Fixed point results for $\mathscr{H}$-contractions in fuzzy metric spaces via admissible

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Abstract

In the present work, we come out with the introduction of α - ψ - fuzzy $\mathscr{H}$ - contraction mapping in the setting of fuzzy metric spaces. We establish fixed point results for such contraction mappings in a complete fuzzy metric space. An example is bestowed to illustrate the applicability of the obtained result.

Keywords

t-norm, fuzzy metric space, α - admissible, fuzzy $\mathscr{H}$ - contractive mapping.

AMS Subject Classification

47H10, 54H25.

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Contents

1. Introduction

The introduction of fuzzy sets in the groundbreaking paper of Zadeh[22] to represent the vagueness in everyday life, exclusively led to the inception of fuzzy mathematics. The study of fuzzy sets activated an immense fuzzification of various mathematical topics and has its usability in various branches like image processing, gaming, coding theory, etc. The concept of a fuzzy metric space was introduced by Kramosil and Michalek[8] which was later altered by George and Veeramani[2]. The study of fixed point theory in fuzzy metric spaces which is analogous to fixed point theory in probabilistic metric space was first initiated by Grabiec[4] in the year 1988.

Gregori and Sapena[7] studied fuzzy contractions and developed Banach Contraction principle in several classes of complete fuzzy metric spaces. Over the years, Many authors extended this concept by introducing and studying different types of fuzzy contractive mappings. For more allusions on the development of fixed point theory in fuzzy metric spaces, see also[[5], [6], [9], [10], [11], [12], [13], [14], [20]].

Recently, Wardowski[21] in the year 2013, proposed fuzzy $\mathscr{H}$ - contractive mapping and established fixed point theorems for such contraction. Gopal et al.[3], proposed the notion of α-ϕ-fuzzy contractive mapping and established fixed point theorems in the sense of Grabiec[4]. Later, as an extension to this work, I. Beg et al.[1], introduced the notion of α - fuzzy $\mathscr{H}$ - contractive mapping and established some fixed point results for such mappings.

In the present work, we first propose the concept of α - ψ - fuzzy $\mathscr{H}$ - contraction mappings and then prove fixed point results for such contractions and also provide a suitable example to show the applicability of our obtained result. Our results extend and generalize some comparable and related results in the existing literature.

2. Preliminaries

Definition 2.1. *[18] A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is said to be a continuous t - norm if for all* $p,q,r,l \in [0,1]$*, the following conditions are satisfied:*

- $p * 1 = p$

- $p * q = q * p$

- $p * q \leq r * l$ *whenever* $p \leq r$ *and* $q \leq l$

- $p * (q * r) = (p * q) * r$

Common examples of continuous t - norms are:

- $p * q = min\{p,q\}$ *(minimum t-norm)*
- $p * q = pq$ *(product t-norm)*
- $p * q = max\{p+q-1,0\}$ *(lukasiewicz t-norm)*
- $p * q = \begin{cases} min\{p,q\}, & if\, max\{p,q\} = 1 \\ 0 & otherwise \end{cases}$

  (weakest t-norm, the drastic product)

For $p_1, p_2, \dots, p_n \in [0,1]$, $n \in \mathbb{N}$, the product $p_1 * p_2 * \dots * p_n$ is denoted by $\prod_{i=1}^{n} p_i$.

A t-norm $*$ is said to be positive, if $p * q > 0$ for p, $q \in (0, 1]$. A t-norm $*$ is said to be nilpotent, if $*$ is continuous and for each $p \in (0,1)$, there exists $n \in \mathbb{N}$ such that $\prod_{i=1}^{n} p = 0$.

Definition 2.2. *[2] Let X be any non - empty set, $*$ is a continuous t - norm and M is a fuzzy set on $X \times X \times (0,\infty)$ satisfying*

2.2(a). $M(p,q,t) > 0$,

2.2(b). $M(p,q,t) = 1$ if and only if $p = q$,

2.2(c). $M(p,q,t) = M(q,p,t)$,

2.2(d). $M(p,r,t+s) \geq M(p,q,t) * M(q,r,s)$,

2.2(e). $M(p,q,\cdot) : (0,\infty) \to (0,1]$ is continuous

where $p,q,r \in X$ and $t,s > 0$. Then, the 3 - tuple $(X,M,*)$ is called a fuzzy metric space. Here, $M(p,q,t)$ represents the degree of nearness between p and q with respect to t.

Lemma 2.3. *[4] Let $(X,M,*)$ be a fuzzy metric space. For each fixed p,q in X, $M(p,q,\cdot)$ is nondecreasing .*

Remark 2.4. *[16] In a fuzzy metric space $(X,M,*)$, if $p*p \geq p$ for all $p \in [0,1]$ then $p*q = min\{p,q\}$ for all $p,q \in [0,1]$.*

Definition 2.5. *([2], [4], [19]) Let $(X,M,*)$ be a fuzzy metric space and $\{x_n\}$ be a sequence in X. Then*

(i). *$\{x_n\}$ is said to converge to some $x \in X$ whenever*

$$\lim_{n \to \infty} M(x_n,x,t) = 1; \ t > 0$$

(ii). *$\{x_n\}$ is said to be a G - Cauchy sequence in X if*

$$\lim_{n \to \infty} M(x_n,x_{n+p},t) = 1 \text{ for all } t > 0, p > 0$$

(iii). *$\{x_n\}$ is said to be M - Cauchy sequence in X if for all $\varepsilon \in (0, 1)$, $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_m,x_n,t) > 1 - \varepsilon$ for any $m,n \geq n_0$.*

A fuzzy metric space $(X,M,*)$ is said to be G - complete(M - complete) if every G - cauchy(M - cauchy) sequence in X is convergent.

Lemma 2.6. *[15] For any two points x,y in a fuzzy metric space $(X,M,*)$ and $k \in (0,1)$, if $M(x,y,kt) \geq M(x,y,t)$ then $x = y$.*

Definition 2.7. *[7] Let $(X,M,*)$ be a fuzzy metric space. $S : X \to X$ is called a fuzzy contractive mapping if there exist $k \in (0,1)$ such that:*

$$\left(\frac{1}{M(Sx,Sy,t)} - 1\right) \leq k \left(\frac{1}{M(x,y,t)} - 1\right) \ \forall \, x,y \in X, t > 0$$
(2.1)

Here, k is called the contractive constant of S.

Definition 2.8. *[21] Let $\mathscr{H}$ be the family of mappings $\eta : (0,1] \to [0,\infty)$ satisfying the conditions*

(H1). η transforms $(0,1]$ onto $[0,\infty)$,

(H2). η is strictly decreasing.

Note that (H1) and (H2) implies that $\eta(1) = 0$.

Remark 2.9. *[17] Let $\eta \in \mathscr{H}$, then from (H1) and (H2) we can easily see that η is necessarily continuous on $(0, 1)$.*

Definition 2.10. *[21] Let $(X,M,*)$ be a fuzzy metric space. A mapping $S : X \to X$ is said to be fuzzy $\mathscr{H}$ - contractive mapping with respect to $\eta \in \mathscr{H}$ if there exists $k \in (0,1)$ satisfying the following condition*

$$\eta(M(Sx,Sy,t)) \leq k\eta(M(x,y,t)) \ \forall \, x,y \in X, t > 0. \quad (2.2)$$

If $\eta(\delta) = \frac{1}{\delta} - 1$, where $\delta \in (0,1]$, then definition 2.10 reduces to definition 2.7.

Proposition 2.11. *[21] Let $(X,M,*)$ be a fuzzy metric space and $\eta \in \mathscr{H}$. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in X is said to be M - Cauchy if and only if for every $\varepsilon > 0$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $\eta(M(x_m,x_n,t)) < \varepsilon$ where $m,n \geq n_0$.*

Proposition 2.12. *[21] Let $(X,M,*)$ be a fuzzy metric space and $\eta \in \mathscr{H}$. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in X is said to converge to $x \in X \iff \lim_{n \to \infty} \eta(M(x_n,x,t)) = 0, t > 0$.*

Definition 2.13. *[3] Let $(X,M,*)$ be a fuzzy metric space. A mapping $S : X \to X$ is said to be α - admissible if there exists a function $\alpha : X \times X \times (0,\infty) \to [0,\infty)$ such that*

$$\alpha(x,y,t) \geq 1 \implies \alpha(Sx,Sy,t) \geq 1$$

for all $x,y \in X$, $t > 0$.

Let Φ be the family of all right continuous function $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(r) < r, r > 0$.

Definition 2.14. *[3] Let $(X,M,*)$ be a fuzzy metric space. $S : X \to X$ is called an α - ϕ - fuzzy contractive mapping if $\exists$ two functions $\alpha : X \times X \times (0,\infty) \to [0,\infty)$ and $\phi \in \Phi$ such that $\forall \, x,y \in X, t > 0$*

$$\alpha(x,y,t) \left(\frac{1}{M(Sx,Sy,t)} - 1\right) \leq \phi\left(\frac{1}{M(x,y,t)} - 1\right) \quad (2.3)$$

Remark 2.15. *[3] It can be seen that if $\alpha(x,y,t) = 1$ for all $x,y \in X$, $t > 0$ and for some $k \in (0,1)$ if $\phi(r) = kr$, $r > 0$, then definition 2.14 reduces to definition 2.7 but the converse need not be necessarily true.*

Definition 2.16. *[1] Let $(X,M,*)$ be a fuzzy metric space. A mapping $S : X \to X$ is said to be an α - fuzzy $\mathscr{H}$ - contractive mapping with respect to $\eta \in \mathscr{H}$ if there exists a function $\alpha : X \times X \times (0,\infty) \to [0,\infty)$ such that $\forall\, x,y \in X, t > 0$*

$$\alpha(x,y,t)\eta(M(Sx,Sy,t)) \le k\eta(M(x,y,t)) \qquad (2.4)$$

Remark 2.17. *[1] If $\alpha(x,y,t) = 1$ for all $x,y \in X$, $t > 0$, then definition 2.16 reduces to definition 2.10 but converse may not be necessarily true.*

In [1], I. Beg et al., proved the following:

Theorem 2.18. *Let $(X,M,*)$ be a M-complete fuzzy metric space, where $*$ is positive. Let $S : X \to X$ be an α - fuzzy $\mathscr{H}$ - contractive mapping with respect to $\eta \in \mathscr{H}$ satisfying the following conditions:*

(I) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0, t) \ge 1$, $t > 0$,

(II) S is α - admissible,

(III) $\eta(r*s) \le \eta(r) + \eta(s)$, $r,s \in (0,1]$,

(IV) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \ge 1$, $n \in \mathbb{N}$ and $\lim_{n \to \infty} x_n = x$, then $\alpha(x_n, x, t) \ge 1$, $n \in \mathbb{N}$, $t > 0$.

Then, S has a fixed point $u \in X$. Moreover, the sequence $\{S^n x_0\}_{n \in \mathbb{N}}$ converges to u.

Moreover, I. Beg et al.[1] showed that in the above theorem, "if $\forall\, x,y \in X$ and $t > 0$ there exists $p \in X$ such that $\alpha(x,p,t) \ge 1$ and $\alpha(y,p,t) \ge 1$" then we can obtain a unique fixed point of S in X.

3. Main Results

Let Ψ_0 be the family of functions $\psi : [0,\infty) \to [0,\infty)$ such that

(a). ψ is continuous and increasing.

(b). $\{\psi^n(t)\}_{n \in \mathbb{N}}$ converges to 0 as $n \to \infty$ for all $t > 0$ where $\psi^n(t)$ denotes the n-th iterate of ψ.

(c). $\psi(t) < t$ for every $t > 0$ and $\psi(0) = 0$.

Definition 3.1. *Let $(X,M,*)$ be a fuzzy metric space. A mapping $S : X \to X$ is said to be an α - ψ - fuzzy $\mathscr{H}$ - contraction with respect to $\eta \in \mathscr{H}$ if there exists a function $\alpha : X \times X \times (0,\infty) \to [0,\infty)$ such that for all $x,y \in X$, $t > 0$, $k \in (0,1)$ and $\psi \in \Psi_0$,*

$$\alpha(x,y,t)\eta(M(Sx,Sy,kt)) \le \psi(\eta(N(x,y))) \text{ where}$$
$$N(x,y) = max\left\{ \begin{array}{c} M(x,y,t), M(x,Sx,t), M(y,Sy,t), \\ min\{M(x,Sy,2t), M(y,Sx,2t)\} \end{array} \right\}$$
$$(3.1)$$

Remark 3.2. *If $\psi(\tau) = r\tau$ for $r \in (0,1)$, $\tau > 0$ then definition 3.1 can be easily reduced to definition 2.16 for all $x,y \in X$, $k \in (0,1)$ and $t > 0$ which shows α - ψ - fuzzy $\mathscr{H}$ - contraction is a generalization of α - fuzzy $\mathscr{H}$ - contractive mapping. In addition, if $\alpha(x,y,t) = 1$ for all $x,y \in X, t > 0$, then definition 3.1 reduces to definition 2.10 but converse may not be necessarily true(see example 3.4 below).*

Theorem 3.3. *Let $(X,M,*)$ be a M-complete fuzzy metric space where $*$ is minimum t-norm and $\psi \in \Psi_0$. Let $S : X \to X$ be an α - ψ - fuzzy $\mathscr{H}$ - contraction with respect to $\eta \in \mathscr{H}$ satisfying the following conditions:*

(i) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0, t) \ge 1, t > 0$;

(ii) S is α - admissible;

(iii) $\eta(r*s) \le \eta(r) + \eta(s), r,s \in (0,1]$;

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \ge 1$, $n \in \mathbb{N}$ and $\lim_{n \to \infty} x_n = x$, then $\alpha(x_n, x, t) \ge 1, n \in \mathbb{N}, t > 0$

Then, S has a fixed point $u \in X$.

Proof. Let $x_0 \in X \ni \alpha(x_0, Sx_0, t) \ge 1$ where $t > 0$. Let $\{x_n\}$ be a sequence in X defined by $Sx_n = x_{n+1}\ \forall\, n \in \mathbb{N} \cup \{0\}$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$, then $u = x_{n_0}$ is a fixed point of S.
So, let us assume that $x_n \ne x_{n+1}$ for any $n \in \mathbb{N} \cup \{0\}$.
Since S is α - admissible, we have
$\alpha(x_0, x_1, t) = \alpha(x_0, Sx_0, t) \ge 1$
$\Rightarrow \alpha(Sx_0, Sx_1, t) = \alpha(x_1, x_2, t) \ge 1,\ t > 0$
By induction, we get

$$\alpha(x_n, x_{n+1}, t) \ge 1 \text{ for all } n \in \mathbb{N} \cup \{0\}, t > 0. \qquad (3.2)$$

we know that

$$M(x_{n+1}, x_{n+2}, t) \ge M(x_{n+1}, x_{n+2}, kt)$$
$$(k \in (0,1) \text{ and from lemma } 2.3)$$
$$\implies \eta(M(x_{n+1}, x_{n+2}, t)) \le \eta(M(x_{n+1}, x_{n+2}, kt))$$
$$(3.3)$$

In view of (3.2) and (3.1), we have

$$\eta(M(x_{n+1}, x_{n+2}, kt)) = 1.\eta(M(Sx_n, Sx_{n+1}, kt))$$
$$\le \alpha(x_n, x_{n+1}, t)\eta(M(Sx_n, Sx_{n+1}, kt))$$
$$\le \psi(\eta(N(x_n, x_{n+1})))\ n \in \mathbb{N} \cup \{0\}, t > 0 \quad (3.4)$$

where $N(x_n, x_{n+1})$

$$= max\left\{ \begin{array}{c} M(x_n, x_{n+1}, t), M(x_n, Sx_n, t), M(x_{n+1}, Sx_{n+1}, t), \\ min\{M(x_n, Sx_{n+1}, 2t), M(x_{n+1}, Sx_n, 2t)\} \end{array} \right\}$$

$$= max\left\{ \begin{array}{c} M(x_n, x_{n+1}, t), M(x_n, x_{n+1}, t), M(x_{n+1}, x_{n+2}, t), \\ min\{M(x_n, x_{n+2}, 2t), M(x_{n+1}, x_{n+1}, 2t)\} \end{array} \right\}$$

$$= max\left\{ \begin{array}{c} M(x_n, x_{n+1}, t), M(x_n, x_{n+1}, t), M(x_{n+1}, x_{n+2}, t), \\ M(x_n, x_{n+2}, 2t) \end{array} \right\}$$

$$\geq \max\left\{\begin{array}{c} M(x_n,x_{n+1},t), M(x_{n+1},x_{n+2},t),\\ M(x_n,x_{n+1},t)*M(x_{n+1},x_{n+2},t) \end{array}\right\}$$

$$= \max\{M(x_n,x_{n+1},t),M(x_{n+1},x_{n+2},t)\}$$

$$\Rightarrow N(x_n,x_{n+1}) \geq \max\{M(x_n,x_{n+1},t),M(x_{n+1},x_{n+2},t)\},$$
$n \in \mathbb{N}\cup\{0\}, t > 0$

If $\max\{M(x_n,x_{n+1},t),M(x_{n+1},x_{n+2},t)\} = M(x_{n+1},x_{n+2},t)$,
then $N(x_n,x_{n+1}) \geq M(x_{n+1},x_{n+2},t)$, $\quad n \in \mathbb{N}\cup\{0\}, t > 0$

$$\implies \eta(N(x_n,x_{n+1})) \leq \eta(M(x_{n+1},x_{n+2},t)) \qquad (3.5)$$

from the properties of ψ, η and using (3.5), the inequality (3.4) gives

$$\begin{aligned} \eta(M(x_{n+1},x_{n+2},kt)) &\leq \psi(\eta(N(x_n,x_{n+1})))\\ &< \eta(N(x_n,x_{n+1}))\\ &\leq \eta(M(x_{n+1},x_{n+2},t))\\ \implies \eta(M(x_{n+1},x_{n+2},kt)) &\leq \eta(M(x_{n+1},x_{n+2},t))\\ \implies M(x_{n+1},x_{n+2},kt) &\geq M(x_{n+1},x_{n+2},t) \end{aligned}$$

which shows that $x_{n+1} = x_{n+2}$ for all $n \in \mathbb{N}\cup\{0\}$ from lemma 2.6, which is a contradiction to our assumption.
Thus, $\max\{M(x_n,x_{n+1},t),M(x_{n+1},x_{n+2},t)\} = M(x_n,x_{n+1},t)$,
$n \in \mathbb{N}\cup\{0\}, t > 0$.

$$\begin{aligned} \therefore \ N(x_n,x_{n+1}) &\geq M(x_n,x_{n+1},t)\\ \implies \eta(N(x_n,x_{n+1})) &\leq \eta(M(x_n,x_{n+1},t))\\ & \qquad n \in \mathbb{N}\cup\{0\}, \ t > 0 \end{aligned}$$

and since ψ is increasing, we get

$$\psi(\eta(N(x_n,x_{n+1}))) \leq \psi(\eta(M(x_n,x_{n+1},t)))$$

Now, (3.4) implies

$$\eta(M(x_{n+1},x_{n+2},kt)) \leq \psi(\eta(M(x_n,x_{n+1},t))) \qquad (3.6)$$

In view of (3.3), (3.6), we obtain

$$\begin{aligned} \eta(M(x_n,x_{n+1},t)) &\leq \eta(M(x_n,x_{n+1},kt))\\ &\leq \psi(\eta(M(x_{n-1},x_n,t))\\ \implies \eta(M(x_n,x_{n+1},t)) &\leq \psi(\eta(M(x_{n-1},x_n,t)))\\ \implies \psi(\eta(M(x_n,x_{n+1},t))) &\leq \psi^2(\eta(M(x_{n-1},x_n,t)))\\ & \qquad\qquad\qquad (3.7) \end{aligned}$$

On repeated application of (3.7), inequality (3.6) gives

$$\begin{aligned} \eta(M(x_{n+1},x_{n+2},kt)) &\leq \psi(\eta(M(x_n,x_{n+1},t)))\\ &\leq \psi^2(\eta(M(x_{n-1},x_n,t)))\\ &\cdots\cdots\\ &\leq \psi^{n+1}(\eta(M(x_0,x_1,t))), t > 0\\ & \qquad\qquad\qquad (3.8) \end{aligned}$$

$$\eta(M(x_{n+1},x_{n+2},kt)) \leq \psi^{n+1}(\eta(M(x_0,x_1,t))),$$
$$n \in \mathbb{N}\cup\{0\}, t > 0 \qquad (3.9)$$

Consider any $m,n \in \mathbb{N}, n_0 \leq m < n$, and let $\{a_i\}_{i\in\mathbb{N}}$ be any strictly decreasing sequence of positive numbers such that
$$\sum_{i=1}^{\infty} a_i = 1.$$
we have

$$M(x_m,x_n,t) \geq M\left(x_m,x_m,t-t\sum_{i=m}^{n-1}a_i\right) * M\left(x_m,x_n,t\sum_{i=m}^{n-1}a_i\right)$$

$$= M\left(x_m,x_n,t\sum_{i=m}^{n-1}a_i\right)$$

$$M(x_m,x_n,t) \geq M(x_m,x_{m+1},a_m t) * M(x_{m+1},x_{m+2},a_{m+1}t)*$$
$$\cdots * M(x_{n-1},x_n,a_{n-1}t)$$

$$\implies M(x_m,x_n,t) \geq \prod_{i=m}^{n-1} M(x_i,x_{i+1},a_i t)$$

$$\implies \eta(M(x_m,x_n,t)) \leq \eta\left(\prod_{i=m}^{n-1} M(x_i,x_{i+1},a_i t)\right)$$

$$\leq \sum_{i=m}^{n-1} \eta(M(x_i,x_{i+1},a_i t))$$

$$\leq \sum_{i=m}^{n-1} \psi^i(\eta(M(x_0,x_1,t)))$$

$$< \varepsilon$$

Thus, $\eta(M(x_m,x_n,t)) < \varepsilon \ \forall \ m,n \geq n_0, m < n, t > 0$.
$$(3.10)$$

Thus from propositon 2.11, it follows that $\{x_n\}_{n\in\mathbb{N}}$ is a M-Cauchy sequence in X. Since X is M-complete, we can find a $u \in X$ such that $x_n \to u$ as $n \to \infty$.
From proposition 2.12, we obtain

$$\lim_{n\to\infty} \eta(M(x_n,u,t)) = 0, t > 0. \qquad (3.11)$$

In view of (3.3), *(iv)* and (3.1), we get

$$\begin{aligned} \eta(M(x_{n+1},Su,t)) &= \eta(M(Sx_n,Su,t))\\ &\leq \eta(M(Sx_n,Su,kt))\\ &= 1.\eta(M(Sx_n,Su,kt))\\ &\leq \alpha(x_n,u,t)\eta(M(Sx_n,Su,kt))\\ &\leq \psi(\eta(N(x_n,u))),\\ & \qquad n \in \mathbb{N}\cup\{0\}, t > 0 \quad (3.12) \end{aligned}$$

where

$$N(x_n,u) = \max\left\{\begin{array}{c} M(x_n,u,t),M(x_n,Sx_n,t),M(u,Su,t),\\ \min\{M(x_n,Su,2t),M(u,Sx_n,2t)\} \end{array}\right\},$$
$$n \in \mathbb{N}\cup\{0\}, t > 0$$

$$= \max\left\{\begin{array}{c} M(x_n,u,t),M(x_n,x_{n+1},t),M(u,Su,t),\\ \min\{M(x_n,Su,2t),M(u,x_{n+1},2t)\} \end{array}\right\}$$

$$= \max \left\{ \begin{array}{c} M(x_n,u,t), M(x_n,x_{n+1},t), M(u,Su,t), \\ M(x_n,Su,2t) * M(u,x_{n+1},2t) \end{array} \right\}$$

$$\geq \max \left\{ \begin{array}{c} M(x_n,u,t), M(x_n,x_{n+1},t), M(u,Su,t), \\ M(x_n,u,t) * M(u,Su,t)* \\ M(u,x_n,t) * M(x_n,x_{n+1},t) \end{array} \right\}$$

$$\therefore \ N(x_n,u) \geq \max \left\{ \begin{array}{c} M(x_n,u,t), M(x_n,x_{n+1},t), \\ M(u,Su,t) \end{array} \right\}$$

$$n \in \mathbb{N} \cup \{0\}, t > 0$$
(3.13)

Let $\mathscr{M} = \max\{M(x_n,u,t), M(x_n,x_{n+1},t), M(u,Su,t)\}$
Case - I:
If $\mathscr{M} = M(x_n,u,t), n \in \mathbb{N} \cup \{0\}, t > 0$
then (3.13) implies

$$N(x_n,u) \geq M(x_n,u,t))$$
$$\implies \eta(N(x_n,u)) \leq \eta(M(x_n,u,t))$$
$$\implies \psi(\eta(N(x_n,u))) \leq \psi(\eta(M(x_n,u,t)))$$

Now (3.12) gives

$$\eta(M(x_{n+1},Su,t)) \leq \psi(\eta(N(x_n,u))$$
$$\leq \psi(\eta(M(x_n,u,t)))$$
$$\implies \eta(M(x_{n+1},Su,t)) \leq \psi(\eta(M(x_n,u,t)))$$

Letting $n \to \infty$ in the above inequality and using properties of ψ and (3.11), we get

$$\lim_{n \to \infty} \eta(M(x_{n+1},Su,t)) \leq \lim_{n \to \infty} \psi(\eta(M(x_n,u,t)))$$
$$= \psi(\lim_{n \to \infty} \eta(M(x_n,u,t)))$$
$$= \psi(0) = 0.$$

i.e., $Su = \lim_{n \to \infty} x_{n+1} = u$. Therefore, u is a fixed point of S in this case.
Case - II:
If $\mathscr{M} = M(x_n,x_{n+1},t), n \in \mathbb{N} \cup \{0\}, t > 0$
then (3.13) implies
$N(x_n,u) \geq M(x_n,x_{n+1},t) \geq M\left(x_n,u,\frac{t}{2}\right) * M\left(u,x_{n+1},\frac{t}{2}\right)$

$$\eta(N(x_n,u)) \leq \eta\left(M\left(x_n,u,\frac{t}{2}\right) * M\left(u,x_{n+1},\frac{t}{2}\right)\right)$$
$$\leq \eta\left(M\left(x_n,u,\frac{t}{2}\right)\right) + \eta\left(M\left(u,x_{n+1},\frac{t}{2}\right)\right)$$

$$\therefore \ \eta(N(x_n,u)) \leq \eta\left(M\left(x_n,u,\frac{t}{2}\right)\right) + \eta\left(M\left(u,x_{n+1},\frac{t}{2}\right)\right)$$

$\psi(\eta(N(x_n,u))) \leq \psi\left(\eta\left(M\left(x_n,u,\frac{t}{2}\right)\right) + \eta\left(M\left(u,x_{n+1},\frac{t}{2}\right)\right)\right)$
Now (3.12) gives $\eta(M(x_{n+1},Su,t)) \leq \psi(\eta(N(x_n,u)) \leq$
$\psi\left(\eta\left(M\left(x_n,u,\frac{t}{2}\right)\right) + \eta\left(M\left(u,x_{n+1},\frac{t}{2}\right)\right)\right)$
$\eta(M(x_{n+1},Su,t)) \leq \psi\left(\eta\left(M\left(x_n,u,\frac{t}{2}\right)\right) + \eta\left(M\left(u,x_{n+1},\frac{t}{2}\right)\right)\right)$
Letting $n \to \infty$ in the above inequality and using properties of ψ and (3.11), we get

$$\lim_{n \to \infty} \eta(M(x_{n+1},Su,t))$$

$$\leq \lim_{n \to \infty} \psi\left(\eta\left(M\left(x_n,u,\frac{t}{2}\right)\right) + \eta\left(M\left(u,x_{n+1},\frac{t}{2}\right)\right)\right)$$
$$= \psi\left(\lim_{n \to \infty} \eta\left(M\left(x_n,u,\frac{t}{2}\right)\right) + \lim_{n \to \infty} \eta\left(M\left(u,x_{n+1},\frac{t}{2}\right)\right)\right)$$

$$= \psi(0) = 0.$$
i.e., $Su = \lim_{n \to \infty} x_{n+1} = u$.
Therefore, u is a fixed point of S in this case.
Case - III:
If $\mathscr{M} = M(u,Su,t), n \in \mathbb{N} \cup \{0\}, t > 0$
then the inequality (3.13) implies

$$N(x_n,u) \geq M(u,Su,t) \tag{3.14}$$

Now, we need to show that u is a fixed point of S in this case i.e., we need to show that $M(u,Su,t) = 1$ for all $t > 0$. Let us suppose that $M(u,Su,t_0) < 1$ for some $t_0 > 0$ then $\eta(M(u,Su,t_0)) > 0$.
For such t_0, (3.14) gives

$$N(x_n,u) \geq M(u,Su,t_0)$$
$$\implies \eta(N(x_n,u)) \leq \eta(M(u,Su,t_0))$$
$$\implies \psi(\eta(N(x_n,u))) \leq \psi(\eta(M(u,Su,t_0))) \tag{3.15}$$

Now using (3.15) and properties of ψ, the inequality (3.12) gives
$\eta(M(x_{n+1},Su,t_0)) \leq \psi(\eta(N(x_n,u))) \leq \psi(\eta(M(u,Su,t_0)))$

$$\implies \eta(M(x_{n+1},Su,t_0)) \leq \psi(\eta(M(u,Su,t_0))) \tag{3.16}$$
$$< \eta(M(u,Su,t_0))$$
$$\implies \eta(M(x_{n+1},Su,t_0)) < \eta(M(u,Su,t_0)) \tag{3.17}$$

Also for all $m \in \mathbb{N}$, we have

$$M\left(u,Su,t_0+\frac{t_0}{m}\right) \geq M\left(u,x_{n+1},\frac{t_0}{m}\right) * M(x_{n+1},Su,t_0)$$
$$\eta\left(M\left(u,Su,t_0+\frac{t_0}{m}\right)\right)$$
$$\leq \eta\left(M\left(u,x_{n+1},\frac{t_0}{m}\right) * M(x_{n+1},Su,t_0)\right)$$
$$\leq \eta\left(M\left(u,x_{n+1},\frac{t_0}{m}\right)\right) + \eta(M(x_{n+1},Su,t_0))$$
$$< \eta\left(M\left(u,x_{n+1},\frac{t_0}{m}\right)\right) + \eta(M(u,Su,t_0))$$
$$\eta\left(M\left(u,Su,t_0+\frac{t_0}{m}\right)\right)$$
$$\leq \eta\left(M\left(u,x_{n+1},\frac{t_0}{m}\right)\right) + \eta(M(x_{n+1},Su,t_0))$$
$$< \eta\left(M\left(u,x_{n+1},\frac{t_0}{m}\right)\right) + \eta(M(u,Su,t_0))$$

Since $\lim_{n \to \infty} \eta(M(x_n,u,t)) = 0$ for all $t > 0$ (from (3.11)) and the above inequalities are true for all $m \in \mathbb{N}$, we must therefore have

$$\lim_{n \to \infty} \eta(M(x_{n+1},Su,t_0)) = \eta(M(u,Su,t_0)) \tag{3.18}$$

In view of (3.18) and from properties of ψ, letting $n \to \infty$ in (3.16) gives

$$\eta(M(u,Su,t_0)) \leq \psi(\eta(M(u,Su,t_0)) < \eta(M(u,Su,t_0)$$

which is a contradiction. Thus, $M(u,Su,t) = 1$ for all $t > 0$. Therefore, $Su = u$ i.e., u is a fixed point of S in this case. $\square$

Example 3.4. *Let $X = \mathbb{R}$, $*$ be a continuous t-norm defined by $a * b = \min\{a,b\}$, $a,b \in [0,1]$ and $M(x,y,t) = \frac{t}{t+|x-y|}$, for all $x,y \in X$, $t > 0$. Clearly, $(X,M,*)$ is a M-complete fuzzy metric space.*
Define $S : X \to X$ by

$$Sx = \begin{cases} \frac{1}{4} & \text{if } x,y \in [0,1] \\ 2 & \text{otherwise} \end{cases},$$

$\alpha : X \times X \times (0,\infty) \to [0,\infty)$ by

$$\alpha(x,y,t) = \begin{cases} 2 & \text{if } x,y \in [0,1] \\ 0 & \text{otherwise} \end{cases},$$

$\eta(s) = \frac{1}{s} - 1$, $s \in (0,1]$ and $\psi: [0,\infty) \to [0,\infty)$ by $\psi(\tau) = \frac{\tau}{2}$
Clearly, S is an α - ψ - fuzzy $\mathscr{H}$ - contraction mapping.

Now, let $x,y \in X$ such that $\alpha(x,y,t) \geq 1$, $t > 0$, this implies that $x,y \in [0,1]$ and by the definitions of S and α, we have $Sx = \frac{1}{4} \in [0,1]$, $Sy = \frac{1}{4} \in [0,1]$ and $\alpha(Sx,Sy,t) = 2 \geq 1$, $t > 0$, i.e., S is α - admissible. Further, there exists $x_0 \in X$ such that $\alpha(x_0,Sx_0,t) \geq 1$, $t > 0$. Indeed for any $x_0 \in [0,1]$, we have $\alpha(x_0,Sx_0,t) = 2 \geq 1, t > 0$. Finally, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n,x_{n+1},t) \geq 1, n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$. From the definition of the function α, it follows that except in the case where $x_n \in [0,1], n \in \mathbb{N}$, in all the other cases the result holds easily. Suppose $x_n \in [0,1], n \in \mathbb{N}$ then $x \in [0,1]$. Therefore, $\alpha(x_n,x,t) \geq 1, n \in \mathbb{N}$. So, all the hypothesis of theorem 3.3 are satisfied. Here, $x = \frac{1}{4}$, 2 are two fixed points of S.

S is not fuzzy $\mathscr{H}$ - contractive mapping. To show this, for any $k \in (0,1)$, let us consider $x = 2$, $y = 1$ then $Sx = 2$, $Sy = \frac{1}{4}$, gives $\eta(M(Sx,Sy,t)) = \frac{7}{4} \not\leq k = k\eta(M(x,y,t)), t > 0$ since $k \in (0,1)$.

Theorem 3.5. *If the condition, " for all $u,v \in X$ and $t > 0$ there exists $p \in X$ such that $\alpha(u,p,t) \geq 1$ and $\alpha(v,p,t) \geq 1$" is added to the hypothesis of the theorem 3.3, we can obtain a unique fixed point of S in X.*

Proof. Suppose that u and v are two fixed points of S in X. If $\alpha(u,v,t) \geq 1$ for some $t > 0$ then by (3.1), we can easily see that $u = v$.
Now let us assume that $\alpha(u,v,t) < 1, t > 0$.
Then, by hypothesis there exists $p \in X$ such that

$$\alpha(u,p,t) \geq 1 \text{ and } \alpha(v,p,t) \geq 1, t > 0. \qquad (3.19)$$

Since S is α - admissible and by induction, we get

$$\alpha(u,S^n p,t) \geq 1; \ \alpha(v,S^n p,t) \geq 1 \qquad (3.20)$$

In view of (3.3), (3.20) and (3.1), we get

$$\eta(M(u,S^n p,t)) = \eta(M(Su,S(S^{n-1})p,t))$$
$$\leq \eta(M(Su,S(S^{n-1})p,kt))$$
$$= 1.\eta(M(Su,S(S^{n-1})p,kt))$$

$$\eta(M(u,S^n p,t)) \leq \alpha(u,S^{n-1}p,t)\eta(M(Su,S(S^{n-1})p,kt))$$

$$\eta(M(u,S^n p,t)) \leq \psi(\eta(N(u,S^{n-1}p))) \qquad (3.21)$$

where
$N(u,S^{n-1}p)$

$$= \max \begin{cases} M(u,S^{n-1}p,t), M(u,Su,t), \\ M(S^{n-1}p,S(S^{n-1})p,t), \\ \min\{M(u,S(S^{n-1})p,2t), M(S^{n-1}p,Su,2t)\} \end{cases}$$

$$= \max \begin{cases} M(u,S^{n-1}p,t), 1, M(S^{n-1}p,S(S^{n-1})p,t), \\ \min\{M(u,S(S^{n-1})p,2t), M(S^{n-1}p,Su,2t)\} \end{cases}$$

$$\implies N(u,S^{n-1}p) = 1 \text{ which gives } \eta(N(u,S^{n-1}p)) = 0$$

Thus, (3.21) gives $\eta(M(u,S^n p,t)) \leq \psi(0) = 0$.
Letting $n \to \infty$, we get $\lim_{n\to\infty} \eta(M(u,S^n p,t)) = 0 \implies \lim_{n\to\infty} S^n p = u$.
Similarly, we can show that $\lim_{n\to\infty} S^n p = v$ and from uniqueness of limit, we get $u = v$. $\square$

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