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Characterization of fuzzy number fuzzy measure using fuzzy integral

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Abstract

By using the concepts of fuzzy number fuzzy measures [2] and fuzzy valued functions [3] a theory of fuzzy integrals is investigated. In this paper we have established the fuzzy version of Generalised monotone Convergence theorem and generalised Fatous lemma.

Keywords

Fuzzy number, Fuzzy-valued functions, Fuzzy integral, Fuzzy number fuzzy measure.

AMS Subject Classification

26E50, 03E72.

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Contents

1. Introduction

In this paper [2], we have introduced a concept of fuzzy number fuzzy measures, defined the fuzzy integral of a function with respect to a fuzzy number fuzzy measure and shown some properties and generalized convergence theorems. It is well-known that a fuzzy–valued function [3, 4] is an extension of a function (point-valued), and the fuzzy integral of fuzzy-valued functions with respect fuzzy measures(point-valued) has been studied [3]; so it is natural to ask whether we can establish a theory about fuzzy integrals of fuzzy valued function with respect to fuzzy number fuzzy measures, the answer is just the paper's purpose. In fact, it is also a continued work of [3]. Since what we will discuss in the following is a generalization of works in [2, 3].

Throughout the paper, R^+ will denote the interval $[0,\infty]$, X is an arbitrary fixed set, $\bar{A}$ is a fuzzy σ-algebra [1] formed by the fuzzy-subsets of X, $(X,\bar{A})$ is a fuzzy measurable space, $\mu : \bar{A} \to R^+$ is a fuzzy measure in Sugeno's sense, $\int_{\bar{A}} f d\mu$ is the resulting fuzzy integral [1]. Operation $E\{+,.,\hat{,}\wedge\}$, $F(x)$ is the set of all $\bar{A}$-measurable functions from x to R^+, $M(x)$ denotes the set of all fuzzy measures, (R^+) denotes the set of interval-numbers, R^+ denote the set of fuzzy numbers [2, 3], $\bar{F}(x)$ denotes the set of all $\bar{A}$-measurable interval-valued functions [3]. $\bar{F}(x)$ denotes the set of all $\bar{A}$-measurable fuzzy valued functions [3]. $\bar{M}(x)$ denotes the set of interval number fuzzy measures [2], $\bar{M}(x]$ denotes the set of fuzzy Number fuzzy Measures [2], we will adopt the preliminaries in [2–4]. Here we omit them for brevity, for more details see [2–4].

2. Definitions and Properties

Definition 2.1. *Let $\bar{f} \in \bar{F}(x)$, $A \in \mathscr{A}$, $\bar{\mu} \in \bar{M}(x)$. Then the fuzzy integral of f and A with respect to $\bar{\mu}$ is defined as $\int_A f d\mu = [\int_A f^- d\mu^- \int_A f^+ d\mu^+]$ where $\bar{f}(x) = \inf \bar{f}(x)$ and $f^+(x) = \sup f^+(x) \bar{\mu}(x) = \inf \bar{\mu}(x)$ and $\mu\mu^+(x) = \sup{}^+(x)$.*

Definition 2.2. *Let $\bar{f} \in \bar{F}(x)$, $A \in \mathscr{A}$, $\bar{\mu} \in \bar{M}(x)$. Then the fuzzy integral of $\bar{f}$ and A with respect to μ is defined as $\int_A f d\mu(r) = \sup\{\lambda \in (0,1] : r \in \int_A f d\mu\}$, where $f_\lambda x = \{r \in (0,1] : f(x)(r) > \lambda\}$ and μ_λ is similar.*

Theorem 2.3. *Let $\varepsilon \in \bar{f}\bar{F}(x), AA, \bar{\mu}\varepsilon\bar{M}(x)$. Then $\varepsilon \int_A f^- d\mu^- R^+$ and the following equation holds:*

$$\left(\int_A \bar{f} d\bar{\mu}\right)_\lambda = \int_A f_\lambda d\mu_\lambda \quad for \quad (0,1]. \tag{2.1}$$

Proof. The condition is sufficient. To prove that the condition is necessary it is enough to verity equation (2.1).

For a fixed $\lambda \in (0,1]$ let $\lambda_n = (1-1/n+1)\lambda$ then $\lambda_n \uparrow \lambda$.

It is easy to see that
$\bar{f}_\lambda(x) = \bigcap_{\lambda'<\lambda} \bar{f}_{\lambda'}(x) = \bigcap_{n=1}^{\infty} \bar{f}_{\lambda n}(x) = \lim_{n\to\infty} \bar{f}_{\lambda n}(x)$
Then we have $\bar{f}_{\bar{\lambda}n}{}^{-} \uparrow f_\lambda{}^{-}$, $\bar{f}_{\lambda n}{}^{+} \uparrow f_\lambda{}^{+}$.
Similarly $\mu_{\lambda n}{}^{-} \uparrow \mu_\lambda{}^{-}$, $\mu_{\lambda n}{}^{+} \uparrow \mu_\lambda{}^{+}$.
We have $\int_A \bar{f}_{\bar{\lambda}n} d\bar{\mu}_n \uparrow \int_A \bar{f}_{\bar{\lambda}} d\bar{\mu}^{+} \downarrow \int_A f^{+} d\mu^{+}$.
Hence

$$\left(\int_A \bar{f} d\bar{\mu}\right)_\lambda = \bigcap_{n=1}^{\infty} \int_A \bar{f}_{\bar{n}} d\bar{\mu}_n \lambda$$
$$= \lim_{n\to\infty} \int_A \bar{f}_{\bar{n}} d\bar{\mu}_n$$
$$= \int_A \bar{f}_{\bar{\lambda}} d\bar{\mu}.$$

Hence the theorem. □

Theorem 2.4. *Fuzzy integral of fuzzy valued functions with respect to fuzzy number fuzzy measures have the following property:*
$f_1 \leqslant f_2 \leqslant \mu_1\mu_2 \Rightarrow \int_A f_1 d\mu_1 \leqslant \int_A f_2 d\mu_2$.

Proof. $\lambda \in (0,1]$. Let $\lambda_n = (1-1/n+1)\lambda$ then $\lambda_n \uparrow \lambda$.
It is easy to see that
$(\bar{f}_1)(x) = \bigcap_{\lambda'<\lambda} \bar{f}_{1\lambda'}(x) = \bigcap_{n=1}^{\infty} \bar{f}_{1_{\lambda n}(x)} = \lim_{n\to\infty} \bar{f}_{1_{\lambda n}(x)}$.
Then we have $(\bar{f}_1)_{\lambda n}{}^{-} \uparrow (\bar{f}_1)_\lambda{}^{-}$, $(\bar{f}_1)_{\lambda n}{}^{+} \uparrow (\bar{f}_1)_\lambda{}^{+}$.
By generalised monotone convergence theorem
$\int_A (\bar{f}_1)_{\bar{\lambda}n} d\bar{\mu}_{1n} \uparrow \int_A \bar{f}_{1\bar{\lambda}} d\bar{\mu}_1 \int_A (f_1)_{\lambda n}{}^{+} d\mu_{1n}{}^{+} \downarrow \int_A f_1{}^{+} d\mu_1{}^{+}$.
Hence

$$\left(\int_A (\bar{f}_1) d\bar{\mu}_1\right) = \bigcap_{n=1}^{\infty} \int_A^{(} \bar{f}_1)_{\bar{\lambda}n} d\bar{\mu}_{1n}$$
$$= \lim_{n\to\infty} \int_A \bar{f}_{1\bar{\lambda}n} d\bar{\mu}_{1n}$$
$$= \int_A \bar{f}_{1\bar{\lambda}n} \bar{d}(\bar{\mu}_{1\lambda})$$
$$= \int_A f_1 d\mu_1$$
$$\leqslant \int_A f_2 d\mu_2.$$

Hence the proof. □

Theorem 2.5. *Fuzzy integral of fuzzy valued functions with respect to fuzzy number fuzzy measure $A \subset B \Rightarrow \int_A f d\mu \leqslant \int_B f d\mu$.*

Proof. For a fixed $\lambda \in (0,1]$ let λ_n =(1-1/n + 1)λ then $\lambda_n \uparrow \lambda$.
It is easy to see that
$\bar{f}_\lambda(x) = \bigcap_{\lambda'<\lambda} \bar{f}_{\lambda'}(x) = \bigcap_{n=1}^{\infty} \bar{f}_{\lambda n}(x) = \lim_{n\to\infty} \bar{f}_{\lambda n}(x)$.
Then we have $\lambda \bar{f}_{\lambda n}{}^{-} \uparrow f_\lambda{}^{-}$, $\bar{f}_{\lambda n}{}^{+} \uparrow f_\lambda{}^{+}$.

By Generalised monotone convergence theorem

$$\int_A (\bar{f}_1)_{\bar{\lambda}n} d\bar{\mu}_n \uparrow \int_A \bar{f}_{\bar{\lambda}} d\bar{\mu}$$
$$= \int_A f_{\lambda n}{}^{+} d\mu_n{}^{+} \downarrow \int_A f_\lambda{}^{+} d\mu^{+}$$
$$= \left(\int_A (f d\mu\right)_\lambda$$
$$= \bigcap_{n=1}^{\infty} \int_A f_{\lambda n} d\mu_n$$
$$= \lim_{n\to\infty} \int_A f_{\lambda n} d\mu_{\lambda n}$$
$$= \int_A f_\lambda d\mu_\lambda$$
$$= \int_A \bigcup_{\lambda\in(0,1]} \lambda f_\lambda d\mu_\lambda$$
$$= \int_A \bar{f} d\mu \leqslant \int_B \bar{f} d\mu \ (A \subset B).$$

Hence the theorem. □

3. Convergence theorems

In this section we canvass the convergence of sequences of fuzzy integrals.

Theorem 3.1 (Generalised Monotone Convergence theorem)**.**

Let $\{\bar{f}_n\ (n \geqslant 1), \bar{f}\} \subset \bar{F}(x)$, $\{\mu_n\ (n \geqslant 1), \mu\} \subset \bar{M}(x)$.

Then

$$(i) \quad \bar{f}_n^{-} \uparrow f^{-} \text{ on } A,\ \lambda \bar{\mu}^{+} \uparrow \bar{\mu}$$
$$\Rightarrow \int_A f_n d\mu_n \uparrow \int_A f_d \mu \tag{3.1}$$
$$(ii) \quad \lambda \bar{f}^{+} \downarrow \bar{f}^{+} \text{ on } A,\ \mu_n^{+} \downarrow \mu^{+}$$
$$\Rightarrow \int_A f_n^{+} d\mu_n \downarrow \int_A f_d^{+} \mu^{+}. \tag{3.2}$$

Proof. To prove (i) it is sufficient to verify equation (3.1). For $\lambda_k = (1-1/1+k)\lambda$ then $\lambda_k \uparrow \lambda$. By the proof of Theorem 2.3 we obtain

$$\bar{f}_\lambda = \lim_{n\to\infty} \lim_{k\to\infty} \bar{f}_{n\lambda k}$$
$$\bar{\mu}_\lambda = \lim_{n\to\infty} \lim_{k\to\infty} \bar{\mu}_{n\lambda k}.$$

Then

$$\left(\lim_{n\to\infty}\int_A f_n d\mu_n\right)_{\lambda_k}$$
$$=\bigcap_{n=1}^{\infty}\lim_{n\to\infty}(\int_A \bar{f}_n d\bar{\mu}_n)_{\lambda k}$$
$$=\lim_{k\to\infty}\lim_{n\to\infty}\int_A(\bar{f}_n)_{\lambda k}d(\mu_n)_{\lambda k}$$
$$=\int_A \lim_{k\to\infty}\lim_{n\to\infty}(\bar{f}_n)_{\lambda k}d(\lim_{k\to\infty}\lim_{n\to\infty}(\mu_n)_{\lambda k}$$
$$=\int_A f_\lambda d\mu_\lambda$$
$$=\int_A (f_d\mu).$$

This proves (i) and (ii) is similar. □

Theorem 3.2 (Generalised Fatous lemma)**.** *Let* $\{\bar{f}_n\ (n\geqslant 1),\bar{f}\}\subset\bar{F}(x)$, $\{\mu_n\ (n\geqslant 1),\underline{\lim}\mu_n\bar{\lim}\mu_n\subset\bar{M}(x)$.

Then (i) $\int_A\underline{\lim}\bar{f}_n d\underline{\lim}\mu_n\leqslant\underline{\lim}\int_A\bar{f}d\bar{\mu}_n$

ii) $\overline{\lim}\int_{\bar{A}}\bar{f}d\mu_n\leqslant\int_A(\overline{\lim}\bar{f}_n)d(\overline{\lim}\mu_n)$.

Proof. To prove(i), for $\lambda\in(0,1]$ let $\lambda_k=(1-1/1+k)\lambda$ then $\lambda_k\uparrow\lambda$.

$$f_\lambda=\lim_{k\to\infty}\lim_{n\to\infty}\bar{f}_{n\lambda k}$$
$$\mu_\lambda=\lim_{k\to\infty}\lim_{n\to\infty}\bar{\mu}_{n\lambda k}.$$

Then

$$\left(\lim_{n\to\infty}\int_A f_n d\mu_n\right)_{\lambda}$$
$$=\bigcap_{k=1}^{\infty}\lim_{n\to\infty}(\int_A f_n d\mu_n)_{\lambda k}$$
$$=\int_A\lim_{k\to\infty}\lim_{n\to\infty}(f_n)_{\lambda k}d\lim_{k\to\infty}\lim_{n\to\infty}(\mu_n)_{\lambda k}$$
$$=\int_A\lim_{k\to\infty}\lim_{n\to\infty}\inf(f_n)_{\lambda k}d\lim_{k\to\infty}\lim_{n\to\infty}\inf(\mu_n)_{\lambda k}$$
$$=\int_A\lim_{k\to\infty}\lim_{n\to\infty}(\underline{\lim}f_n)_{\lambda k}d\lim_{k\to\infty}\lim_{n\to\infty}\underline{\lim}(\mu_n)_{\lambda k}$$
$$\leqslant\underline{\lim}\int_A\lim_{k\to\infty}\lim_{n\to\infty}(f_n)_{\lambda k}d\lim_{k\to\infty}\lim_{n\to\infty}(\mu_n)_{\lambda k}$$
$$\leqslant\underline{\lim}\int_A\bigcup_{(0,1]}(f_n)_d(\mu_n)_\lambda$$
$$=\underline{\lim}\int_A f_n d\mu_n.$$

(ii) $$\left(\lim_{n\to\infty}\int_A f_n d\mu_n\right)_{\lambda}$$
$$=\bigcap_{k=1}^{\infty}\lim_{n\to\infty}(\int_A f_n d\mu_n)_{\lambda k}$$
$$=\lambda\lim_{k\to\infty}\lim_{n\to\infty}(\int_A f_n d\mu_n)_{\lambda k}$$
$$=\lim_{k\to\infty}\lim_{n\to\infty}(\sup\int_A f_n d\mu_n\lambda)_{\lambda k}$$
$$=\lambda\lim_{k\to\infty}\lim_{n\to\infty}(\bar{\lim}\int_A f_n d\mu_n)_{\lambda k}$$
$$\leqslant\overline{\lim}\int_A\lim_{k\to\infty}\lim_{n\to\infty}(f_n)d\mu_n)$$
$$\leqslant\overline{\lim}\int_A\lim_{n\to\infty}(f_n)d(\mu_n)$$
$$=\int_A\overline{\lim}(f_n)d(\bar{\lim}\mu_n).$$

Hence the theorem. □

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