_k \Gamma(n-\nu_k)\Gamma(n+\eta_k) \left(t^{n-1} - t_0^{n-1}\right)}{(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))} \\
& \times \int_0^1 \left(\begin{array}{c} \frac{(1-s)^{\alpha_k-\nu_k-1} s^{-\delta_k}}{\Gamma(\alpha_k-\nu_k)} \\ + \frac{(1-s)^{\alpha_k+\eta_k-1} s^{-\delta_k}}{\Gamma(\alpha_k+\eta_k)} \end{array} \right) \left| s^{\delta_k} F_k(s) \right| ds \\
\le\ & \left(\begin{array}{c} \frac{G_k \Gamma(1-\delta_k) \left(t^{\alpha_k-\delta_k} - t_0^{\alpha_k-\delta_k}\right)}{\Gamma(\alpha_k+1-\delta_k)} + \sum_{j=1}^{n-2} \frac{\left|\omega_j^k\right| \left(t^j - t_0^j\right)}{j!} \\ + \sum_{j=0}^{n-2} \frac{\left|\omega_j^k\right| \Gamma(n-\nu_k)\Gamma(n+\eta_k) \left(t^{n-1} - t_0^{n-1}\right)}{\Gamma(j+1+\eta_k)(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))} \\ + \frac{G_k \Gamma(n-\nu_k)\Gamma(n+\eta_k) \left(t^{n-1} - t_0^{n-1}\right)}{(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))} \\ \times \left(\frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k-\nu_k+1-\delta_k)} + \frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k+\eta_k+1-\delta_k)} \right) \end{array} \right) \\
\to\ & 0, \text{ as } t \to t_0.
\end{aligned}$$

Case 3: For $t_0\ (0,1]$ and for all $t \in [0, t_0]$, the proof is the same as in case 2, we ometed it. This completes the proof. □

Lemma 2.2. *For $k = 1,2,...,n,\ n \in \mathbb{N}^* \setminus \{1\}$, let $n-1 < \alpha_k < n,\ i-1 < \alpha_k^i < i,\ i = 1,2,...,n-1$. Assume that $f_k : (0,1] \times \mathbb{R}^{n^2} \to \mathbb{R}$ are continuous, $\lim_{t \to 0^+} f_k(t,...) = \infty$, $0 < \delta_k < 1$, and $t^{\delta_k} f_k(t,...)$ are continuous on $[0,1] \times \mathbb{R}^{n^2}$. Then, for all $k = 1,2,...,n$ and $i = 1,2,...,n-1$, $D^{\alpha_k^i} T_k(u_1, u_2, ..., u_n)$ are* *continuous on* $[0,1]\times\mathbb{R}^{n^2}$.

$$
\begin{aligned}
&D^{\alpha_k^i}T_k\left(u_1,...,u_n\right)(t)=\\
&\int_0^t \frac{(t-s)^{\alpha_k-\alpha_k^i-1}}{\Gamma\left(\alpha_k-\alpha_k^i\right)} f_k\begin{pmatrix} s,u_1(s),\\ ...,u_n(s),\\ D^{\alpha_1^1}u_1(s),...,\\ D^{\alpha_1^{n-1}}u_1(s),\\ ...,D^{\alpha_n^1}u_n(s),\\ ...,D^{\alpha_n^{n-1}}u_n(s)\end{pmatrix} ds\\
&+\sum_{j=i}^{n-2}\frac{\omega_j^k}{\Gamma\left(j+1-\alpha_k^i\right)}t^{j-\alpha_k^i}\\
&-\sum_{j=0}^{n-2}\frac{\omega_j^k\Gamma(n-\nu_k)\Gamma(n+\eta_k)t^{n-1-\alpha_k^i}}{\Gamma(j+1+\eta_k)\Gamma\left(n-\alpha_k^i\right)(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}\\
&-\frac{\Gamma(n-\nu_k)\Gamma(n+\eta_k)t^{n-1-\alpha_k^i}}{\Gamma\left(n-\alpha_k^i\right)(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}\\
&\times\int_0^1\begin{pmatrix}\frac{(1-s)^{\alpha_k-\nu_k-1}}{\Gamma(\alpha_k-\nu_k)}+\\ \\ \frac{(1-s)^{\alpha_k+\eta_k-1}}{\Gamma(\alpha_k+\eta_k)}\end{pmatrix} f_k\begin{pmatrix} s,u_1(s),\\ ...,u_n(s),\\ D^{\alpha_1^1}u_1(s),...,\\ D^{\alpha_1^{n-1}}u_1(s),\\ ...,D^{\alpha_n^1}u_n(s),\\ ...,D^{\alpha_n^{n-1}}u_n(s)\end{pmatrix} ds,
\end{aligned}
\tag{2.1}
$$

where $i=1,2,...,n-2,$

and

$$
\begin{aligned}
&D^{\alpha_k^{n-1}}T_k\left(u_1,...,u_n\right)(t)=\\
&\int_0^t \frac{(t-s)^{\alpha_k-\alpha_k^{n-1}-1}}{\Gamma\left(\alpha_k-\alpha_k^{n-1}\right)} f_k\begin{pmatrix} s,u_1(s),\\ ...,u_n(s),\\ D^{\alpha_1^1}u_1(s),...,\\ D^{\alpha_1^{n-1}}u_1(s),\\ ...,D^{\alpha_n^1}u_n(s),\\ ...,D^{\alpha_n^{n-1}}u_n(s),\end{pmatrix} ds\\
&-\sum_{j=0}^{n-2}\frac{\omega_j^k\Gamma(n-\nu_k)\Gamma(n+\eta_k)t^{n-1-\alpha_k^{n-1}}}{\Gamma(j+1+\eta_k)\Gamma\left(n-\alpha_k^{n-1}\right)(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}\\
&-\frac{\Gamma(n-\nu_k)\Gamma(n+\eta_k)t^{n-1-\alpha_k^{n-1}}}{\Gamma\left(n-\alpha_k^{n-1}\right)(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}\\
&\times\int_0^1\begin{pmatrix}\frac{(1-s)^{\alpha_k-\nu_k-1}}{\Gamma(\alpha_k-\nu_k)}\\ \\ +\frac{(1-s)^{\alpha_k+\eta_k-1}}{\Gamma(\alpha_k+\eta_k)}\end{pmatrix} f_k\begin{pmatrix} s,u_1(s),\\ ...,u_n(s),\\ D^{\alpha_1^1}u_1(s),...,\\ D^{\alpha_1^{n-1}}u_1(s),\\ ...,D^{\alpha_n^1}u_n(s),\\ ...,D^{\alpha_n^{n-1}}u_n(s),\end{pmatrix} ds.
\end{aligned}
\tag{2.2}
$$

Proof. Let $(u_1,u_2,...,u_n)\in B$, so $u_k(t)\in C([0,1])$, and

$$D^{\alpha_k^i}u_k(t)\in C([0,1]);k=1,2,...,n,$$

where $i=1,2,...,n-1$, $n\in\mathbb{N}^*\setminus\{1\}$. Then, there exist n^2 nonnegative constants c_k,c_k^i, such that $|u_k(t)|\le c_k$, $\left|D^{\alpha_k^i}u_k(t)\right|\le c_k^i$, for each $t\in[0,1]$. Similarly, since $t^{\delta_k}f_k(t,...)$ are continuous on $[0,1]\times\mathbb{R}^{n^2}$, there exist $A_k>0$:

$$
A_k=\left\|t^{\delta_k}f_k\begin{pmatrix} t,u_1(t),...,\\ u_n(t),D^{\alpha_1^1}u_1(t),\\ ...,D^{\alpha_1^{n-1}}u_1(t),\\ D^{\alpha_2^1}u_2(t),...,\\ D^{\alpha_2^{n-1}}u_2(t),...,\\ D^{\alpha_n^1}u_n(t),...,\\ D^{\alpha_n^{n-1}}u_n(t)\end{pmatrix}\right\|_\infty,
$$

where $-c_k\le u_k\le c_k$, $-c_k^i\le D^{\alpha_k^i}u_k(t)\le c_k^i$. Then,

$$
\begin{aligned}
&\left|D^{\alpha_k^i}T_k\left(u_1,...,u_n\right)(t)\right|\le\\
&\frac{A_k}{\Gamma\left(\alpha_n-\alpha_k^i\right)}\int_0^t(t-s)^{\alpha_k-\alpha_k^i-1}s^{-\delta_k}ds\\
&+\sum_{j=i}^{n-2}\frac{\left|\omega_j^k\right|}{\Gamma\left(j+1-\alpha_k^i\right)}t^{j-\alpha_k^i}\\
&+\sum_{j=0}^{n-2}\frac{\left|\omega_j^k\right|\Gamma(n-\nu_k)\Gamma(n+\eta_k)t^{n-1-\alpha_k^i}}{\Gamma(j+1+\eta_k)\Gamma\left(n-\alpha_k^i\right)(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}\\
&+\frac{A_k\Gamma(n-\nu_k)\Gamma(n+\eta_k)t^{n-1-\alpha_k^i}}{\Gamma\left(n-\alpha_k^i\right)(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}\\
&\times\int_0^1\left(\frac{(1-s)^{\alpha_k-\nu_k-1}s^{-\delta_k}}{\Gamma(\alpha_k-\nu_k)}+\frac{(1-s)^{\alpha_k+\eta_k-1}s^{-\delta_k}}{\Gamma(\alpha_k+\eta_k)}\right)ds,
\end{aligned}
$$

where $i=1,2,...,n-2$.

Therefore,

$$
\left|D^{\alpha_k^i}T_k\left(u_1,...,u_n\right)(t)\right|\le
$$

$$
\begin{pmatrix}
\frac{A_k\Gamma(1-\delta_k)t^{\alpha_k-\alpha_k^i-\delta_k}}{\Gamma\left(\alpha_n-\alpha_k^i+1-\delta_k\right)}+\sum_{j=i}^{n-2}\frac{\left|\omega_j^k\right|}{\Gamma\left(j+1-\alpha_k^j\right)}t^{j-\alpha_k^i}\\
\\
+\sum_{j=0}^{n-2}\frac{\left|\omega_j^k\right|\Gamma(n-\nu_k)\Gamma(n+\eta_k)t^{n-1-\alpha_k^i}}{\Gamma(j+1+\eta_k)\Gamma\left(n-\alpha_k^i\right)(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}\\
\\
+\frac{A_k\Gamma(n-\nu_k)\Gamma(n+\eta_k)t^{n-1-\alpha_k^i}}{\Gamma\left(n-\alpha_k^i\right)(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}\\
\times\left(\frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k-\nu_k+1-\delta_k)}+\frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k+\eta_k+1-\delta_k)}\right)
\end{pmatrix},
\tag{2.3}
$$

where $i=1,2,...,n-2$.

And for $i=n-1$,

$$
\left|D^{\alpha_k^{n-1}}T_k\left(u_1,...,u_n\right)(t)\right|\le
$$

$$
\begin{pmatrix}
\frac{A_k\Gamma(1-\delta_k)t^{\alpha_k-\alpha_k^{n-1}-\delta_k}}{\Gamma\left(\alpha_n-\alpha_k^{n-1}+1-\delta_k\right)}\\
+\sum_{j=0}^{n-2}\frac{\left|\omega_j^k\right|\Gamma(n-\nu_k)\Gamma(n+\eta_k)t^{n-1-\alpha_k^{n-1}}}{\Gamma(j+1+\eta_k)\Gamma\left(n-\alpha_k^{n-1}\right)(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}\\
+\frac{A_k\Gamma(n-\nu_k)\Gamma(n+\eta_k)t^{n-1-\alpha_k^{n-1}}}{\Gamma\left(n-\alpha_k^{n-1}\right)(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}\\
\times\left(\frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k-\nu_k+1-\delta_k)}+\frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k+\eta_k+1-\delta_k)}\right)
\end{pmatrix}.
\tag{2.4}
$$

From the inequalities (2.3) and (2.4), we see that $t^{\alpha_k-\alpha_k^i-\delta_k}$, $t^{j-\alpha_k^i}$, and $t^{n-1-\alpha_k^i}$ are continuous on $[0,1]$. Hence, by the same method as in Lemma 2.1, we can show that for $n\in\mathbb{N}^*\setminus\{1\}$, $k=1,2,...,n$, and for all $i=1,2,...,n-1$, $D^{\alpha_k^i}T_k(u_1,...,u_n)(t)$ are continuous on $[0,1]$. □

Lemma 2.3. *Let $n-1<\alpha_k<n$, $n\in\mathbb{N}^*\setminus\{1\}$, $f_k:(0,1]\times\mathbb{R}^{n^2}\to\mathbb{R}$ are continuous, $\lim_{t\to 0^+} f_k(t,...)=\infty$, $0<\delta_k<1$, and $t^{\delta_k}f_k(t,...)$ are continuous on $[0,1]\times\mathbb{R}^{n^2}$. Then, $T:B\to B$ is completely continuous.*

Proof. Thanks to Lemma 2.1 and Lemma 2.2, we obtain $T:B\to B$. Let given the proof into three steps.

Step 1 : We will show that $T:B\to B$ is a continuous operator.

Let

$$\left(u_1^0,...,u_n^0\right)\in B:\left\|\left(u_1^0,...,u_n^0\right)\right\|_B=a_0,$$

and $(u_1,...,u_n)\in B$;

$$\left\|(u_1,...,u_n)-\left(u_1^0,...,u_n^0\right)\right\|_B<1.$$

Then,

$$\|(u_1,...,u_n)\|_B<1+a_0=a.$$

From the continuity of $t^{\delta_k}f_k(t,...)$ on $[0,1]\times\mathbb{R}^{n^2}$, we can see that $t^{\delta_k}f_k(t,...)$ are uniformly continuous on $[0,1]\times[-a,a]^{n^2}$.

Thus, $\forall t\in[0,1]$, $\forall\varepsilon>0$, there exist $\xi>0$ $(\xi<1)$;

$$\left|\begin{array}{l} t^{\delta_k}f_k\begin{pmatrix} t,u_1(t),...,u_n(t),\\ D^{\alpha_1^1}u_1(t),...,\\ D^{\alpha_1^{n-1}}u_1(t),D^{\alpha_2^1}u_2(t),\\ ...,D^{\alpha_2^{n-1}}u_2(t),\\ ...,D^{\alpha_n^1}u_n(t),\\ ...,D^{\alpha_n^{n-1}}u_n(t)\end{pmatrix}\\ -t^{\delta_k}f_k\begin{pmatrix} t,u_1^0(t),...,u_n^0(t),\\ D^{\alpha_1^1}u_1^0(t),...,\\ D^{\alpha_1^{n-1}}u_1^0(t),D^{\alpha_2^1}u_2^0(t),\\ ...,D^{\alpha_2^{n-1}}u_2^0(t),\\ ...,D^{\alpha_n^1}u_n^0(t),\\ ...,D^{\alpha_n^{n-1}}u_n^0(t)\end{pmatrix}\end{array}\right|<\varepsilon, \quad (2.5)$$

where $(u_1,...,u_n)\in B$ and $\left\|(u_1,...,u_n)-\left(u_1^0,...,u_n^0\right)\right\|_B<\xi$. Using (2.5), yields

$$\begin{aligned}
&\left\|T_k(u_1,...,u_n)-T_k\left(u_1^0,...,u_n^0\right)\right\|_\infty\\
&\le \frac{\varepsilon}{\Gamma(\alpha_k)}\sup_{t\in[0,1]}\int_0^t(t-s)^{\alpha_k-1}s^{-\delta_k}ds\\
&\quad+\frac{\varepsilon\Gamma(n-\nu_k)\Gamma(n+\eta_k)}{(n-1)!\,(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}\\
&\quad\times\sup_{t\in[0,1]}t^{n-1}\int_0^1\begin{pmatrix}\frac{(1-s)^{\alpha_k-\nu_k-1}s^{-\delta_k}}{\Gamma(\alpha_k-\nu_k)}\\ +\frac{(1-s)^{\alpha_k+\eta_k-1}s^{-\delta_k}}{\Gamma(\alpha_k+\eta_k)}\end{pmatrix}ds\\
&\le \frac{\varepsilon\Gamma(1-\delta_k)}{\Gamma(\alpha_k+1-\delta_k)}\sup_{t\in[0,1]}t^{\alpha_k-\delta_k}\\
&\quad+\frac{\varepsilon\Gamma(n-\nu_k)\Gamma(n+\eta_k)}{(n-1)!\,(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}\\
&\quad\times\begin{pmatrix}\frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k-\nu_k+1-\delta_k)}\\ +\frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k+\eta_k+1-\delta_k)}\end{pmatrix}.
\end{aligned}$$

For all $k=1,2,...,n$, we pose:

$$\begin{aligned}\Upsilon_k&:=\frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k+1-\delta_k)}+\frac{\Gamma(n-\nu_k)\Gamma(n+\eta_k)}{(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}\\ &\times\left(\frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k-\nu_k+1-\delta_k)}+\frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k+\eta_k+1-\delta_k)}\right).\end{aligned} \quad (2.6)$$

Therefore,

$$\left\|T_k(u_1,...,u_n)-T_k\left(u_1^0,...,u_n^0\right)\right\|_\infty\le\varepsilon\Upsilon_k. \quad (2.7)$$

By the same arguments, we get

$$\left\|D^{\alpha_k^i}\left(T_k(u_1,...,u_n)-T_k\left(u_1^0,...,u_n^0\right)\right)\right\|_\infty\le\varepsilon\Upsilon_k^i, \quad (2.8)$$

where $k=1,2,...,n$, $i=1,2,...,n-1$,

and

$$\begin{aligned}\Upsilon_k^i&:=\frac{\Gamma(1-\delta_k)}{\Gamma\left(\alpha_k-\alpha_k^i+1-\delta_k\right)}+\frac{\Gamma(n-\nu_k)\Gamma(n+\eta_k)}{\Gamma\left(n-\alpha_k^i\right)(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}\\ &\times\left(\frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k-\nu_k+1-\delta_k)}+\frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k+\eta_k+1-\delta_k)}\right).\end{aligned} \quad (2.9)$$

The inequalities (2.7) and (2.8) implies that

$$\left\|T(u_1,...,u_n)-T\left(u_1^0,...,u_n^0\right)\right\|_B\le\varepsilon\Upsilon,$$

$$\Upsilon=\max_{1\le k\le n}\left(\Upsilon_k,\Upsilon_k^i\right),\ 1\le i\le n-1.$$

So,

$$\left\|T(u_1,...,u_n)-T\left(u_1^0,...,u_n^0\right)\right\|_B\to 0$$

as

$$\left\|(u_1,...,u_n)-\left(u_1^0,...,u_n^0\right)\right\|_B\to 0.$$

That is $T:B\to B$ is continuous.

Step 2 : For $\lambda>0$, we consider

$$\Delta:=\left\{(u_1,...,u_n)\in B:\|(u_1,...,u_n)\|_B\le\lambda\right\}$$

and we show that $T(\Delta)$ is bounded.

Since $t^{\delta_k} f_k(t,...)$ are continuous on $[0,1]\times[-\lambda,\lambda]^{n^2}$, there exist $V_k>0$:

$$\left| t^{\delta_k} f_k \begin{pmatrix} t, u_1(t), ..., \\ u_n(t), D^{\alpha_1^1} u_1(t), \\ ..., D^{\alpha_1^{n-1}} u_1(t), \\ D^{\alpha_2^1} u_2(t), ..., \\ D^{\alpha_2^{n-1}} u_2(t), ..., \\ D^{\alpha_n^1} u_n(t), ..., \\ D^{\alpha_n^{n-1}} u_n(t) \end{pmatrix} \right| \le V_k, \tag{2.10}$$

$\forall t\in[0,1], \forall (u_1,...,u_n)\in\Delta$. Then, it follows (2.10) that

$$\begin{aligned} &\|T_k(u_1,...,u_n)\|_\infty \le V_k\Upsilon_k + \sum_{j=0}^{n-2}\frac{|\omega_j^k|}{j!} \\ &+\sum_{j=0}^{n-2}\frac{|\omega_j^k|\Gamma(n-\nu_k)\Gamma(n+\eta_k)}{\Gamma(j+1+\eta_k)(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}, \end{aligned}$$

$$\begin{aligned} &\left\|D^{\alpha_k^i} T_k(u_1,...,u_n)\right\|_\infty \le V_k\Upsilon_k^i \\ &+\sum_{j=i}^{n-2}\frac{|\omega_j^k|}{\Gamma(j+1-\alpha_k^i)} \\ &+\sum_{j=0}^{n-2}\frac{|\omega_j^k|\Gamma(n-\nu_k)\Gamma(n+\eta_k)}{\Gamma(j+1+\eta_k)\Gamma(n-\alpha_k^i)(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}, \\ &i=1,2,...,n-2, \end{aligned} \tag{2.11}$$

$$\begin{aligned} &\left\|D^{\alpha_k^{n-1}} T_k(u_1,...,u_n)\right\|_\infty \le V_k\Upsilon_k^{n-1} \\ &+\sum_{j=0}^{n-2}\frac{|\omega_j^k|\Gamma(n-\nu_k)\Gamma(n+\eta_k)}{\Gamma(j+1+\eta_k)\Gamma(n-\alpha_k^{n-1})(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}. \end{aligned}$$

Therefore,

$$\|T(u_1,...,u_n)\|_B \le$$

$$\max_{1\le k\le n}\begin{pmatrix} V_k\Upsilon_k+\sum_{j=0}^{n-2}\frac{|\omega_j^k|}{j!} \\ +\sum_{j=0}^{n-2}\frac{|\omega_j^k|\Gamma(n-\nu_k)\Gamma(n+\eta_k)}{\Gamma(j+1+\eta_k)(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}, \\ V_k\Upsilon_k^i+\sum_{j=i}^{n-2}\frac{|\omega_j^k|}{\Gamma(j+1-\alpha_k^i)} \\ +\sum_{j=0}^{n-2}\frac{|\omega_j^k|\Gamma(n-\nu_k)\Gamma(n+\eta_k)}{\Gamma(j+1+\eta_k)\Gamma(n-\alpha_k^i)(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}, \\ V_k\Upsilon_k^{n-1} \\ +\sum_{j=0}^{n-2}\frac{|\omega_j^k|\Gamma(n-\nu_k)\Gamma(n+\eta_k)}{\Gamma(j+1+\eta_k)\Gamma(n-\alpha_k^{n-1})(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))} \end{pmatrix}. \tag{2.12}$$

By (2.12), we state that $T(\Delta)$ is bounded.

Step 3 : We show that $T(\Delta)$ is equicontinuous.

So, for $(u_1,...,u_n)\in\Delta$, and $t_1,t_2\in[0,1]: t_1<t_2$, we obtain

$$\|T_k(u_1,...,u_n)(t_2)-T_k(u_1,...,u_n)(t_1)\|_\infty$$

$$\le\begin{pmatrix} \frac{V_k\Gamma(1-\delta_k)\left(t_2^{\alpha_k-\delta_k}-t_1^{\alpha_k-\delta_k}\right)}{\Gamma(\alpha_k+1-\delta_k)}+\sum_{j=0}^{n-2}\frac{|\omega_j^k|\left(t_2^j-t_1^j\right)}{j!} \\ +\sum_{j=0}^{n-2}\frac{|\omega_j^k|\Gamma(n-\nu_k)\Gamma(n+\eta_k)\left(t_2^{n-1}-t_1^{n-1}\right)}{\Gamma(j+1+\eta_k)(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))} \\ +\frac{\Gamma(n-\nu_k)\Gamma(n+\eta_k)\left(t_2^{n-1}-t_1^{n-1}\right)}{(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))} \\ \times\left(\frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k-\nu_k+1-\delta_k)}+\frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k+\eta_k+1-\delta_k)}\right) \end{pmatrix}, \tag{2.13}$$

where $k=1,2,...,n$,

$$\left\|D^{\alpha_k^i}T_k(u_1,...,u_n)(t_2)-D^{\alpha_k^i}T_k(u_1,...,u_n)(t_1)\right\|_\infty$$

$$\le\begin{pmatrix} \frac{V_k\Gamma(1-\delta_k)\left(t_2^{\alpha_k-\alpha_k^i-\delta_k}-t_1^{\alpha_k-\alpha_k^i-\delta_k}\right)}{\Gamma\left(\alpha_k-\alpha_k^i+1-\delta_k\right)} \\ +\sum_{j=i}^{n-2}\frac{|\omega_j^k|\left(t_2^{j-\alpha_k^i}-t_1^{j-\alpha_k^i}\right)}{\Gamma\left(j+1-\alpha_k^j\right)} \\ +\sum_{j=0}^{n-2}\frac{|\omega_j^k|\Gamma(n-\nu_k)\Gamma(n+\eta_k)\left(t_2^{n-1-\alpha_k^i}-t_1^{n-1-\alpha_k^i}\right)}{\Gamma(j+1+\eta_k)\Gamma\left(n-\alpha_k^i\right)(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))} \\ +\frac{\Gamma(n-\nu_k)\Gamma(n+\eta_k)\left(t_2^{n-1-\alpha_k^i}-t_1^{n-1-\alpha_k^i}\right)}{\Gamma\left(n-\alpha_k^i\right)(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))} \\ \times\left(\frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k-\nu_k+1-\delta_k)}+\frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k+\eta_k+1-\delta_k)}\right) \end{pmatrix}, \tag{2.14}$$

where $k=1,2,...,n$, and $i=1,2,...,n-2$,

$$\left\|D^{\alpha_k^{n-1}}T_k(u_1,...,u_n)(t_2)-D^{\alpha_k^{n-1}}T_k(u_1,...,u_n)(t_1)\right\|_\infty$$

$$\le\begin{pmatrix} \frac{V_k\Gamma(1-\delta_k)}{\Gamma\left(\alpha_k-\alpha_k^{n-1}+1-\delta_k\right)} \\ \times\left(t_2^{\alpha_k-\alpha_k^{n-1}-\delta_k}-t_1^{\alpha_k-\alpha_k^{n-1}-\delta_k}\right) \\ +\sum_{j=0}^{n-2}\frac{|\omega_j^k|\Gamma(n-\nu_k)\Gamma(n+\eta_k)}{\Gamma(j+1+\eta_k)\Gamma\left(n-\alpha_k^{n-1}\right)(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))} \\ \times\left(t_2^{n-1-\alpha_k^{n-1}}-t_1^{n-1-\alpha_k^{n-1}}\right) \\ +\frac{\Gamma(n-\nu_k)\Gamma(n+\eta_k)\left(t_2^{n-1-\alpha_k^{n-1}}-t_1^{n-1-\alpha_k^{n-1}}\right)}{\Gamma\left(n-\alpha_k^{n-1}\right)(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))} \\ \times\left(\frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k-\nu_k+1-\delta_k)}+\frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k+\eta_k+1-\delta_k)}\right) \end{pmatrix}, \tag{2.15}$$

where $k=1,2,...,n$, and $i=n-1$.

The right-hand sides of (2.13), (2.14), and (2.15), are independent of $(u_1,...,u_n)$ and tend to zero as $t_1\to t_2$. Indeed, $T(\Delta)$ is equicontinuous. Using Arzela-Ascoli theorem, we deduce that T is completely continuous. □

Theorem 2.4. *Assume that:*

(H_1) : *There exist nonnegative constants* $\left(\mu_j^k\right)_{j=1,\dots,n^2}^{k=1,\dots,n}$ *such that:*

$$\begin{aligned} & t^{\delta_k}\left|f_k\left(t,x_1,\dots,x_{n^2}\right)-f_k\left(t,y_1,\dots,y_{n^2}\right)\right| \\ \leq & \sum_{j=1}^{n^2}\mu_j^k\left|x_j-y_j\right|, \end{aligned}$$

$\forall t\in[0,1]$, *and* $\forall\left(x_1,\dots,x_{n^2}\right),\left(y_1,\dots,y_{n^2}\right)\in\mathbb{R}^{n^2}$.

(H_2) : $\Upsilon:=\max_{1\leq k\leq n}\Sigma_k\left(\Upsilon_k,\Upsilon_k^i\right)<1$, *where* $\Sigma_k=\sum_{j=1}^{n^2}\mu_j^k$.

Then, the problem (1.1) has a unique solution on $[0,1]$.

Proof. Our aim is to show that T is a contractive operator on B. Then, $\forall\left(u_1,\dots,u_n\right),\left(v_1,\dots,v_n\right)\in B$ and $\forall t\in[0,1]$, we get

$$\left\|T_k\left(u_1,\dots,u_n\right)-T_k\left(v_1,\dots,v_n\right)\right\|_\infty\leq$$

$$\sup_{t\in[0,1]}\int_0^t\frac{(t-s)^{\alpha_k-1}s^{-\delta_k}}{\Gamma(\alpha_k)}s^{\delta_k}\left|f_k\begin{pmatrix}s,u_1(s),\\ \dots,u_n(s),\\ D^{\alpha_1^1}u_1(s),\dots,\\ D^{\alpha_1^{n-1}}u_1(s),\dots,\\ D^{\alpha_n^1}u_n(s),\dots,\\ D^{\alpha_n^{n-1}}u_n(s)\end{pmatrix}-f_k\begin{pmatrix}s,v_1(s),\\ \dots,v_n(s),\\ D^{\alpha_1^1}v_1(s),\dots,\\ D^{\alpha_1^{n-1}}v_1(s),\dots,\\ D^{\alpha_n^1}v_n(s),\dots,\\ D^{\alpha_n^{n-1}}v_n(s)\end{pmatrix}\right|ds$$

$$+\frac{\Gamma(n-\nu_k)\Gamma(n+\eta_k)}{(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}\sup_{t\in[0,1]}t^{n-1}$$

$$\times\int_0^1\begin{pmatrix}\frac{(1-s)^{\alpha_k-\nu_k-1}}{\Gamma(\alpha_k-\nu_k)}\\ +\frac{(1-s)^{\alpha_k+\eta_k-1}}{\Gamma(\alpha_k+\eta_k)}\end{pmatrix}s^{-\delta_k}s^{\delta_k}\left|f_k\begin{pmatrix}s,u_1(s),\\ \dots,u_n(s),\\ D^{\alpha_1^1}u_1(s),\\ \dots,\\ D^{\alpha_1^{n-1}}u_1(s),\\ \dots,\\ D^{\alpha_n^1}u_n(s),\\ \dots,\\ D^{\alpha_n^{n-1}}u_n(s)\end{pmatrix}-f_k\begin{pmatrix}s,v_1(s),\\ \dots,\\ v_n(s),\\ D^{\alpha_1^1}v_1(s),\\ \dots,\\ D^{\alpha_1^{n-1}}v_1(s),\\ \dots,\\ D^{\alpha_n^1}v_n(s),\\ \dots,\\ D^{\alpha_n^{n-1}}v_n(s)\end{pmatrix}\right|ds.$$

Using the hypothesis (H_1), yields

$$\left\|T_k\left(u_1,\dots,u_n\right)-T_k\left(v_1,\dots,v_n\right)\right\|_\infty\leq$$

$$\begin{pmatrix}\frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k+1-\delta_k)}\sup_{t\in[0,1]}t^{\alpha_k-\delta_k}\\ +\frac{\Gamma(n-\nu_k)\Gamma(n+\eta_k)}{(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}\\ \times\left(\frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k-\nu_k+1-\delta_k)}+\frac{\Gamma(1-\delta_k)}{\Gamma(\alpha_k+\eta_k+1-\delta_k)}\right)\end{pmatrix}$$

$$\times\begin{pmatrix}\mu_1^k\left\|u_1-v_1\right\|_\infty+\dots\\ +\mu_n^k\left\|u_n-v_n\right\|_\infty\\ +\mu_{n+1}^k\left\|D^{\alpha_1^1}\left(u_1-v_1\right)\right\|_\infty+\dots\\ +\mu_{2n-1}^k\left\|D^{\alpha_1^{n-1}}\left(u_1-v_1\right)\right\|_\infty+\\ \dots+\mu_{2n}^k\left\|D^{\alpha_2^1}\left(u_2-v_2\right)\right\|_\infty\\ +\dots+\mu_{3n-2}^k\left\|D^{\alpha_2^{n-1}}\left(u_2-v_2\right)\right\|_\infty\\ +\dots\\ +\mu_{n+(n-1)^2+1}^k\left\|D^{\alpha_n^1}\left(u_n-v_n\right)\right\|_\infty\\ +\dots+\mu_{n^2}^k\left\|D^{\alpha_n^{n-1}}\left(u_n-v_n\right)\right\|_\infty\end{pmatrix}.$$

Then,

$$\begin{aligned} & \left\|T_k\left(u_1,\dots,u_n\right)-T_k\left(v_1,\dots,v_n\right)\right\|_\infty \\ \leq & \sum_{j=1}^{n^2}\mu_j^k\Upsilon_k\max\begin{pmatrix}\left\|u_1-v_1\right\|_\infty,\dots,\\ \left\|u_n-v_n\right\|_\infty,\\ \left\|D^{\alpha_1^1}\left(u_1-v_1\right)\right\|_\infty,\\ \dots,\left\|D^{\alpha_1^{n-1}}\left(u_1-v_1\right)\right\|_\infty,\\ \dots,\left\|D^{\alpha_n^1}\left(u_n-v_n\right)\right\|_\infty,\\ \dots,\left\|D^{\alpha_n^{n-1}}\left(u_n-v_n\right)\right\|_\infty\end{pmatrix}\end{aligned}$$

So,

$$\begin{aligned} & \left\|T_k\left(u_1,\dots,u_n\right)-T_k\left(v_1,\dots,v_n\right)\right\|_\infty \\ & \leq\Sigma_k\Upsilon_k\left\|u_1-v_1,\dots,u_n-v_n\right\|_B. \end{aligned}\tag{2.16}$$

Similarly on the other hand, we get

$$\begin{aligned} & \left\|D^{\alpha_k^i}T_k\left(u_1,\dots,u_n\right)-D^{\alpha_k^i}T_k\left(v_1,\dots,v_n\right)\right\|_\infty \\ \leq & \Sigma_k\Upsilon_k^i\left\|u_1-v_1,\dots,u_n-v_n\right\|_B, \end{aligned}\tag{2.17}$$

where $k=1,2,\dots,n$, and $i=1,2,\dots,n-1$.

Thanks to (2.16) and (2.17), we obtain

$$\begin{aligned} & \left\|T\left(u_1,\dots,u_n\right)-T\left(v_1,\dots,v_n\right)\right\|_B \\ \leq & \max_{1\leq k\leq n}\Sigma_k\left(\Upsilon_k,\Upsilon_k^i\right)\left\|u_1-v_1,\dots,u_n-v_n\right\|_B. \end{aligned}$$

By the hypothesis (H_2), we state that T is a contractive operator. Using Banach fixed point theorem, we conclude that T has a fixed point which is the unique solution of the problem (1.1). This completes the proof. □

Example 2.5. *Consider the singular fractional coupled system:*

$$
\begin{cases}
D^{\frac{11}{5}}u_1(t) = \\
\dfrac{\left[\begin{array}{c}\cos\left(u_1(t)+u_2(t)+u_3(t)\right) \\ +\sin\left(\begin{array}{c} D^{\frac{3}{5}}u_1(t)+D^{\frac{8}{5}}u_1(t) \\ +D^{\frac{2}{3}}u_2(t)+D^{\frac{5}{3}}u_2(t) \\ +D^{\frac{1}{2}}u_3(t)+D^{\frac{3}{2}}u_3(t)\end{array}\right)\end{array}\right]}{126\pi t^{\frac{2}{5}}}, \\
0<t\le 1, \\
D^{\frac{8}{3}}u_2(t) = \\
\dfrac{\left|\begin{array}{c} u_1(t)+u_2(t)+u_3(t) \\ +D^{\frac{3}{5}}u_1(t)+D^{\frac{8}{5}}u_1(t) \\ +D^{\frac{2}{3}}u_2(t)+D^{\frac{5}{3}}u_2(t) \\ +D^{\frac{1}{2}}u_3(t)+D^{\frac{3}{2}}u_3(t)\end{array}\right|}{135\pi t^{\frac{1}{3}}\left(1+\left|\begin{array}{c} u_1(t)+u_2(t)+u_3(t) \\ +D^{\frac{3}{5}}u_1(t)+D^{\frac{8}{5}}u_1(t) \\ +D^{\frac{2}{3}}u_2(t)+D^{\frac{4}{3}}u_2(t) \\ +D^{\frac{1}{2}}u_3(t)+D^{\frac{3}{2}}u_3(t)\end{array}\right|\right)}, \\
0<t\le 1, \\
D^{\frac{5}{2}}u_3(t) = \dfrac{\sin\left(\begin{array}{c} u_1(t)+u_2(t)+u_3(t) \\ +D^{\frac{3}{5}}u_1(t)+D^{\frac{8}{5}}u_1(t) \\ +D^{\frac{2}{3}}u_2(t)+D^{\frac{5}{3}}u_2(t) \\ +D^{\frac{1}{2}}u_3(t)+D^{\frac{3}{2}}u_3(t)\end{array}\right)}{90\pi t^{\frac{1}{4}}} \\
0<t\le 1, \\
u_1(0)=\sqrt{3},\ u_1^{'}(0)=-1, \\
D^{\frac{6}{5}}u_1(1)=J^{\frac{33}{5}}u_1(1), \\
u_2(0)=2\sqrt{3},\ u_2^{'}(0)=1, \\
D^{\frac{4}{3}}u_2(1)=J^{\frac{17}{3}}u_2(1), \\
u_3(0)=\sqrt{2},\ u_3^{'}(0)=2\sqrt{5}, \\
D^{\frac{7}{4}}u_3(1)=J^{\frac{25}{2}}u_3(1).
\end{cases}
\tag{2.18}
$$

Then, $n=3, \alpha_1=\frac{11}{5}, \alpha_2=\frac{8}{3}, \alpha_3=\frac{5}{2}, \alpha_1^1=\frac{3}{5}, \alpha_1^2=\frac{8}{5}, \alpha_2^1=\frac{2}{3}, \alpha_2^2=\frac{5}{3}$,

$\alpha_3^1=\frac{1}{2}, \alpha_3^2=\frac{3}{2}, \nu_1=\frac{6}{5}, \nu_2=\frac{4}{3}, \nu_3=\frac{7}{4}, \eta_1=\frac{33}{5}, \eta_2=\frac{17}{3}, \eta_3=\frac{25}{2}$.

Taking: $\delta_1=\frac{4}{5}$, $\delta_2=\frac{2}{3}$, $\delta_3=\frac{1}{2}$, $t\in[0,1]$

and $(x_1,...,x_9), (y_1,...,y_9)\in\mathbb{R}^9$, *we obtain:*

$$
t^{\frac{4}{5}}\left|\begin{array}{c} f_1(t,x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9) \\ -f_1(t,y_1,y_2,y_3,y_4,y_5,y_6,y_7,y_8,y_9)\end{array}\right|
$$

$$
\le \frac{t^{\frac{2}{5}}}{126\pi}\left(\begin{array}{c} |x_1-y_1|+|x_2-y_2| \\ +|x_3-y_3|+|x_4-y_4| \\ +|x_5-y_5|+|x_6-y_6| \\ +|x_7-y_7|+|x_8-y_8| \\ +|x_9-y_9|\end{array}\right),
$$

$$
t^{\frac{2}{3}}\left|\begin{array}{c} f_2(t,x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9) \\ -f_2(t,y_1,y_2,y_3,y_4,y_5,y_6,y_7,y_8,y_9)\end{array}\right|
$$

$$
\le \frac{t^{\frac{1}{3}}}{135\pi}\left(\begin{array}{c} |x_1-y_1|+|x_2-y_2| \\ +|x_3-y_3|+|x_4-y_4| \\ +|x_5-y_5|+|x_6-y_6| \\ +|x_7-y_7|+|x_8-y_8| \\ +|x_9-y_9|\end{array}\right),
$$

and

$$
t^{\frac{1}{2}}\left|\begin{array}{c} f_3(t,x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9) \\ -f_3(t,y_1,y_2,y_3,y_4,y_5,y_6,y_7,y_8,y_9)\end{array}\right|
$$

$$
\le \frac{t^{\frac{1}{4}}}{90\pi}\left(\begin{array}{c} |x_1-y_1|+|x_2-y_2| \\ +|x_3-y_3|+|x_4-y_4| \\ +|x_5-y_5|+|x_6-y_6| \\ +|x_7-y_7|+|x_8-y_8| \\ +|x_9-y_9|\end{array}\right).
$$

Indeed,

$$
\begin{aligned}
\left(\mu_j^1\right)_{j=1,2,...,9} &= \frac{1}{126\pi}, \left(\mu_j^2\right)_{j=1,2,...,9} = \frac{1}{135\pi}, \\
\left(\mu_j^3\right)_{j=1,2,...,9} &= \frac{1}{90\pi},
\end{aligned}
$$

$$
\Sigma_1=\frac{1}{14\pi},\ \Sigma_2=\frac{1}{15\pi},\ \Sigma_3=\frac{1}{10\pi}.
$$

$$
\begin{aligned}
\Upsilon_1 &= 6.0242,\ \Upsilon_1^1=8.6780,\ \Upsilon_1^2=9.1917, \\
\Upsilon_2 &= 2.6789,\ \Upsilon_2^1=4.5001,\ \Upsilon_2^2=6.0000, \\
\Upsilon_3 &= 1.7725,\ \Upsilon_3^1=2.6666,\ \Upsilon_3^2=4.0001.
\end{aligned}
$$

$$
\begin{aligned}
\Sigma_1\Upsilon_1 &= 0.1370, \\
\Sigma_1\Upsilon_1^1 &= 0.1973,\ \Sigma_1\Upsilon_1^2=0.2090, \\
\Sigma_2\Upsilon_2 &= 0.0568,\ \Sigma_2\Upsilon_2^1=0.0955, \\
\Sigma_2\Upsilon_2^2 &= 0.1273, \\
\Sigma_3\Upsilon_3 &= 0.0564,\ \Sigma_3\Upsilon_3^1=0.0849, \\
\Sigma_3\Upsilon_3^2 &= 0.1273.
\end{aligned}
$$

Thus, (2.18) has a unique solution.

Theorem 2.6. *Let* $n-1<\alpha_k<n$, $n\in\mathbb{N}^*\setminus\{1\}$, $f_k:(0,1]\times\mathbb{R}^{n^2}\to\mathbb{R}$ *are continuous,* $\lim_{t\to 0^+} f_k(t,...)=\infty$, $0<\delta_k<1$, $k=1,2,...,n$, *and* $t^{\delta_k}f_k(t,...)$ *are continuous on* $[0,1]\times\mathbb{R}^{n^2}$. *Then, problem (1.1) has at least one solution on* $[0,1]$.

Proof. Let

$$
P_k=\sup_{t\in[0,1]}\left|t^{\delta_k}f_k\left(\begin{array}{c} t,u_1(t),...,u_n(t), \\ D^{\alpha_1^1}u_1(t),..., \\ D^{\alpha_1^{n-1}}u_1(t),D^{\alpha_2^1}u_2(t), \\ ...,D^{\alpha_2^{n-1}}u_2(t), \\ ...,D^{\alpha_n^1}u_n(t), \\ ...,D^{\alpha_n^{n-1}}u_n(t)\end{array}\right)\right|, \tag{2.19}
$$

and

$$
\Omega:=\{(u_1,...,u_n)\in B:\|(u_1,...,u_n)\|_B\le r\},
$$

such that

$$r = \max_{\substack{1 \le k \le n \\ 1 \le i \le n-1}} \left(\begin{array}{l} P_k \Upsilon_k + \sum_{j=0}^{n-2} \frac{\left|\omega_j^k\right|}{j!} \\ + \sum_{j=0}^{n-2} \frac{\left|\omega_j^k\right| \Gamma(n-\nu_k)\Gamma(n+\eta_k)}{[\Gamma(j+1+\eta_k)(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))]}, \\ P_k \Upsilon_k^i + \sum_{j=i}^{n-2} \frac{\left|\omega_j^k\right|}{\Gamma\left(j+1-\alpha_k^i\right)} \\ + \sum_{j=0}^{n-2} \frac{\left|\omega_j^k\right| \Gamma(n-\nu_k)\Gamma(n+\eta_k)}{\left[\Gamma(j+1+\eta_k)\Gamma\left(n-\alpha_k^i\right)(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))\right]}, \\ P_k \Upsilon_k^{n-1} \\ + \sum_{j=0}^{n-2} \frac{\left|\omega_j^k\right| \Gamma(n-\nu_k)\Gamma(n+\eta_k)}{\left[\Gamma(j+1+\eta_k)\Gamma\left(n-\alpha_k^{n-1}\right)(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))\right]} \end{array} \right). \tag{2.20}$$

To show that $T : \Omega \to \Omega$, let $t \in [0,1]$, and $(u_1, ..., u_n) \in \Omega$.

Then, taking into account (2.12) and thanks to (2.19), we obtain

$$\|T(u_1, ..., u_n)\|_B \le r. \tag{2.21}$$

Furthermore, from Lemma 2.1 and Lemma 2.2, we get

$$T_k(u_1, ..., u_n), D^{\alpha_k^i} T_1(u, v, w) \in C([0,1]).$$

So, we deduce that $T : \Omega \to \Omega$.

On the other hand, Lemma 2.3 show that T is a completely continuous operator. Using Lemma 1.1, we deduce that problem (1.1) has at least one solution on $[0,1]$. This completes the proof. □

Example 2.7. *Consider the singular fractional system*

$$\left\{ \begin{array}{l} D^{\frac{15}{4}} u_1(t) = \\ \dfrac{\left[\begin{array}{l} \sin\left(\begin{array}{l} u_1(t) + D^{\frac{1}{3}} u_1(t) \\ + D^{\frac{5}{3}} u_1(t) + D^{\frac{8}{3}} u_1(t) \end{array} \right) \\ + \sin\left(\begin{array}{l} u_2(t) + D^{\frac{2}{5}} u_2(t) \\ + D^{\frac{5}{4}} u_2(t) + D^{\frac{5}{2}} u_2(t) \end{array} \right) \end{array} \right]}{\pi e^t t^{\frac{1}{4}} \left[\begin{array}{l} \cos\left(\begin{array}{l} u_3(t) + D^{\frac{1}{4}} u_3(t) \\ + D^{\frac{3}{2}} u_3(t) + D^{\frac{9}{4}} u_3(t) \end{array} \right) \\ \times \cos\left(\begin{array}{l} u_4(t) + D^{\frac{1}{9}} u_4(t) \\ + D^{\frac{5}{3}} u_4(t) + D^{\frac{15}{7}} u_4(t) \end{array} \right) \end{array} \right]}, \\ D^{\frac{10}{3}} u_2(t) = \\ \dfrac{\cos\left(\begin{array}{l} u_1(t) D^{\frac{1}{3}} u_1(t) D^{\frac{5}{3}} u_1(t) D^{\frac{8}{3}} u_1(t) \\ + u_3(t) D^{\frac{1}{4}} u_3(t) D^{\frac{3}{2}} u_3(t) D^{\frac{9}{4}} u_3(t) \end{array} \right)}{t^{\frac{4}{9}} \left(\pi e - \sin\left(\begin{array}{l} u_2(t) D^{\frac{2}{5}} u_2(t) \\ \times D^{\frac{5}{4}} u_2(t) D^{\frac{5}{2}} u_2(t) \\ - u_4(t) D^{\frac{1}{9}} u_4(t) \\ \times D^{\frac{5}{3}} u_4(t) D^{\frac{15}{7}} u_4(t) \end{array} \right) \right)}, \\ 0 < t \le 1, \\ D^{\frac{7}{2}} u_3(t) = \\ \dfrac{\left[\begin{array}{l} \sin\left(\begin{array}{l} u_1(t) + D^{\frac{1}{3}} u_1(t) \\ + D^{\frac{5}{3}} u_1(t) + D^{\frac{8}{3}} u_1(t) \end{array} \right) \\ \times \cos\left(\begin{array}{l} u_3(t) + D^{\frac{1}{4}} u_3(t) \\ + D^{\frac{3}{2}} u_3(t) + D^{\frac{9}{4}} u_3(t) \end{array} \right) \end{array} \right]}{t^{\frac{1}{2}} \left[\begin{array}{l} \pi + \sin\left(\begin{array}{l} u_2(t) + D^{\frac{2}{5}} u_2(t) \\ + D^{\frac{5}{4}} u_2(t) + D^{\frac{5}{2}} u_2(t) \end{array} \right) \\ \times \sin\left(\begin{array}{l} u_4(t) + D^{\frac{1}{9}} u_4(t) \\ + D^{\frac{5}{3}} u_4(t) + D^{\frac{15}{7}} u_4(t) \end{array} \right) \end{array} \right]}, \\ D^{\frac{17}{5}} u_4(t) \\ = \dfrac{e^t \sin\left(\begin{array}{l} u_3(t) D^{\frac{1}{4}} u_3(t) D^{\frac{3}{2}} u_3(t) D^{\frac{9}{4}} u_3(t) \\ + u_4(t) D^{\frac{1}{9}} u_4(t) \\ \times D^{\frac{5}{3}} u_4(t) D^{\frac{15}{7}} u_4(t) \end{array} \right)}{t^{\frac{2}{5}} \left(e + \sin\left(\begin{array}{l} u_1(t) D^{\frac{1}{3}} u_1(t) \\ \times D^{\frac{5}{3}} u_1(t) D^{\frac{8}{3}} u_1(t) \\ - u_2(t) D^{\frac{2}{5}} u_2(t) \\ \times D^{\frac{5}{4}} u_2(t) D^{\frac{5}{2}} u_2(t) \end{array} \right) \right)}, \\ u_1(0) = \sqrt[3]{2},\ u_1'(0) = 1,\ u_1''(0) = \sqrt{3}, \\ D^{\frac{5}{2}} u_1(1) = J^{\frac{10}{9}} u_1(1),\ u_2(0) = 5\sqrt{3}, \\ u_2'(0) = \pi, u_2''(0) = 1, D^{\frac{11}{5}} u_2(1) = J^{\frac{43}{7}} u_2(1), \\ u_3(0) = \sqrt{2}, u_3'(0) = -1,\ u_3''(0) = \sqrt{3}, \\ D^{\frac{7}{3}} u_3(1) = J^{\frac{11}{3}} u_3(1), u_4(0) = \sqrt{2}, u_4'(0) = -1, \\ u_4''(0) = \sqrt{3}, D^{\frac{13}{6}} u_4(1) = J^{\frac{17}{4}} u_4(1). \end{array} \right. \tag{2.22}$$

We have: $n = 4$, $\alpha_1 = \frac{15}{4}$, $\alpha_2 = \frac{10}{3}$, $\alpha_3 = \frac{7}{2}$, $\alpha_4 = \frac{17}{5}$, $\alpha_1^1 = \frac{1}{3}$, $\alpha_1^2 = \frac{5}{3}$,
$\alpha_1^3 = \frac{8}{3}$, $\alpha_2^1 = \frac{2}{5}$, $\alpha_2^2 = \frac{5}{4}$, $\alpha_2^3 = \frac{5}{2}$, $\alpha_3^1 = \frac{1}{4}$, $\alpha_3^2 = \frac{3}{2}$, $\alpha_3^3 =$

$\frac{9}{4}$,

$\alpha_4^1 = \frac{1}{9}$, $\alpha_4^2 = \frac{5}{3}$, $\alpha_4^3 = \frac{15}{7}$, $\nu_1 = \frac{5}{2}$, $\nu_2 = \frac{11}{5}$, $\nu_3 = \frac{7}{3}$, $\nu_4 = \frac{13}{6}$,

$\eta_1 = \frac{10}{9}$, $\eta_2 = \frac{43}{7}$, $\eta_3 =, \frac{11}{3}$ $\eta_4 = \frac{17}{4}$.

We take $\delta_1 = \frac{1}{2}$, $\delta_2 = \frac{5}{9}$, $\delta_3 = \frac{3}{4}$, and $\delta_4 = \frac{3}{5}$.

Then, (2.22) has at least one solution.

3. Ulam-Hyers Stability

The present section is build to define and discuss the Ulam-Hyers stability and the generalized Ulam-Hyers stability for the singular fractional coupled system (1.1).

Definition 3.1. *The system (1.1) is Ulam-Hyers stable, if there exists $\varphi > 0$, such that for all $(\varepsilon_1, ..., \varepsilon_n) > 0$, and for all solution $(u_1, ..., u_n) \in B$ of*

$$\begin{cases} \left| \begin{array}{c} D^{\alpha_1} u_1(t) \\ -f_1 \begin{pmatrix} t, u_1(t), ..., u_n(t), \\ D^{\alpha_1^1} u_1(t), ..., D^{\alpha_1^{n-1}} u_1(t), \\ D^{\alpha_2^1} u_2(t), ..., D^{\alpha_2^{n-1}} u_2(t), \\ ..., \\ D^{\alpha_n^1} u_n(t), ..., D^{\alpha_n^{n-1}} u_n(t) \end{pmatrix} \end{array} \right| < \varepsilon_1, \\ 0 < t \leq 1, \\ \vdots \\ \left| \begin{array}{c} D^{\alpha_n} u_n(t) \\ -f_n \begin{pmatrix} t, u_1(t), ..., u_n(t), \\ D^{\alpha_1^1} u_1(t), ..., D^{\alpha_1^{n-1}} u_1(t), \\ D^{\alpha_2^1} u_2(t), ..., D^{\alpha_2^{n-1}} u_2(t), \\ ..., \\ D^{\alpha_n^1} u_n(t), ..., D^{\alpha_n^{n-1}} u_n(t) \end{pmatrix} \end{array} \right| < \varepsilon_n, \\ 0 < t \leq 1, \end{cases} \tag{3.1}$$

there exists $(w_1, ..., w_n) \in B$ of

$$\begin{cases} D^{\alpha_1} w_1(t) \\ = f_1 \begin{pmatrix} t, w_1(t), ..., w_n(t), \\ D^{\alpha_1^1} w_1(t), ..., D^{\alpha_1^{n-1}} w_1(t), \\ D^{\alpha_2^1} w_2(t), ..., D^{\alpha_2^{n-1}} w_2(t), ..., \\ D^{\alpha_n^1} w_n(t), ..., D^{\alpha_n^{n-1}} w_n(t) \end{pmatrix}, \\ 0 < t \leq 1, \\ \vdots \\ D^{\alpha_n} w_n(t) \\ = f_n \begin{pmatrix} t, w_1(t), ..., w_n(t), \\ D^{\alpha_1^1} w_1(t), ..., D^{\alpha_1^{n-1}} w_1(t), \\ D^{\alpha_2^1} w_2(t), ..., D^{\alpha_2^{n-1}} w_2(t), ..., \\ D^{\alpha_n^1} w_n(t), ..., D^{\alpha_n^{n-1}} w_n(t) \end{pmatrix}, \\ 0 < t \leq 1, \\ n-1 < \alpha_k < n, \; k = 1, 2, ..., n, \\ i-1 < \alpha_k^i < i, \; i = 1, 2, ..., n-1, \\ w_k^{(j)}(0) = \omega_j^k, \; D^{\nu_k} w_k(1) + J^{\eta_k} w_k(1) = 0, \\ n-2 < \nu_k < n-1, \; \eta_k > 0, \end{cases} \tag{3.2}$$

such that

$$\|(u_1 - w_1, ..., u_n - w_n)\|_B < \varphi \varepsilon, \; \varepsilon > 0.$$

Definition 3.2. *The coupled system (1.1) is generalized Ulam-Hyers stable if there exist $\psi \in C(\mathbb{R}^+, \mathbb{R}^+)$, such that for all $\varepsilon > 0$, and for each solution $(u_1, ..., u_n) \in B$ of (3.1), there exists $(w_1, ..., w_n) \in B$ of (3.2), where*

$$\|(u_1 - w_1, ..., u_n - w_n)\|_B < \psi(\varepsilon), \varepsilon > 0.$$

Theorem 3.3. *For $n-1 < \alpha_k < n$, $n \in \mathbb{N}^* \setminus \{1\}$, and $0 < \delta_k < 1$, $k = 1, 2, ..., n$, let the following assumptions hold*

(E_1) : $f_k : (0,1] \times \mathbb{R}^{n^2} \to \mathbb{R}$ are continuous, $\lim_{t \to 0^+} f_k(t, ...) = \infty$, and $t^{\delta_k} f_k(t, ...)$ are continuous on $[0,1] \times \mathbb{R}^{n^2}$.

(E_2) : The following inequality holds:

$$\left\| t^{\delta_k} D^{\alpha_k} u_k \right\|_\infty \geq$$

$$\max_{1 \leq k \leq n} \begin{pmatrix} P_k \Upsilon_k + \sum_{j=0}^{n-2} \frac{\left|\omega_j^k\right|}{j!} \\ + \sum_{j=0}^{n-2} \frac{\left|\omega_j^k\right| \Gamma(n-\nu_k) \Gamma(n+\eta_k)}{\Gamma(j+1+\eta_k)(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}, \\ P_k \Upsilon_k^i + \sum_{j=i}^{n-2} \frac{\left|\omega_j^k\right|}{\Gamma\left(j+1-\alpha_k^i\right)} \\ + \sum_{j=0}^{n-2} \frac{\left|\omega_j^k\right| \Gamma(n-\nu_k) \Gamma(n+\eta_k)}{\Gamma(j+1+\eta_k) \Gamma\left(n-\alpha_k^i\right)(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}, \\ P_k \Upsilon_k^{n-1} \\ + \sum_{j=0}^{n-2} \frac{\left|\omega_j^k\right| \Gamma(n-\nu_k) \Gamma(n+\eta_k)}{\Gamma(j+1+\eta_k) \Gamma\left(n-\alpha_k^{n-1}\right)(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))} \end{pmatrix}.$$

where $1 \leq i \leq n-1$.

(E_3) : All the hypotheses of Theorem 2.4 hold.

(E_4) : $\max_{1 \leq k \leq n} \Sigma_k < 1$.

Then, the singular fractional system (1.1) is generalized Ulam-Hyers stable in B.

Proof. Thanks to (E_1), we receive (2.21) and we can write

$$\|(u_1, ..., u_n)\|_B$$

$$\leq \max_{1 \leq k \leq n} \begin{pmatrix} P_k \Upsilon_k + \sum_{j=0}^{n-2} \frac{\left|\omega_j^k\right|}{j!} + \\ \sum_{j=0}^{n-2} \frac{\left|\omega_j^k\right| \Gamma(n-\nu_k) \Gamma(n+\eta_k)}{\Gamma(j+1+\eta_k)(n-1)!(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}, \\ P_k \Upsilon_k^i + \sum_{j=i}^{n-2} \frac{\left|\omega_j^k\right|}{\Gamma\left(j+1-\alpha_k^i\right)} + \\ \sum_{j=0}^{n-2} \frac{\left|\omega_j^k\right| \Gamma(n-\nu_k) \Gamma(n+\eta_k)}{\Gamma(j+1+\eta_k) \Gamma\left(n-\alpha_k^i\right)(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))}, \\ P_k \Upsilon_k^{n-1} + \\ \sum_{j=0}^{n-2} \frac{\left|\omega_j^k\right| \Gamma(n-\nu_k) \Gamma(n+\eta_k)}{\Gamma(j+1+\eta_k) \Gamma\left(n-\alpha_k^{n-1}\right)(\Gamma(n-\nu_k)+\Gamma(n+\eta_k))} \end{pmatrix}. \tag{3.3}$$

where $1 \leq i \leq n-1$, and $(u_1,...,u_n) \in B$ solution of (3.1). Then, it follows (E_2) and (3.3), that

$$\|(u_1,...,u_n)\|_B \leq \max_{1\leq k\leq n} \left\| t^{\delta_k} D^{\alpha_k} u_k \right\|_\infty. \tag{3.4}$$

By (E_3), $(w_1,...,w_n) \in B$ is a solution of (3.2).

Therefore, (3.4) yields

$$\begin{aligned} &\|(u_1-w_1,...,u_n-w_n)\|_B \\ \leq\ & \max_{1\leq k\leq n} \left\| t^{\delta_k} \left(D^{\alpha_k} u_k - D^{\alpha_k} w_k \right) \right\|_\infty \qquad (3.5) \\ \leq\ & \max_{1\leq k\leq n} \left\| \begin{array}{l} t^{\delta_k} \left(\begin{array}{l} D^{\alpha_k} u_k \\ -f_k \left(\begin{array}{l} t, u_1(t),...,u_n(t), \\ D^{\alpha_1^1} u_1(t),..., \\ D^{\alpha_1^{n-1}} u_1(t), \\ D^{\alpha_2^1} u_2(t),..., \\ D^{\alpha_2^{n-1}} u_2(t),..., \\ ..., D^{\alpha_n^1} u_n(t), \\ ..., D^{\alpha_n^{n-1}} u_n(t) \end{array} \right) \end{array} \right) \\ -t^{\delta_k} \left(\begin{array}{l} D^{\alpha_k} w_k \\ -f_k \left(\begin{array}{l} t, w_1(t),..., \\ w_n(t), D^{\alpha_1^1} w_1(t), \\ ..., D^{\alpha_1^{n-1}} w_1(t), \\ D^{\alpha_2^1} w_2(t),..., \\ D^{\alpha_2^{n-1}} w_2(t),..., \\ D^{\alpha_n^1} w_n(t),..., \\ D^{\alpha_n^{n-1}} w_n(t) \end{array} \right) \end{array} \right) \\ +t^{\delta_k} \left(\begin{array}{l} f_k \left(\begin{array}{l} t, u_1(t),...,u_n(t), \\ D^{\alpha_1^1} u_1(t),..., \\ D^{\alpha_1^{n-1}} u_1(t), \\ D^{\alpha_2^1} u_2(t),..., \\ D^{\alpha_2^{n-1}} u_2(t),..., \\ D^{\alpha_n^1} u_n(t),..., \\ D^{\alpha_n^{n-1}} u_n(t) \end{array} \right) \\ -f_k \left(\begin{array}{l} t, w_1(t),...,w_n(t), \\ D^{\alpha_1^1} w_1(t),..., \\ D^{\alpha_1^{n-1}} w_1(t), \\ D^{\alpha_2^1} w_2(t),..., \\ D^{\alpha_2^{n-1}} w_2(t),..., \\ D^{\alpha_n^1} w_n(t),..., \\ D^{\alpha_n^{n-1}} w_n(t) \end{array} \right) \end{array} \right) \end{array} \right\|_\infty . \end{aligned}$$

Then,

$$\max \|(u_1-w_1,...,u_n-w_n)\|_B \leq \left(\begin{array}{l} \left\| t^{\delta_k} \right\|_\infty \\ \times \left\| \begin{array}{l} D^{\alpha_k} u_k \\ -f_k \left(\begin{array}{l} t, u_1(t),...,u_n(t), \\ D^{\alpha_1^1} u_1(t),..., \\ D^{\alpha_1^{n-1}} u_1(t), \\ D^{\alpha_2^1} u_2(t),..., \\ D^{\alpha_2^{n-1}} u_2(t), \\ ..., D^{\alpha_n^1} u_n(t),..., \\ D^{\alpha_n^{n-1}} u_n(t) \end{array} \right) \end{array} \right\|_\infty \\ + \left\| t^{\delta_k} \right\|_\infty \\ \times \left\| \begin{array}{l} D^{\alpha_k} w_k \\ -f_k \left(\begin{array}{l} t, w_1(t),...,w_n(t), \\ D^{\alpha_1^1} w_1(t),..., \\ D^{\alpha_1^{n-1}} w_1(t), \\ D^{\alpha_2^1} w_2(t),..., \\ D^{\alpha_2^{n-1}} w_2(t),..., \\ D^{\alpha_n^1} w_n(t),..., \\ D^{\alpha_n^{n-1}} w_n(t) \end{array} \right) \end{array} \right\|_\infty \\ + \left\| t^{\delta_k} \left(\begin{array}{l} f_k \left(\begin{array}{l} t, u_1(t),...,u_n(t), \\ D^{\alpha_1^1} u_1(t),..., \\ D^{\alpha_1^{n-1}} u_1(t), \\ D^{\alpha_2^1} u_2(t),..., \\ D^{\alpha_2^{n-1}} u_2(t),..., \\ D^{\alpha_n^1} u_n(t),..., \\ D^{\alpha_n^{n-1}} u_n(t) \end{array} \right) \\ -f_k \left(\begin{array}{l} t, w_1(t),...,w_n(t), \\ D^{\alpha_1^1} w_1(t),..., \\ D^{\alpha_1^{n-1}} w_1(t), \\ D^{\alpha_2^1} w_2(t),..., \\ D^{\alpha_2^{n-1}} w_2(t),..., \\ D^{\alpha_n^1} w_n(t),..., \\ D^{\alpha_n^{n-1}} w_n(t) \end{array} \right) \end{array} \right) \right\|_\infty \end{array} \right).$$

Using (3.1), (3.2) and (E_3), we get

$$\begin{aligned} &\|(u_1-w_1,...,u_n-w_n)\|_B \\ \leq\ & \max_{1\leq k\leq n} \left(\varepsilon_k + \Sigma_k \|(u_1-w_1,...,u_n-w_n)\|_B \right). \end{aligned}$$

Hence,

$$\|(u_1-w_1,...,u_n-w_n)\|_B \leq \frac{\varepsilon}{1-\max\limits_{1\leq k\leq n} \Sigma_k} := Q\varepsilon,\ \varepsilon = \max_{1\leq k\leq n} \varepsilon_k. \tag{3.6}$$

From (E_4), we see that $Q>0$. So, by (3.6) we deduce that problem (1.1) is Ulam-Hyers stable. Putting $\Upsilon(\varepsilon) = Q\varepsilon$, allow us to state that problrm (1.1) is also generalized Ulam-Hyers stable. This completes the proof. □

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