Some ranking indexes of stochastic orders and their applications

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Abstract

In this paper, we have recalled some of the known stochastic orders and the shifted version of them, and discussed their four relations and its properties. Also, we obtained some applications of proportional likelihood ratio ordering fuzzy hazard rate ordering and mean inactivity ordering and its applications.

Keywords

Fuzzy random variables, Fuzzy likelihood ratio order, Fuzzy Hazard rate order, Mean inactivity time order and their Shifted orders.

AMS Subject Classification

60E15, 62F07.

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1. Introduction

Stochastic orders have been proven to be very useful in applied probability, statistics, reliability, operation research, economics and other fields. Various types of stochastic orders and associate properties have been developed rapidly over the years. A lot of research works have done on, hazard rate and reversed hazard rate orders due to their properties and applications in the various sciences, for example hazard rate order is a well known and useful tool in reliability theory and reversed hazard rate order is defined via stochastic comparison of inactivity time. We can refer reader to the papers such as, Chandra and Roy [6], Gupta and Nanda [8], Nanda and Shaked [11], Kayid and Ahmad [10] and Shaked and Shanthikumar [13]. Ramos-Romero and Sordo-Diaz [12] introduced a new stochastic order between two absolutely continuous random variables and called it proportional Hazard Rate order (*PHR*) order, which is closely related to the usual Hazard Rate order. The proportional Hazard Rate order can be used to characterize random variables whose logarithms have log-concave (log-convex) densities. Many income random variables satisfy this property and they are said to have the increasing proportional Hazard Rate order (*IPHR*) and decreasing proportional Hazard Rate Order (*DPHR*) properties. As an application, they showed that the *IPHR* and *DPHR* properties are sufficient conditions for the Lorenz ordering of truncated distributions.

Jarrahiferiz et al. [9] studied some other properties of the proportional Hazard Rate Order, then extended hazard rate and reversed hazard rate orders to proportional state similar to proportional Hazard Rate order called them proportional (reversed) hazard rate orders, and studied their properties and relations.

Shifted stochastic orders that are useful tools for establishing interesting inequalities that have been introduced and studied. Also, they have been studied in detail four shifted stochastic orders, namely the up likelihood ratio order, the

down likelihood ratio order, the up hazard rate order and the down hazard rate order. For more details about this concept, we refer to the readers [2–5, 7, 14]. They have compared them and obtained some basic and closure properties of them and have shown how those can be used for stochastic comparisons of order statistics. Recently, Aboukalam and Kayid [1] obtained some new results about shifted hazard and shifted likelihood ratio orders. In this paper we recall the proportional state of stochastic orders and the shifted version of them and so obtained some applications of proportional Hazard Rate order.

2. Preliminaries

2.1 Fuzzy Numbers

Let X be a universal set and $S_X = \{x \varepsilon X : f(x; \theta) > 0\}$ be the support of X A fuzzy subset (briefly, a fuzzy set) $\tilde{x}$ of S_X is defined by its membership function $\mu_{\tilde{x}} : S_X \to [0,1]$. We denote the α–cuts of $\tilde{x}$ by $\tilde{x}_\alpha = \{x : \mu_{\tilde{x}}(x) \geqslant \alpha$, and $\tilde{x}_0$ is the closure of the set $\{x : \mu_{\tilde{x}}(x) > 0\}$. then $f(x; \theta)$ is a fuzzy set.

The fuzzy set $\tilde{x}$ is called a normal fuzzy set if there exists $x \in S_X$ such that $\mu_{\tilde{x}}(x) = 1$, and called convex fuzzy sets $\mu_{\tilde{x}}(\lambda x + (1 - \lambda)y) \geqslant \min\{\mu_{\tilde{x}}(x), \mu_{\tilde{x}}(y)\}$ for every $xy \in S_X$ and $\lambda \in [0,1]$. The fuzzy set $\tilde{x}$ is called fuzzy number if it is normal and convex fuzzy set and its α–cuts are bounded for all $\alpha \in [0,1]$. In addition, if $\tilde{x}$ is a fuzzy number and the support of its membership function $\mu_{\tilde{x}}$ is compact, then we called $\tilde{x}$ as a bounded fuzzy numbers.

If $\tilde{x}$ is a closed and bounded fuzzy number with $\tilde{x}_\alpha^L = \min\{x : x \in \tilde{x}_\alpha\}$ and $\tilde{x}_\alpha^U = \max\{x : x \in \tilde{x}_\alpha\}$ and its membership function be strictly increasing on the interval $[x_\alpha^L x_1^L]$ and strictly decreasing on the interval $[x_\alpha^U x_1^U]$, then $\tilde{x}$ is called a canonical fuzzy number.

2.2 Fuzzy Random Variable

The fuzzy number $\tilde{x}$ with membership function $\mu_{\tilde{x}}(r)$ can be induced by any real number $x \in S_X$ such that $\mu_{\tilde{x}}(x) = 1$ and $\mu_{\tilde{x}}(r) < 1$ for $r \neq x$. We denote the set of all fuzzy real numbers induced by real number $x \in S_X$ by F(S_X).

The relation $\sim$ on F(S_X) define as $\tilde{x}_1 \sim \tilde{x}_2$ if and only if $\tilde{x}_1$ and $\tilde{x}_2$ are induced by the same real number x. Then $\sim$ is an equivalence relation, which induce the equivalence classes $[\tilde{x}] = \{\tilde{a}\tilde{a} \sim \tilde{x}\}$. The set (F($S_X$)/$\sim$) called a fuzzy real number system. In practice, we take only one element $\tilde{x}$ from each equivalence class $[\tilde{x}]$ to form the fuzzy real number system (F(S_X)/$\sim$). If the fuzzy real number system (F(S_X)/$\sim$) consists all of the canonical fuzzy real numbers, then we call (F(S_X)/$\sim$) as the canonical fuzzy real number system.

Let X be a random variable with support S_X and $\mathscr{F}(S_X)$ is the set of all canonical fuzzy numbers induce the real numbers in S_X. A fuzzy random variable is a function $X : \psi \to \mathscr{F}(S_X)$ where for all $\alpha \in [0,1]$.

$$\{(\omega, x) : \omega \in \psi, x \in \tilde{X}_\alpha(\omega)\} \in f \times B$$

Noting that F(S_X) is the support of the fuzzy random variable

$\tilde{X}$ and hence, each α–cut set of $\tilde{X}$ depends on the random variable X.

Definition 2.1. *Let $F(R)$ be a canonical fuzzy real number system. Then $\tilde{X}$ is a fuzzy random variables if and only if $\tilde{X}_\alpha^L$ and $\tilde{X}_\alpha^U$ are ordinary random variables for all $\alpha \in [0,1]$.*

Let X be a non negative fuzzy random variable with density function $f(\tilde{x})$ and cumulative distribution function $\bar{F}(\tilde{x})$ respectively, and $\tilde{x}$ be a fuzzy random variables induced by X. The fuzzy function $\tilde{\mu}_{\tilde{f}}(\tilde{x})$ is a likelihood ratio order of fuzzy random variables $\tilde{x}$, whenever its membership function is given by

$$\mu_{\tilde{f}\tilde{x}}(y) = \sup_{0 \leqslant \alpha \leqslant 1} \alpha^I \tilde{r}(\tilde{x})_\alpha^L, \tilde{r}(\tilde{x})_\alpha^U(y)$$

where,

$$f(\tilde{x})_\alpha^L = \min \left\{ \left\{ \min_{\alpha \leqslant \beta \leqslant 1} f(\tilde{x}) ; x = \tilde{x}_\beta^L \right\}, \right.$$
$$\left. \left\{ \min_{\alpha \leqslant \beta \leqslant 1} f(\tilde{x}) ; x = \tilde{x}_\beta^U \right\} \right\},$$
$$f(\tilde{x})_\alpha^U = \max \left\{ \left\{ \max_{\alpha \leqslant \beta \leqslant 1} f(\tilde{x}) ; x = \tilde{x}_\beta^L \right\}, \right.$$
$$\left. \left\{ \max_{\alpha \leqslant \beta \leqslant 1} f(\tilde{x}) ; x = \tilde{x}_\beta^U \right\} \right\},$$

Such that the interval, $f(\tilde{x})_\alpha^L$ and $f(\tilde{x})_\alpha^U$ will contain all of the cumulative distribution function for $\beta \geqslant \alpha$.

3. Fuzzy Likelihood Ratio Order

Let X and Y are continuous non negative fuzzy random variables with density functions f and g respectively, we propose four relations to compare fuzzy likelyhood ratio order of X and Y are as follows.

1. $X \leqslant_{FLR1} Y$ if,

$$\min \left\{ \min_{\alpha \leqslant \beta \leqslant 1} f(\tilde{x})_\alpha^L, \min_{\alpha \leqslant \beta \leqslant 1} g(\tilde{x})_\alpha^L \right\}$$
$$\leqslant \min \left\{ \min_{\alpha \leqslant \beta \leqslant 1} f(\tilde{y})_\alpha^U, \min_{\alpha \leqslant \beta \leqslant 1} g(\tilde{y})_\alpha^U \right\},$$

and

$$\max \left\{ \max_{\alpha \leqslant \beta \leqslant 1} f(\tilde{x})_\alpha^L, \max_{\alpha \leqslant \beta \leqslant 1} g(\tilde{x})_\alpha^L \right\}$$
$$\leqslant \max \left\{ \max_{\alpha \leqslant \beta \leqslant 1} f(\tilde{y})_\alpha^U, \max_{\alpha \leqslant \beta \leqslant 1} g(\tilde{y})_\alpha^U \right\}$$

2. $X \leqslant_{FLR2} Y$ if,

$$\min \left\{ \min_{\alpha \leqslant \beta \leqslant 1} f(\tilde{x})_\alpha^L, \min_{\alpha \leqslant \beta \leqslant 1} g(\tilde{x})_\alpha^U \right\}$$
$$\leqslant \min \left\{ \min_{\alpha \leqslant \beta \leqslant 1} f(\tilde{y})_\alpha^L, \min_{\alpha \leqslant \beta \leqslant 1} g(\tilde{y})_\alpha^U \right\},$$

and

$$\max\left\{\max_{\alpha\leqslant\beta\leqslant1}f(\tilde{x})_\alpha^L, \max_{\alpha\leqslant\beta\leqslant1}g(\tilde{x})_\alpha^U\right\}$$
$$\leqslant\max\left\{\max_{\alpha\leqslant\beta\leqslant1}f(\tilde{y})_\alpha^L, \max_{\alpha\leqslant\beta\leqslant1}g(\tilde{y})_\alpha^U\right\}$$

3. $X\leqslant_{FLR3}Y$ if,

$$\min\left\{\min_{\alpha\leqslant\beta\leqslant1}f(\tilde{x})_\alpha^U, \min_{\alpha\leqslant\beta\leqslant1}g(\tilde{x})_\alpha^U\right\}$$
$$\leqslant\min\left\{\min_{\alpha\leqslant\beta\leqslant1}f(\tilde{y})_\alpha^U, \min_{\alpha\leqslant\beta\leqslant1}g(\tilde{y})_\alpha^U\right\},$$

and

$$\max\left\{\max_{\alpha\leqslant\beta\leqslant1}f(\tilde{x})_\alpha^U, \max_{\alpha\leqslant\beta\leqslant1}g(\tilde{x})_\alpha^U\right\}$$
$$\leqslant\max\left\{\max_{\alpha\leqslant\beta\leqslant1}f(\tilde{y})_\alpha^U, \max_{\alpha\leqslant\beta\leqslant1}g(\tilde{y})_\alpha^U\right\}$$

4. $X\leqslant_{FLR4}Y$ if,

$$\min\left\{\min_{\alpha\leqslant\beta\leqslant1}f(\tilde{x})_\alpha^L, \min_{\alpha\leqslant\beta\leqslant1}g(\tilde{x})_\alpha^L\right\}$$
$$\leqslant\min\left\{\min_{\alpha\leqslant\beta\leqslant1}f(\tilde{y})_\alpha^L, \min_{\alpha\leqslant\beta\leqslant1}g(\tilde{y})_\alpha^L\right\},$$

and

$$\max\left\{\max_{\alpha\leqslant\beta\leqslant1}f(\tilde{x})_\alpha^L, \max_{\alpha\leqslant\beta\leqslant1}g(\tilde{x})_\alpha^L\right\}$$
$$\leqslant\max\left\{\max_{\alpha\leqslant\beta\leqslant1}f(\tilde{y})_\alpha^L, \max_{\alpha\leqslant\beta\leqslant1}g(\tilde{y})_\alpha^L\right\}$$

for all $X\leq Y$, $\alpha\in[0,1]$.

3.1 Up Proportional Fuzzy Likelihood Ratio Order

Suppose that X and Y are two continuous non negative fuzzy random variables with density functions f and g respectively, we propose four relations to compare Up proportional fuzzy likelyhood ratio order of X and Y are follows.

1. $X\leqslant_{PFLR1}Y$ if,
$$\min\left\{\min_{\alpha\leqslant\beta\leqslant1}f[\lambda\tilde{x}_\alpha^L-t/\lambda\tilde{x}_\alpha^L\geqslant t], \min_{\alpha\leqslant\beta\leqslant1}g[\lambda\tilde{x}_\alpha^L-t/\lambda\tilde{x}_\alpha^L\geqslant t]\right\}$$
$$\geqslant\min\left\{\min_{\alpha\leqslant\beta\leqslant1}f(\tilde{y})_\alpha^U, \min_{\alpha\leqslant\beta\leqslant1}g(\tilde{y})_\alpha^U\right\},$$

and

$$\max\left\{\max_{\alpha\leqslant\beta\leqslant1}f[\lambda\tilde{x}_\alpha^L-t/\lambda\tilde{x}_\alpha^L\geqslant t], \min_{\alpha\leqslant\beta\leqslant1}g[\lambda\tilde{x}_\alpha^L-t/\lambda\tilde{x}_\alpha^L\geqslant t]\right\}$$
$$\geqslant\max\left\{\max_{\alpha\leqslant\beta\leqslant1}f(\tilde{y})_\alpha^U, \max_{\alpha\leqslant\beta\leqslant1}g(\tilde{y})_\alpha^U\right\}.$$

2. $X\leqslant_{PFLR2}Y$ if,
$$\min\left\{\min_{\alpha\leqslant\beta\leqslant1}f[\lambda\tilde{x}_\alpha^L-t/\lambda\tilde{x}_\alpha^L\geqslant t], \min_{\alpha\leqslant\beta\leqslant1}g[\lambda\tilde{x}_\alpha^U-t/\lambda\tilde{x}_\alpha^U\geqslant t]\right\}$$
$$\geqslant\min\left\{\min_{\alpha\leqslant\beta\leqslant1}f(\tilde{y})_\alpha^L, \min_{\alpha\leqslant\beta\leqslant1}g(\tilde{y})_\alpha^U\right\},$$

and

$$\max\left\{\max_{\alpha\leqslant\beta\leqslant1}f[\lambda\tilde{x}_\alpha^L-t/\lambda\tilde{x}_\alpha^L\geqslant t], \min_{\alpha\leqslant\beta\leqslant1}g[\lambda\tilde{x}_\alpha^U-t/\lambda\tilde{x}_\alpha^U\geqslant t]\right\}$$
$$\geqslant\max\left\{\max_{\alpha\leqslant\beta\leqslant1}f(\tilde{y})_\alpha^L, \max_{\alpha\leqslant\beta\leqslant1}g(\tilde{y})_\alpha^U\right\}.$$

3. $X\leqslant_{PFLR3}Y$ if,
$$\min\left\{\min_{\alpha\leqslant\beta\leqslant1}f[\lambda\tilde{x}_\alpha^U-t/\lambda\tilde{x}_\alpha^U\geqslant t], \min_{\alpha\leqslant\beta\leqslant1}g[\lambda\tilde{x}_\alpha^U-t/\lambda\tilde{x}_\alpha^U\geqslant t]\right\}$$
$$\geqslant\min\left\{\min_{\alpha\leqslant\beta\leqslant1}f(\tilde{y})_\alpha^U, \min_{\alpha\leqslant\beta\leqslant1}g(\tilde{y})_\alpha^U\right\},$$

and

$$\max\left\{\max_{\alpha\leqslant\beta\leqslant1}f[\lambda\tilde{x}_\alpha^U-t/\lambda\tilde{x}_\alpha^U\geqslant t],\ \min_{\alpha\leqslant\beta\leqslant1}g[\lambda\tilde{x}_\alpha^U-t/\lambda\tilde{x}_\alpha^U\geqslant t]\right\}$$

$$\geqslant\max\left\{\max_{\alpha\leqslant\beta\leqslant1}f(\tilde{y})_\alpha^U,\ \max_{\alpha\leqslant\beta\leqslant1}g(\tilde{y})_\alpha^U\right\}.$$

4. $X\leqslant_{PFLR3}Y$ if,

$$\min\left\{\min_{\alpha\leqslant\beta\leqslant1}f[\lambda\tilde{x}_\alpha^L-t/\lambda\tilde{x}_\alpha^L\geqslant t],\ \min_{\alpha\leqslant\beta\leqslant1}g[\lambda\tilde{x}_\alpha^L-t/\lambda\tilde{x}_\alpha^L\geqslant t]\right\}$$

$$\geqslant\min\left\{\min_{\alpha\leqslant\beta\leqslant1}f(\tilde{y})_\alpha^L,\ \min_{\alpha\leqslant\beta\leqslant1}g(\tilde{y})_\alpha^L\right\}$$

and

$$\max\left\{\max_{\alpha\leqslant\beta\leqslant1}f[\lambda\tilde{x}_\alpha^L-t/\lambda\tilde{x}_\alpha^L\geqslant t],\ \min_{\alpha\leqslant\beta\leqslant1}g[\lambda\tilde{x}_\alpha^L-t/\lambda\tilde{x}_\alpha^L\geqslant t]\right\}$$

$$\geqslant\max\left\{\max_{\alpha\leqslant\beta\leqslant1}f(\tilde{y})_\alpha^L,\ \max_{\alpha\leqslant\beta\leqslant1}g(\tilde{y})_\alpha^L\right\}$$

for all $X\leq Y$, $\alpha\in[0,1]$.

3.2 Down Proportional Likelihood Ratio Order

Suppose that X and Y are two continuous non negative fuzzy random variables with density functions f and g respectively, we propose four relations to compare Down proportional fuzzy likelyhood ratio order of X and Y are follows.

1. $X\leqslant_{PFLR1}Y$ if,

$$\min\left\{\min_{\alpha\leqslant\beta\leqslant1}f(\tilde{x})_\alpha^L,\ \min_{\alpha\leqslant\beta\leqslant1}g(\tilde{x})_\alpha^L\right\}$$

$$\leqslant\min\left\{\min_{\alpha\leqslant\beta\leqslant1}f[\lambda\tilde{y}_\alpha^U-t/\lambda\tilde{y}_\alpha^U\geqslant t]\ \min_{\alpha\leqslant\beta\leqslant1}g[\lambda\tilde{y}_\alpha^U-t/\lambda\tilde{y}_\alpha^U\geqslant t]\right\},$$

and

$$\max\left\{\max_{\alpha\leqslant\beta\leqslant1}f(\tilde{x})_\alpha^L,\ \max_{\alpha\leqslant\beta\leqslant1}g(\tilde{x})_\alpha^L\right\}$$

$$\leqslant\min\left\{\max_{\alpha\leqslant\beta\leqslant1}f[\lambda\tilde{y}_\alpha^U-t/\lambda\tilde{y}_\alpha^U\geqslant t]\ \max_{\alpha\leqslant\beta\leqslant1}g[\lambda\tilde{y}_\alpha^U-t/\lambda\tilde{y}_\alpha^U\geqslant t]\right\}.$$

2. $X\leqslant_{PFLR2}Y$ if,

$$\min\left\{\min_{\alpha\leqslant\beta\leqslant1}f(\tilde{x})_\alpha^L,\ \min_{\alpha\leqslant\beta\leqslant1}g(\tilde{x})_\alpha^U\right\}$$

$$\leqslant\min\left\{\min_{\alpha\leqslant\beta\leqslant1}f[\lambda\tilde{y}_\alpha^L-t/\lambda\tilde{y}_\alpha^L\geqslant t]\ \min_{\alpha\leqslant\beta\leqslant1}g[\lambda\tilde{y}_\alpha^U-t/\lambda\tilde{y}_\alpha^U\geqslant t]\right\},$$

and

$$\max\left\{\max_{\alpha\leqslant\beta\leqslant1}f(\tilde{x})_\alpha^L,\ \max_{\alpha\leqslant\beta\leqslant1}g(\tilde{x})_\alpha^U\right\}$$

$$\leqslant\min\left\{\max_{\alpha\leqslant\beta\leqslant1}f[\lambda\tilde{y}_\alpha^L-t/\lambda\tilde{y}_\alpha^L\geqslant t]\ \max_{\alpha\leqslant\beta\leqslant1}g[\lambda\tilde{y}_\alpha^U-t/\lambda\tilde{y}_\alpha^U\geqslant t]\right\}.$$

3. $X\leqslant_{PFLR3}Y$ if,

$$\min\left\{\min_{\alpha\leqslant\beta\leqslant1}f(\tilde{x})_\alpha^U,\ \min_{\alpha\leqslant\beta\leqslant1}g(\tilde{x})_\alpha^U\right\}$$

$$\leqslant\min\left\{\min_{\alpha\leqslant\beta\leqslant1}f[\lambda\tilde{y}_\alpha^U-t/\lambda\tilde{y}_\alpha^U\geqslant t]\ \min_{\alpha\leqslant\beta\leqslant1}g[\lambda\tilde{y}_\alpha^U-t/\lambda\tilde{y}_\alpha^U\geqslant t]\right\},$$

and

$$\max\left\{\max_{\alpha\leqslant\beta\leqslant1}f(\tilde{x})_\alpha^U,\ \max_{\alpha\leqslant\beta\leqslant1}g(\tilde{x})_\alpha^U\right\}$$

$$\leqslant \min\left\{\max_{\alpha\leqslant\beta\leqslant1} f[\lambda\tilde{y}_\alpha^U - t/\lambda\tilde{y}_\alpha^U \geqslant t]\ \max_{\alpha\leqslant\beta\leqslant1} g[\lambda\tilde{y}_\alpha^U - t/\lambda\tilde{y}_\alpha^U \geqslant t]\right\}.$$

4. $X \leqslant_{PFLR4} Y$ if,

$$\min\left\{\min_{\alpha\leqslant\beta\leqslant1} f(\tilde{x})_\alpha^L,\ \min_{\alpha\leqslant\beta\leqslant1} g(\tilde{x})_\alpha^L\right\}$$

$$\leqslant \min\left\{\min_{\alpha\leqslant\beta\leqslant1} f[\lambda\tilde{y}_\alpha^L - t/\lambda\tilde{y}_\alpha^L \geqslant t]\ \min_{\alpha\leqslant\beta\leqslant1} g[\lambda\tilde{y}_\alpha^L - t/\lambda\tilde{y}_\alpha^L \geqslant t]\right\},$$

and

$$\max\left\{\max_{\alpha\leqslant\beta\leqslant1} f(\tilde{x})_\alpha^L,\ \max_{\alpha\leqslant\beta\leqslant1} g(\tilde{x})_\alpha^L\right\}$$

$$\leqslant \min\left\{\max_{\alpha\leqslant\beta\leqslant1} f[\lambda\tilde{y}_\alpha^L - t/\lambda\tilde{y}_\alpha^L \geqslant t]\ \max_{\alpha\leqslant\beta\leqslant1} g[\lambda\tilde{y}_\alpha^L - t/\lambda\tilde{y}_\alpha^L \geqslant t]\right\}$$

$$\text{for all } X \leq Y,\ \alpha \in [0,1].$$

Theorem 3.1. *The two fuzzy random variables X and Y satisfies $X \leqslant_{PFLRi} Y$ if and only if $X \leqslant_{FLRi} aY$ for all > 1. (i = 1, 2, 3, 4).*

Proof. Suppose that $X \leqslant_{PFLR2} Y$. Thus we have that,

$$\min\left\{\min_{\alpha\leqslant\beta\leqslant1} f\left[(a)\tilde{y}_\alpha^L - t/(a)\tilde{y}_\alpha^L \geqslant t\right],\ \min_{\alpha\leqslant\beta\leqslant1} g[(a\mu)\tilde{y}_\alpha^U - t/(a)\tilde{y}_\alpha^U \geqslant t]\right\}$$

and

$$\max\left\{\max_{\alpha\leqslant\beta\leqslant1} f\left[(a)\tilde{y}_\alpha^L - t/(a)\tilde{y}_\alpha^L \geqslant t\right],\ \max_{\alpha\leqslant\beta\leqslant1} g[(a)\tilde{y}_\alpha^U - t/(a)\tilde{y}_\alpha^U \geqslant t]\right\}$$

is equal to

$$\min\left\{\min_{\alpha\leqslant\beta\leqslant1} f\left[(1/a)\tilde{y}_\alpha^L - t/(1/a)\tilde{y}_\alpha^L \geqslant t\right],\ \min_{\alpha\leqslant\beta\leqslant1} g[(1/a)\tilde{y}_\alpha^U - t/(1/a)\tilde{y}_\alpha^U \geqslant t]\right\}$$

and

$$\max\left\{\max_{\alpha\leqslant\beta\leqslant1} f\left[(1/a)\tilde{y}_\alpha^L - t/(1/a)\tilde{y}_\alpha^L \geqslant t\right],\ \max_{\alpha\leqslant\beta\leqslant1} g[(1/a)\tilde{y}_\alpha^U - t/(1/a)\tilde{y}_\alpha^U \geqslant t]\right\}$$

put $(1/a) = \lambda$, then equal to

$$\min\left\{\min_{\alpha\leqslant\beta\leqslant1} f[\lambda\tilde{y}_\alpha^L - t/\lambda\tilde{y}_\alpha^L \geqslant t],\ \min_{\alpha\leqslant\beta\leqslant1} g[\lambda\tilde{y}_\alpha^U - t/\lambda\tilde{y}_\alpha^U \geqslant t]\right\}$$

and

$$\max\left\{\max_{\alpha\leqslant\beta\leqslant1} f\left[\lambda\tilde{y}_\alpha^L - t/\lambda\tilde{y}_\alpha^L \geqslant t\right],\ \max_{\alpha\leqslant\beta\leqslant1} g[\lambda\tilde{y}_\alpha^U - t/\lambda\tilde{y}_\alpha^U \geqslant t]\right\}$$

since $X \leqslant_{PFLR2} Y$, then greater than or equal to

$$\min\left\{\min_{\alpha\leqslant\beta\leqslant1} f[\lambda\tilde{x}_\alpha^L - t/\lambda\tilde{x}_\alpha^L \geqslant t],\ \min_{\alpha\leqslant\beta\leqslant1} g[\lambda\tilde{x}_\alpha^U - t/\lambda\tilde{x}_\alpha^U \geqslant t]\right\}$$

and

$$\max\left\{\max_{\alpha\leqslant\beta\leqslant1} f\left[\lambda\tilde{x}_\alpha^L - t/\lambda\tilde{x}_\alpha^L \geqslant t\right],\ \max_{\alpha\leqslant\beta\leqslant1} g[\lambda\tilde{x}_\alpha^U - t/\lambda\tilde{x}_\alpha^U \geqslant t]\right\}$$

equal to

$$\min\left\{\min_{\alpha\leqslant\beta\leqslant1} f(a\tilde{x})_\alpha^L,\ \min_{\alpha\leqslant\beta\leqslant1} g(a\tilde{x})_\alpha^U\right\}$$

and

$$\max\left\{\max_{\alpha\leqslant\beta\leqslant1} f(a\tilde{x})_\alpha^L,\ \max_{\alpha\leqslant\beta\leqslant1} g(a\tilde{x})_\alpha^U\right\}.$$

which implies that $X \leqslant_{PFLR2} Y$. The similar proof are holds for the another ranking indexes. □

where,

$$\bar{F}(\tilde{x})^L_\alpha = \min\left\{ \left\{ \min_{\alpha\leqslant\beta\leqslant1} \bar{G}(\tilde{x}); x = \tilde{x}^L_\beta \right\}, \right.$$

$$\left. \left\{ \min_{\alpha\leqslant\beta\leqslant1} \bar{G}(\tilde{x}); x = \tilde{x}^L_\beta \right\} \right\},$$

$$\bar{F}(\tilde{x})^U_\alpha = \max\left\{ \left\{ \max_{\alpha\leqslant\beta\leqslant1} \bar{G}(\tilde{x}); x = \tilde{x}^L_\beta \right\}, \right.$$

$$\left. \left\{ \max_{\alpha\leqslant\beta\leqslant1} \bar{G}(\tilde{x}); x = \tilde{x}^L_\beta \right\} \right\},$$

Such that the interval, $\bar{F}(\tilde{x})^L_\alpha$ and $\bar{F}(\tilde{x})^U_\alpha$ will contain all of the reversed Hazard rate of each $\tilde{x}^L_\beta$ and $\tilde{x}^U_\beta$ for $\beta \geqslant \alpha$.

4. General Reversed Fuzzy Hazard Rate Order

Let X be a non negative fuzzy random variable with density function $f(\tilde{x})$ and cumulative distribution function $\bar{F}(\tilde{x})$ respectively, and $\tilde{x}$ be a fuzzy random variables induced by X. The fuzzy function $\tilde{r}_{\bar{F}}(\tilde{x})$ is a reversed Hazard rate of fuzzy random variables $\tilde{x}$, whenever its membership function is given by $\mu_{\tilde{r}}(y) = \sup \alpha^I \tilde{r}(\tilde{x})^L_\alpha, \tilde{r}(\tilde{x})^U_\alpha\}(y)$

1. $X \leqslant_{FRH1} Y$ if,$^{0\leqslant\alpha\leqslant1}$

$$\frac{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^L_\beta\right), \min\limits_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^U_\beta\right) \right\}}{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right), \min\limits_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^U_\beta\right) \right\}} \geqslant \frac{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^L_\beta\right), \min\limits_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^U_\beta\right) \right\}}{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^L_\beta\right), \min\limits_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^U_\beta\right) \right\}}$$

and

$$\frac{\max\left\{ \max\limits_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^L_\beta\right), \max\limits_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^U_\beta\right) \right\}}{\max\left\{ \max\limits_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right), \max\limits_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^U_\beta\right) \right\}} \geqslant \frac{\max\left\{ \max\limits_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^L_\beta\right), \max\limits_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^U_\beta\right) \right\}}{\max\left\{ \max\limits_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^L_\beta\right), \max\limits_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^U_\beta\right) \right\}}$$

2. $X \leqslant_{FRH2} Y$ if,

$$\frac{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^L_\beta\right), \min\limits_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^L_\beta\right) \right\}}{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right), \min\limits_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right) \right\}} \geqslant \frac{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^U_\beta\right), \min\limits_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^U_\beta\right) \right\}}{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^U_\beta\right), \min\limits_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(y^U_\beta\right) \right\}}$$

and

$$\frac{\max\left\{ \max\limits_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^L_\beta\right), \max\limits_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^L_\beta\right) \right\}}{\max\left\{ \max\limits_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right), \max\limits_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right) \right\}} \geqslant \frac{\max\left\{ \max\limits_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^U_\beta\right), \max\limits_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^U_\beta\right) \right\}}{\max\left\{ \max\limits_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^U_\beta\right), \max\limits_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^U_\beta\right) \right\}}$$

3. $X \leqslant_{FRH3} Y$ if,

$$\frac{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^L_\beta\right), \min\limits_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^L_\beta\right) \right\}}{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right), \min\limits_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right) \right\}} \geqslant \frac{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^L_\beta\right), \min\limits_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^L_\beta\right) \right\}}{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^L_\beta\right), \min\limits_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^L_\beta\right) \right\}}$$

and

$$\frac{\max\left\{ \max\limits_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^L_\beta\right), \max\limits_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^L_\beta\right) \right\}}{\max\left\{ \max\limits_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right), \max\limits_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right) \right\}} \geqslant \frac{\max\left\{ \max\limits_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^L_\beta\right), \max\limits_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^L_\beta\right) \right\}}{\max\left\{ \max\limits_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(y^L_\beta\right), \max\limits_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(y^L_\beta\right) \right\}}$$

4. $X \leqslant_{FRH4} Y$ if,

$$\frac{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^U_\beta\right), \min\limits_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^U_\beta\right) \right\}}{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right), \min\limits_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right) \right\}} \geqslant \frac{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^U_\beta\right), \min\limits_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^U_\beta\right) \right\}}{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^U_\beta\right), \min\limits_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^U_\beta\right) \right\}}$$

4.1 Fuzzy Hazard Rate Order

Let X and Y are two non negative fuzzy random variables with continuous distribution functions and with Reversed Hazard rate functions $\tilde{r}(\tilde{x})$ and $\tilde{q}(\tilde{x})$ respectively, then X is smaller than Y. We propose four relations

and

$$\frac{\max\left\{\max_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^U_\beta\right), \max_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^U_\beta\right)\right\}}{\max\left\{\max_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^U_\beta\right), \max_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^U_\beta\right)\right\}} \geqslant \frac{\max\left\{\max_{\alpha\leqslant\beta\leqslant1} g\left(y^U_\beta\right), \max_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^U_\beta\right)\right\}}{\max\left\{\max_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^U_\beta\right), \max_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^U_\beta\right)\right\}}$$

For each α, $\beta(0,1]\cap Q$, where $\bar{F} f$ are the survival and density functions of X respectively and $\bar{G}$, g are the survival and density functions of Y respectively.

4.2 Fuzzy Reversed Hazard Rate Order

Let X and Y are two non negative fuzzy random variables with continuous distribution functions and with Reversed Hazard rate functions $\tilde{r}(\tilde{x})$ and $\tilde{q}(\tilde{x})$ respectively, then X is smaller than Y. We propose four relations

1. $X \leqslant_{FRH1} Y$ if,

$$\frac{\min\left\{\min_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^L_\beta\right), \min_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^U_\beta\right)\right\}}{\min\left\{\min_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right), \min_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^U_\beta\right)\right\}} \leqslant \frac{\min\left\{\min_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^L_\beta\right), \min_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^U_\beta\right)\right\}}{\min\left\{\min_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^L_\beta\right), \min_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^U_\beta\right)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^L_\beta\right), \max_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^U_\beta\right)\right\}}{\max\left\{\max_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right), \max_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^U_\beta\right)\right\}} \leqslant \frac{\max\left\{\max_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^L_\beta\right), \max_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^U_\beta\right)\right\}}{\max\left\{\max_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(y^L_\beta\right), \max_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^U_\beta\right)\right\}}$$

2. $X \leqslant_{FRH2} Y$ if,

$$\frac{\min\left\{\min_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^L_\beta\right), \min_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^L_\beta\right)\right\}}{\min\left\{\min_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right), \min_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right)\right\}} \leqslant \frac{\min\left\{\min_{\alpha\leqslant\beta\leqslant1} g\left(y^U_\beta\right), \min_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^U_\beta\right)\right\}}{\min\left\{\min_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^U_\beta\right), \min_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^U_\beta\right)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^L_\beta\right), \max_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^L_\beta\right)\right\}}{\max\left\{\max_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right), \max_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right)\right\}} \leqslant \frac{\max\left\{\max_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^U_\beta\right), \max_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^U_\beta\right)\right\}}{\max\left\{\max_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^U_\beta\right), \max_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^U_\beta\right)\right\}}$$

3. $X \leqslant_{FRH3} Y$ if,

$$\frac{\min\left\{\min_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^L_\beta\right), \min_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^L_\beta\right)\right\}}{\min\left\{\min_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right), \min_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right)\right\}} \leqslant \frac{\min\left\{\min_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^L_\beta\right), \min_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^L_\beta\right)\right\}}{\min\left\{\min_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^L_\beta\right), \min_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^L_\beta\right)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^L_\beta\right), \max_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^L_\beta\right)\right\}}{\max\left\{\max_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right), \max_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right)\right\}} \leqslant \frac{\max\left\{\max_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^L_\beta\right), \max_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^L_\beta\right)\right\}}{\max\left\{\max_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^L_\beta\right), \max_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^L_\beta\right)\right\}}$$

4. $X \leqslant_{FRH4} Y$ if,

$$\frac{\min\left\{\min_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^U_\beta\right), \min_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^U_\beta\right)\right\}}{\min\left\{\min_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right), \min_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right)\right\}} \leqslant \frac{\min\left\{\min_{\alpha\leqslant\beta\leqslant1} g\left(y^U_\beta\right), \min_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^U_\beta\right)\right\}}{\min\left\{\min_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^U_\beta\right), \min_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^U_\beta\right)\right\}}$$

And

$$\frac{\max\left\{\max_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^U_\beta\right), \max_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^U_\beta\right)\right\}}{\max\left\{\max_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^U_\beta\right), \max_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^U_\beta\right)\right\}} \leqslant \frac{\max\left\{\max_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^U_\beta\right), \max_{\alpha\leqslant\beta\leqslant1} g\left(\tilde{y}^U_\beta\right)\right\}}{\max\left\{\max_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^U_\beta\right), \max_{\alpha\leqslant\beta\leqslant1} \bar{G}\left(\tilde{y}^U_\beta\right)\right\}}$$

For each $\alpha, \beta (0,1] \cap Q$, where $\bar{F} f$ are the survival and density functions of X respectively and $\bar{G}, g$ are the survival and density functions of Y respectively.

Lemma 4.1. *A continuous non negative fuzzy random variables X admits Up increasing fuzzy proportional Hazard rate order propose four property denoted by*

1. $X \in UIPHRO1$ *if,*

$$\frac{\min\left\{\min\limits_{\alpha \leqslant \beta \leqslant 1}[f(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t], \min\limits_{\alpha \leqslant \beta \leqslant 1}[f(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t]\right\}}{\min\left\{\min\limits_{\alpha \leqslant \beta \leqslant 1}[\bar{F}(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t], \min\limits_{\alpha \leqslant \beta \leqslant 1}[\bar{F}(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t]\right\}} \geqslant \frac{\min\left\{\min\limits_{\alpha \leqslant \beta \leqslant 1} f\left(\tilde{x}_\beta^L\right), \min\limits_{\alpha \leqslant \beta \leqslant 1} f\left(\tilde{x}_\beta^U\right)\right\}}{\min\left\{\min\limits_{\alpha \leqslant \beta \leqslant 1} \bar{F}\left(\tilde{y}_\beta^L\right), \min\limits_{\alpha \leqslant \beta \leqslant 1} \bar{F}\left(\tilde{x}_\beta^U\right)\right\}}$$

and

$$\frac{\max\left\{\max\limits_{\alpha \leqslant \beta \leqslant 1}[f(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t], \max\limits_{\alpha \leqslant \beta \leqslant 1}[f(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t]\right\}}{\max\left\{\max\limits_{\alpha \leqslant \beta \leqslant 1}[\bar{F}(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t], \max\limits_{\alpha \leqslant \beta \leqslant 1}[\bar{F}(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t]\right\}} \geqslant \frac{\max\left\{\max\limits_{\alpha \leqslant \beta \leqslant 1} f\left(\tilde{x}_\beta^L\right), \max\limits_{\alpha \leqslant \beta \leqslant 1} f\left(\tilde{x}_\beta^U\right)\right\}}{\max\left\{\max\limits_{\alpha \leqslant \beta \leqslant 1} \bar{F}\left(\tilde{x}_\beta^L\right), \max\limits_{\alpha \leqslant \beta \leqslant 1} \bar{F}\left(\tilde{x}_\beta^U\right)\right\}}$$

2. $X \in UIPHRO2$ *if,*

$$\frac{\min\left\{\min\limits_{\alpha \leqslant \beta \leqslant 1}[f(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t], \min\limits_{\alpha \leqslant \beta \leqslant 1}[f(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t]\right\}}{\min\left\{\min\limits_{\alpha \leqslant \beta \leqslant 1}[\bar{F}(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t], \min\limits_{\alpha \leqslant \beta \leqslant 1}[\bar{F}(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t]\right\}} \geqslant \frac{\min\left\{\min\limits_{\alpha \leqslant \beta \leqslant 1} f\left(\tilde{x}_\beta^U\right), \min\limits_{\alpha \leqslant \beta \leqslant 1} f\left(\tilde{x}_\beta^U\right)\right\}}{\min\left\{\min\limits_{\alpha \leqslant \beta \leqslant 1} \bar{F}\left(\tilde{y}_\beta^U\right), \min\limits_{\alpha \leqslant \beta \leqslant 1} \bar{F}\left(\tilde{x}_\beta^U\right)\right\}}$$

and

$$\frac{\max\left\{\max\limits_{\alpha \leqslant \beta \leqslant 1}[f(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t], \max\limits_{\alpha \leqslant \beta \leqslant 1}[f(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t]\right\}}{\max\left\{\max\limits_{\alpha \leqslant \beta \leqslant 1}[\bar{F}(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t], \max\limits_{\alpha \leqslant \beta \leqslant 1}[\bar{F}(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t]\right\}} \geqslant \frac{\max\left\{\max\limits_{\alpha \leqslant \beta \leqslant 1} f\left(\tilde{x}_\beta^U\right), \max\limits_{\alpha \leqslant \beta \leqslant 1} f\left(\tilde{x}_\beta^U\right)\right\}}{\max\left\{\max\limits_{\alpha \leqslant \beta \leqslant 1} \bar{F}\left(\tilde{x}_\beta^U\right), \max\limits_{\alpha \leqslant \beta \leqslant 1} \bar{F}\left(\tilde{x}_\beta^U\right)\right\}}$$

3. $X \in UIPHRO3$ *if,*

$$\frac{\min\left\{\min\limits_{\alpha \leqslant \beta \leqslant 1}[f(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t], \min\limits_{\alpha \leqslant \beta \leqslant 1}[f(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t]\right\}}{\min\left\{\min\limits_{\alpha \leqslant \beta \leqslant 1}[\bar{F}(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t], \min\limits_{\alpha \leqslant \beta \leqslant 1}[\bar{F}(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t]\right\}} \geqslant \frac{\min\left\{\min\limits_{\alpha \leqslant \beta \leqslant 1} f\left(\tilde{x}_\beta^L\right), \min\limits_{\alpha \leqslant \beta \leqslant 1} f\left(\tilde{x}_\beta^L\right)\right\}}{\min\left\{\min\limits_{\alpha \leqslant \beta \leqslant 1} \bar{F}\left(\tilde{y}_\beta^L\right), \min\limits_{\alpha \leqslant \beta \leqslant 1} \bar{F}\left(\tilde{x}_\beta^L\right)\right\}}$$

and

$$\frac{\max\left\{\max\limits_{\alpha \leqslant \beta \leqslant 1}[f(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t], \max\limits_{\alpha \leqslant \beta \leqslant 1}[f(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t]\right\}}{\max\left\{\max\limits_{\alpha \leqslant \beta \leqslant 1}[\bar{F}(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t], \max\limits_{\alpha \leqslant \beta \leqslant 1}[\bar{F}(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t]\right\}} \geqslant \frac{\max\left\{\max\limits_{\alpha \leqslant \beta \leqslant 1} f\left(\tilde{x}_\beta^L\right), \max\limits_{\alpha \leqslant \beta \leqslant 1} f\left(\tilde{x}_\beta^L\right)\right\}}{\max\left\{\max\limits_{\alpha \leqslant \beta \leqslant 1} \bar{F}\left(\tilde{x}_\beta^L\right), \max\limits_{\alpha \leqslant \beta \leqslant 1} \bar{F}\left(\tilde{x}_\beta^L\right)\right\}}$$

4. $X \in UIPHRO4$ *if,*

$$\frac{\min\left\{\min\limits_{\alpha \leqslant \beta \leqslant 1}[f(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t], \min\limits_{\alpha \leqslant \beta \leqslant 1}[f(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t]\right\}}{\min\left\{\min\limits_{\alpha \leqslant \beta \leqslant 1}[\bar{F}(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t], \min\limits_{\alpha \leqslant \beta \leqslant 1}[\bar{F}(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t]\right\}} \geqslant \frac{\min\left\{\min\limits_{\alpha \leqslant \beta \leqslant 1} f\left(\tilde{x}_\beta^U\right), \min\limits_{\alpha \leqslant \beta \leqslant 1} f\left(\tilde{x}_\beta^U\right)\right\}}{\min\left\{\min\limits_{\alpha \leqslant \beta \leqslant 1} \bar{F}\left(\tilde{y}_\beta^U\right), \min\limits_{\alpha \leqslant \beta \leqslant 1} \bar{F}\left(\tilde{x}_\beta^U\right)\right\}}$$

and

$$\frac{\max\left\{\max\limits_{\alpha \leqslant \beta \leqslant 1}[f(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t], \max\limits_{\alpha \leqslant \beta \leqslant 1}[f(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t]\right\}}{\max\left\{\max\limits_{\alpha \leqslant \beta \leqslant 1}[\bar{F}(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t], \max\limits_{\alpha \leqslant \beta \leqslant 1}[\bar{F}(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t]\right\}} \geqslant \frac{\min\left\{\min\limits_{\alpha \leqslant \beta \leqslant 1} f\left(\tilde{x}_\beta^U\right), \min\limits_{\alpha \leqslant \beta \leqslant 1} f\left(\tilde{x}_\beta^U\right)\right\}}{\min\left\{\min\limits_{\alpha \leqslant \beta \leqslant 1} \bar{F}\left(\tilde{y}_\beta^U\right), \min\limits_{\alpha \leqslant \beta \leqslant 1} \bar{F}\left(\tilde{x}_\beta^U\right)\right\}}$$

The next theorem gives the relationship between the fuzzy Likelihood ratio and fuzzy Reversed Hazard rate orders.

Theorem 4.2. *Suppose that X and Y are two non negative fuzzy random variables with fuzzy cumulative distributions functions $\bar{F}$ and $\bar{G}$, and also fuzzy Reversed Hazard rate order functions f and g respectively. The fuzzy likelihood ratio ordering is stronger than the fuzzy Reversed Hazard rate ordering.*

Proof. Suppose that $X \leqslant_{FLR1} Y$. Then for all $\tilde{x} \leqslant \tilde{y}$, we can write

$$\min\left\{\min_{\alpha\leqslant\beta\leqslant1} f(\tilde{x})_\alpha^U, \min_{\alpha\leqslant\beta\leqslant1} g(\tilde{x})_\alpha^U\right\} \leqslant \min\left\{\min_{\alpha\leqslant\beta\leqslant1} f(\tilde{y})_\alpha^U, \min_{\alpha\leqslant\beta\leqslant1} g(\tilde{y})_\alpha^U\right\}$$

and

$$\max\left\{\max_{\alpha\leqslant\beta\leqslant1} f(\tilde{x})_\alpha^U, \max_{\alpha\leqslant\beta\leqslant1} g(\tilde{x})_\alpha^U\right\} \leqslant \max\left\{\max_{\alpha\leqslant\beta\leqslant1} f(\tilde{y})_\alpha^U, \max_{\alpha\leqslant\beta\leqslant1} g(\tilde{y})_\alpha^U\right\}$$

$$= \frac{\min\left\{\min_{\alpha\leqslant\beta\leqslant1}\bar{F}\left(\tilde{x}_\beta^U\right), \min_{\alpha\leqslant\beta\leqslant1}\bar{F}\left(\tilde{x}_\beta^U\right)\right\}}{\min\left\{\min_{\alpha\leqslant\beta\leqslant1}f\left(\tilde{x}_\beta^U\right), \min_{\alpha\leqslant\beta\leqslant1}f\left(\tilde{x}_\beta^U\right)\right\}} \geqslant \frac{\min\left\{\min_{\alpha\leqslant\beta\leqslant1}\bar{G}\left(\tilde{y}_\beta^U\right), \min_{\alpha\leqslant\beta\leqslant1}\bar{G}\left(\tilde{y}_\beta^U\right)\right\}}{\min\left\{\min_{\alpha\leqslant\beta\leqslant1}g\left(\tilde{y}_\beta^U\right), \min_{\alpha\leqslant\beta\leqslant1}g\left(\tilde{y}_\beta^U\right)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\leqslant\beta\leqslant1}\bar{F}\left(\tilde{x}_\beta^U\right), \max_{\alpha\leqslant\beta\leqslant1}\bar{F}\left(\tilde{x}_\beta^U\right)\right\}}{\max\left\{\max_{\alpha\leqslant\beta\leqslant1}f\left(\tilde{x}_\beta^U\right), \max_{\alpha\leqslant\beta\leqslant1}\left(\tilde{x}_\beta^U\right)\right\}} \geqslant \frac{\max\left\{\max_{\alpha\leqslant\beta\leqslant1}\bar{G}\left(\tilde{y}_\beta^U\right), \max_{\alpha\leqslant\beta\leqslant1}\bar{G}\left(\tilde{y}_\beta^U\right)\right\}}{\max\left\{\max_{\alpha\leqslant\beta\leqslant1}g\left(\tilde{y}_\beta^U\right), \max_{\alpha\leqslant\beta\leqslant1}g\left(\tilde{y}_\beta^U\right)\right\}}$$

$$= \int_0^y \frac{\dfrac{\min\left\{\min_{\alpha\leqslant\beta\leqslant1}\bar{F}\left(\tilde{x}_\beta^U\right), \min_{\alpha\leqslant\beta\leqslant1}\bar{F}\left(\tilde{x}_\beta^U\right)\right\}}{\min\left\{\min_{\alpha\leqslant\beta\leqslant1}f\left(\tilde{x}_\beta^U\right), \min_{\alpha\leqslant\beta\leqslant1}f\left(\tilde{x}_\beta^U\right)\right\}}}{\dfrac{\max\left\{\max_{\alpha\leqslant\beta\leqslant1}\bar{F}\left(\tilde{x}_\beta^U\right), \max_{\alpha\leqslant\beta\leqslant1}\bar{F}\left(\tilde{x}_\beta^U\right)\right\}}{\max\left\{\max_{\alpha\leqslant\beta\leqslant1}f\left(\tilde{x}_\beta^U\right), \max_{\alpha\leqslant\beta\leqslant1}\left(\tilde{x}_\beta^U\right)\right\}}} \geqslant \frac{\dfrac{\min\left\{\min_{\alpha\leqslant\beta\leqslant1}\bar{G}\left(\tilde{y}_\beta^U\right), \min_{\alpha\leqslant\beta\leqslant1}\bar{G}\left(\tilde{y}_\beta^U\right)\right\}}{\min\left\{\min_{\alpha\leqslant\beta\leqslant1}g\left(\tilde{y}_\beta^U\right), \min_{\alpha\leqslant\beta\leqslant1}g\left(\tilde{y}_\beta^U\right)\right\}}}{\dfrac{\max\left\{\max_{\alpha\leqslant\beta\leqslant1}\bar{G}\left(\tilde{y}_\beta^U\right), \max_{\alpha\leqslant\beta\leqslant1}\bar{G}\left(\tilde{y}_\beta^U\right)\right\}}{\max\left\{\max_{\alpha\leqslant\beta\leqslant1}g\left(\tilde{y}_\beta^U\right), \max_{\alpha\leqslant\beta\leqslant1}g\left(\tilde{y}_\beta^U\right)\right\}}} dx$$

By Definition 2.1

$$\geqslant \int_0^y \frac{\dfrac{\min\left\{\min_{\alpha\leqslant\beta\leqslant1}\bar{F}\left(\tilde{x}_\beta^L\right), \min_{\alpha\leqslant\beta\leqslant1}\bar{F}\left(\tilde{x}_\beta^L\right)\right\}}{\min\left\{\min_{\alpha\leqslant\beta\leqslant1}f\left(\tilde{x}_\beta^L\right), \min_{\alpha\leqslant\beta\leqslant1}f\left(\tilde{x}_\beta^L\right)\right\}}}{\dfrac{\max\left\{\max_{\alpha\leqslant\beta\leqslant1}\bar{F}\left(\tilde{x}_\beta^L\right), \max_{\alpha\leqslant\beta\leqslant1}\bar{F}\left(\tilde{x}_\beta^L\right)\right\}}{\max\left\{\max_{\alpha\leqslant\beta\leqslant1}f\left(\tilde{x}_\beta^L\right), \max_{\alpha\leqslant\beta\leqslant1}\left(\tilde{x}_\beta^L\right)\right\}}} \geqslant \frac{\dfrac{\min\left\{\min_{\alpha\leqslant\beta\leqslant1}\bar{G}\left(\tilde{y}_\beta^L\right), \min_{\alpha\leqslant\beta\leqslant1}\bar{G}\left(\tilde{y}_\beta^L\right)\right\}}{\min\left\{\min_{\alpha\leqslant\beta\leqslant1}g\left(\tilde{y}_\beta^L\right), \min_{\alpha\leqslant\beta\leqslant1}g\left(\tilde{y}_\beta^L\right)\right\}}}{\dfrac{\max\left\{\max_{\alpha\leqslant\beta\leqslant1}\bar{G}\left(\tilde{y}_\beta^L\right), \max_{\alpha\leqslant\beta\leqslant1}\bar{G}\left(\tilde{y}_\beta^L\right)\right\}}{\max\left\{\max_{\alpha\leqslant\beta\leqslant1}g\left(\tilde{y}_\beta^L\right), \max_{\alpha\leqslant\beta\leqslant1}g\left(\tilde{y}_\beta^L\right)\right\}}} dx$$

is equal to

$$\min\left\{\min_{\alpha\leqslant\beta\leqslant1} f(\tilde{x})_\alpha^L, \min_{\alpha\leqslant\beta\leqslant1} g(\tilde{x})_\alpha^L\right\} \leqslant \min\left\{\min_{\alpha\leqslant\beta\leqslant1} f(\tilde{y})_\alpha^L, \min_{\alpha\leqslant\beta\leqslant1} g(\tilde{y})_\alpha^L\right\}$$

And

$$\max\left\{\max_{\alpha\leqslant\beta\leqslant1} f(\tilde{x})_\alpha^L, \max_{\alpha\leqslant\beta\leqslant1} g(\tilde{x})_\alpha^L\right\} \leqslant \max\left\{\max_{\alpha\leqslant\beta\leqslant1} f(\tilde{y})_\alpha^L, \max_{\alpha\leqslant\beta\leqslant1} g(\tilde{y})_\alpha^L\right\}$$

Now using Definition 2.1 proof is complete. □

Suppose that $X_1, X_2, X_3, \ldots, X_n$ and $Y_1, Y_2, Y_3, \ldots, Y_n$ are two independent fuzzy random samples of size n.

Induced by $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots, \tilde{x}_n$ (with cumulative distribution functions F) and $\tilde{y}_1 \tilde{y}_2 \tilde{y}_3, \ldots, \tilde{y}_n$ (with cumulative distribution functions G), respectively. We denote $\tilde{r}_{\bar{F}_k}(\tilde{x})$ as the fuzzy reversed Hazard rate of the k-th fuzzy statistic $X_{k:n}$ as the following:

$$\mu_{\tilde{r}_{\bar{F}_k}(\tilde{x})}(y) = \max_{\alpha\leqslant\beta\leqslant1} \tilde{r}_{\bar{F}_k}(\tilde{x})_\alpha^L, \max_{\alpha\leqslant\beta\leqslant1} \tilde{r}_{\bar{F}_k}(\tilde{x})_\alpha^U\}(y)$$

where,

$$\tilde{r}_{\bar{F}_k}(\tilde{x})_\alpha^L = \min\left\{\min_{\alpha\leqslant\beta\leqslant1}\left[\frac{\left(\frac{n!}{(k-1)!(n-k)!}\right)\left(\frac{f(x)}{F(x)}\right)}{\sum_k^n \frac{n!}{j!(n-j)!}\left[\frac{F(x)}{f(x)}\right](k-j)}\right]\right\} : x = \tilde{x}_\beta^L,$$

$$\min\left\{\min_{\alpha\leqslant\beta\leqslant 1}\left[\frac{\left(\frac{n!}{(k-1)!(n-k)!}\right)\left(\frac{f(x)}{\bar{F}(x)}\right)}{\sum\limits_{k}^{n}\frac{n!}{j!(n-j)!}\left[\frac{\bar{F}(x)}{f(x)}\right](k-j)}\right]\right\}: x=\tilde{x}_{\beta}^{U}$$

and

$$\tilde{r}_{\tilde{F}_k}(\tilde{x})_{\alpha}^{U}=\max\left\{\min_{\alpha\leqslant\beta\leqslant 1}\left[\frac{\left(\frac{n!}{(k-1)!(n-k)!}\right)\left(\frac{f(x)}{\bar{F}(x)}\right)}{\sum\limits_{k}^{n}\frac{n!}{j!(n-j)!}\left[\frac{\bar{F}(x)}{f(x)}\right](k-j)}\right]\right\}: x=\tilde{x}_{\beta}^{L},$$

$$\max\left\{\min_{\alpha\leqslant\beta\leqslant 1}\left[\frac{\left(\frac{n!}{(k-1)!(n-k)!}\right)\left(\frac{f(x)}{\bar{F}(x)}\right)}{\sum\limits_{k}^{n}\frac{n!}{j!(n-j)!}\left[\frac{\bar{F}(x)}{f(x)}\right](k-j)}\right]\right\}: x=\tilde{x}_{\beta}^{U}.$$

Theorem 4.3. *Suppose that* $X_i \leqslant_{FRH1} Y_i$ *for all* $i=1,2,3,\ldots,n$ *and*

$$\left\{\frac{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant 1} f\left(\tilde{x}_{\beta}^{L}\right),\min\limits_{\alpha\leqslant\beta\leqslant 1} f\left(\tilde{x}_{\beta}^{U}\right)\right\}}{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant 1}\bar{F}\left(\tilde{x}_{\beta}^{L}\right),\min\limits_{\alpha\leqslant\beta\leqslant 1}\bar{F}\left(\tilde{x}_{\beta}^{U}\right)\right\}}\leqslant\frac{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant 1} g\left(\tilde{y}_{\beta}^{L}\right),\min\limits_{\alpha\leqslant\beta\leqslant 1} g\left(\tilde{y}_{\beta}^{U}\right)\right\}}{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant 1}\bar{G}\left(\tilde{y}_{\beta}^{L}\right),\min\limits_{\alpha\leqslant\beta\leqslant 1}\bar{G}\left(\tilde{y}_{\beta}^{U}\right)\right\}}\right\}$$

$$\geqslant\left\{\frac{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant 1} f\left(\tilde{x}_{\beta}^{L}\right),\max\limits_{\alpha\leqslant\beta\leqslant 1} f\left(\tilde{x}_{\beta}^{U}\right)\right\}}{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant 1}\bar{F}\left(\tilde{x}_{\beta}^{L}\right),\max\limits_{\alpha\leqslant\beta\leqslant 1}\bar{F}\left(\tilde{x}_{\beta}^{U}\right)\right\}}\leqslant\frac{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant 1} g\left(\tilde{y}_{\beta}^{L}\right),\max\limits_{\alpha\leqslant\beta\leqslant 1} g\left(\tilde{y}_{\beta}^{U}\right)\right\}}{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant 1}\bar{G}\left(y_{\beta}^{L}\right),\max\limits_{\alpha\leqslant\beta\leqslant 1}\bar{G}\left(\tilde{y}_{\beta}^{U}\right)\right\}}\right\}$$

Then $X_{i:n}\leqslant_{FRH1} Y_{i:n}$.

Proof. We can prove easily that the function $[\sum\limits_{k}^{n}\frac{n!}{j!(n-j)!}\left[\frac{1-\tilde{x}}{\tilde{x}}\right](k-j)]^{-1}$ is non decreasing function in x. Since that $X_i \leqslant_{FRHR1} Y_i$, we can conclude that

$$\frac{1}{\left[\sum\limits_{k}^{n}\frac{n!}{j!(n-j)!}\left[\frac{1-F(\tilde{x})_{\alpha}^{L}}{f}(\tilde{x})_{\alpha}^{L}\right](k-j)\right]}\geqslant\frac{1}{\left[\sum\limits_{k}^{n}\frac{n!}{j!(n-j)!}\left[\frac{1-G(\tilde{x})_{\alpha}^{L}}{g}(\tilde{x})_{\alpha}^{L}\right](k-j)\right]}$$

Let us suppose that

$$A=\left\{\frac{n!}{j!(n-1)!}\frac{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant 1} f\left(\tilde{x}_{\beta}^{L}\right),\min\limits_{\alpha\leqslant\beta\leqslant 1} f\left(\tilde{x}_{\beta}^{U}\right)\right\}}{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant 1}\bar{F}\left(\tilde{x}_{\beta}^{L}\right),\min\limits_{\alpha\leqslant\beta\leqslant 1}\bar{F}\left(\tilde{x}_{\beta}^{U}\right)\right\}}\right\}$$

$$B=\left\{\frac{n!}{j!(n-1)!}\frac{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant 1} g\left(\tilde{x}_{\beta}^{L}\right),\min\limits_{\alpha\leqslant\beta\leqslant 1} g\left(\tilde{x}_{\beta}^{U}\right)\right\}}{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant 1}\bar{G}\left(\tilde{x}_{\beta}^{L}\right),\min\limits_{\alpha\leqslant\beta\leqslant 1}\bar{G}\left(\tilde{x}_{\beta}^{U}\right)\right\}}\right\}$$

$$C=\left\{\frac{n!}{j!(n-1)!}\frac{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant 1} g\left(\tilde{x}_{\beta}^{L}\right),\max\limits_{\alpha\leqslant\beta\leqslant 1} g\left(\tilde{x}_{\beta}^{U}\right)\right\}}{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant 1}\bar{G}\left(\tilde{x}_{\beta}^{L}\right),\max\limits_{\alpha\leqslant\beta\leqslant 1}\bar{G}\left(\tilde{x}_{\beta}^{U}\right)\right\}}\right\}$$

$$D=\left\{\frac{n!}{j!(n-1)!}\frac{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant 1} f\left(\tilde{x}_{\beta}^{L}\right),\max\limits_{\alpha\leqslant\beta\leqslant 1} f\left(\tilde{x}_{\beta}^{U}\right)\right\}}{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant 1}\bar{F}\left(\tilde{x}_{\beta}^{L}\right),\max\limits_{\alpha\leqslant\beta\leqslant 1}\bar{F}\left(\tilde{x}_{\beta}^{U}\right)\right\}}\right\}$$

Now by using the minimum and maximum property and inequality 3, we have

$$\min\left\{\min_{\alpha\leqslant\beta\leqslant 1}\left[\frac{\left(\frac{n!}{(k-1)!(n-k)!}\right)\left(\frac{f(x)}{\bar{F}(x)}\right)}{\sum\limits_{k}^{n}\frac{n!}{j!(n-j)!}\left[\frac{\bar{F}(x)}{f(x)}\right](k-j)}\right]\right\}: x=\tilde{x}_{\beta}^{L}$$

$$\geqslant\max\left\{\min_{\alpha\leqslant\beta\leqslant 1}\left[\frac{\left(\frac{n!}{(k-1)!(n-k)!}\right)\left(\frac{f(x)}{\bar{F}(x)}\right)}{\sum\limits_{k}^{n}\frac{n!}{j!(n-j)!}\left[\frac{\bar{F}(x)}{f(x)}\right](k-j)}\right]\right\}: x=\tilde{x}_{\beta}^{U}$$

or equivalently $\tilde{r}_{\tilde{F}_k}(\tilde{x})_{\alpha}^{L}\geqslant\tilde{r}_{\tilde{F}_k}(\tilde{x})_{\alpha}^{U}$

And hence $X_{i:n}\leqslant_{FRH1} Y_{i:n}$. $\qquad\square$

5. Definition Fuzzy Mean Inactivity Time Order

Let X be a non negative fuzzy random variable with density function $f(\tilde{t})$ and cumulative distribution function $\bar{F}(\tilde{t})$ respectively, and $\tilde{x}$ be a fuzzy random variables induced by

X. The fuzzy function $\tilde{m}_{\bar{F}}(t)$ is said a fuzzy Mean inactivity time of fuzzy random variables $\tilde{x}$, whenever its membership function is given by

$$\mu_{\tilde{m}}(y) = \sup_{0 \leqslant \alpha \leqslant 1} \alpha^I \tilde{m}(\tilde{t})_{\alpha}^L, \tilde{m}(\tilde{t})_{\alpha}^U \}(y) \text{ where,}$$

$$\int_0^t \tilde{m}(\tilde{t})_{\alpha}^L dx = \min\left\{ \left\{ \min_{\alpha \leqslant \beta \leqslant 1} m(\tilde{t}) ; x = \tilde{t}_{\beta}^L \right\}, \left\{ \min_{\alpha \leqslant \beta \leqslant 1} m(\tilde{t}) ; x = \tilde{t}_{\beta}^L \right\} \right\},$$

$$\int_0^t \tilde{m}(\tilde{t})_{\alpha}^U dx = \max\left\{ \left\{ \max_{\alpha \leqslant \beta \leqslant 1} m(\tilde{t}) ; x = \tilde{t}_{\beta}^L \right\}, \left\{ \max_{\alpha \leqslant \beta \leqslant 1} m(\tilde{t}) ; x = \tilde{t}_{\beta}^L \right\} \right\}.$$

Such that the interval, $\tilde{m}(\tilde{t})_{\alpha}^L$ and $\tilde{m}(\tilde{t})_{\alpha}^U$ will contain all of the mean inactivity time of each $\tilde{t}_{\beta}^L$ and $\tilde{t}_{\beta}^U$ for $\beta \geqslant \alpha$.

Definition 5.1. *Let X and Y are two non negative fuzzy random variables with continuous distribution functions an fuzzy Mean inactivity time with functions ($\tilde{x}$) and ($\tilde{y}$) respectively, then X is smaller than Y. We propose four relations*
(1) $X \leqslant_{FMIT1} Y$ if,

$$\frac{\int_0^t \min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_{\beta}^L, \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_{\beta}^U \right\} dx}{\min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_{\beta}^L, \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_{\beta}^U \right\}} \leqslant \frac{\int_0^t \min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y})_{\beta}^L, \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y})_{\beta}^U \right\} dx}{\min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_{\beta}^L, \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_{\beta}^U \right\}}$$

and

$$\frac{\int_0^t \max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_{\beta}^L, \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_{\beta}^U \right\} dx}{\max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_{\beta}^L, \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_{\beta}^U \right\}} \leqslant \frac{\int_0^t \max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y})_{\beta}^L, \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y})_{\beta}^U \right\} dx}{\max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_{\beta}^L, \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_{\beta}^U \right\}}$$

(2) $X \leqslant_{FMIT2} Y$ if,

$$\frac{\int_0^t \min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_{\beta}^L, \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_{\beta}^L \right\} dx}{\min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_{\beta}^L, \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_{\beta}^L \right\}} \leqslant \frac{\int_0^t \min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y})_{\beta}^U, \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y})_{\beta}^U \right\} dx}{\min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_{\beta}^U, \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_{\beta}^U \right\}}$$

and

$$\frac{\int_0^t \max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_{\beta}^L, \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_{\beta}^L \right\} dx}{\max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_{\beta}^L, \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_{\beta}^L \right\}} \leqslant \frac{\int_0^t \max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y})_{\beta}^U, \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y})_{\beta}^U \right\} dx}{\max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_{\beta}^U, \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_{\beta}^U \right\}}$$

(3) $X \leqslant_{FMIT3} Y$ if,

$$\frac{\int_0^t \min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_{\beta}^L, \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_{\beta}^L \right\} dx}{\min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_{\beta}^L, \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_{\beta}^L \right\}} \leqslant \frac{\int_0^t \min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y})_{\beta}^L, \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y})_{\beta}^L \right\} dx}{\min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_{\beta}^L, \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_{\beta}^L \right\}}$$

and

$$\frac{\int_0^t \max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_{\beta}^L, \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_{\beta}^L \right\} dx}{\max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_{\beta}^L, \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_{\beta}^L \right\}} \leqslant \frac{\int_0^t \max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y})_{\beta}^L, \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y})_{\beta}^L \right\} dx}{\max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_{\beta}^L, \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_{\beta}^L \right\}}$$

(4) $X \leqslant_{FMIT3} Y$ if,

$$\frac{\int_0^t \min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_{\beta}^U, \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_{\beta}^U \right\} dx}{\min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_{\beta}^U, \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_{\beta}^U \right\}} \leqslant \frac{\int_0^t \min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y})_{\beta}^U, \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y})_{\beta}^U \right\} dx}{\min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_{\beta}^U, \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_{\beta}^U \right\}}$$

and

$$\frac{\int_0^t \max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_\beta^U, \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_\beta^U \right\} dx}{\max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_\beta^U, \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_\beta^U \right\}} \leqslant \frac{\int_0^t \max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{y})_\beta^U, \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{y})_\beta^U \right\} dx}{\max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_\beta^U, \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_\beta^U \right\}}$$

For each α, β (0,1] ∩ Q, where F G are the survival and density functions of X and Y respectively.

Lemma 5.2 (Decreasing Mean Inactivity). *Suppose that X and Y are two fuzzy random variables with fuzzy cumulative distribution functions F(x̃) and G(ỹ) respectively then, we propose four relations*
(1) X ≤$_{FMIT1}$ Y if,

$$\frac{\int_0^t \min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_\beta^L, \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_\beta^U \right\} dx}{\min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_\beta^L, \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_\beta^U \right\}} \leqslant \frac{\int_0^t \min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y}-t)_\beta^L, \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y}-t)_\beta^U \right\} dx}{\min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_\beta^L, \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_\beta^U \right\}}$$

and

$$\frac{\int_0^t \max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_\beta^L, \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_\beta^U \right\} dx}{\max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_\beta^L, \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_\beta^U \right\}} \leqslant \frac{\int_0^t \max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y}-t)_\beta^L, \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y}-t)_\beta^U \right\} dx}{\max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_\beta^L, \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_\beta^U \right\}}$$

(2) X ≤$_{FMIT2}$ Y if,

$$\frac{\int_0^t \min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_\beta^L, \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_\beta^L \right\} dx}{\min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_\beta^L, \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_\beta^L \right\}} \leqslant \frac{\int_0^t \min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y}-t)_\beta^U, \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y})_\beta^U \right\} dx}{\min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_\beta^U, \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_\beta^U \right\}}$$

and

$$\frac{\int_0^t \max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_\beta^L, \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_\beta^L \right\} dx}{\max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_\beta^L, \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_\beta^L \right\}} \leqslant \frac{\int_0^t \max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y}-t)_\beta^U, \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y}-t)_\beta^U \right\} dx}{\max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_\beta^U, \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_\beta^U \right\}}$$

(3) X ≤$_{FMIT3}$ Y if,

$$\frac{\int_0^t \min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_\beta^L, \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_\beta^L \right\} dx}{\min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_\beta^L, \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_\beta^L \right\}} \leqslant \frac{\int_0^t \min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y}-t)_\beta^L, \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y}-t)_\beta^L \right\} dx}{\min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_\beta^L, \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_\beta^L \right\}}$$

and

$$\frac{\int_0^t \max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_\beta^L, \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_\beta^L \right\} dx}{\max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_\beta^L, \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_\beta^L \right\}} \leqslant \frac{\int_0^t \max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y}-t)_\beta^L, \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y}-t)_\beta^L \right\} dx}{\max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_\beta^L, \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_\beta^L \right\}}$$

(4) X ≤$_{FMIT4}$ Y if,

$$\frac{\int_0^t \min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_\beta^U, \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_\beta^U \right\} dx}{\min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_\beta^U, \min_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_\beta^U \right\}} \leqslant \frac{\int_0^t \min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y}-t)_\beta^U, \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{y}-t)_\beta^U \right\} dx}{\min\left\{ \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_\beta^U, \min_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_\beta^U \right\}}$$

and

$$\frac{\int_0^t \max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_\beta^U, \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{x})_\beta^U \right\} dx}{\max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_\beta^U, \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{t})_\beta^U \right\}} \leqslant \frac{\int_0^t \max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{y}-t)_\beta^U, \max_{\alpha \leqslant \beta \leqslant 1} F(\tilde{y}-t)_\beta^U \right\} dx}{\max\left\{ \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_\beta^U, \max_{\alpha \leqslant \beta \leqslant 1} G(\tilde{t})_\beta^U \right\}}$$

is decreasing in x > 0.

 The following theorems gives the relationship between the fuzzy mean inactivity time and fuzzy reversed hazard rate order.

Theorem 5.3. *Let X and Y are two fuzzy random variables with fuzzy cumulative distribution functions F and G respectively*

1. *If $X \leqslant_{FRHR1} Y$, then $X \leqslant_{FMIT2} Y$.*

2. *If $X \leqslant_{FRHR1} Y$, then $X \leqslant_{FMIT3} Y$.*

3. *If $X \leqslant_{FRHR1} Y$, then $X \leqslant_{FMIT4} Y$.*

4. *If $X \leqslant_{FRHR1} Y$, then $X \leqslant_{FMIT1} Y$.*

Proof. Suppose that $X \leqslant_{FRHR1} Y$. Based on lemma,

$$\Rightarrow \frac{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1}[f(\tilde{x}^L_\alpha - t)/\tilde{x}^L_\alpha \geqslant t], \min\limits_{\alpha\leqslant\beta\leqslant1}[f(\tilde{x}^U_\alpha - t)/\tilde{x}^U_\alpha \geqslant t]\right\}}{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1}[\bar{F}(\tilde{x}^L_\alpha - t)/\tilde{x}^L_\alpha \geqslant t], \min\limits_{\alpha\leqslant\beta\leqslant1}[\bar{F}(\tilde{x}^U_\alpha - t)/\tilde{x}^U_\alpha \geqslant t]\right\}}$$

and

$$\frac{\max\left\{ \max\limits_{\alpha\leqslant\beta\leqslant1}[f(\tilde{x}^L_\alpha - t)/\tilde{x}^L_\alpha \geqslant t], \max\limits_{\alpha\leqslant\beta\leqslant1}[f(\tilde{x}^U_\alpha - t)/\tilde{x}^U_\alpha \geqslant t]\right\}}{\max\left\{ \max\limits_{\alpha\leqslant\beta\leqslant1}[\bar{F}(\tilde{x}^L_\alpha - t)/\tilde{x}^L_\alpha \geqslant t], \max\limits_{\alpha\leqslant\beta\leqslant1}[\bar{F}(\tilde{x}^U_\alpha - t)/\tilde{x}^U_\alpha \geqslant t]\right\}} \text{ is increasing } x > 0,$$

and hence

$$\frac{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1}[f(\tilde{x}^L_\alpha - t)/\tilde{x}^L_\alpha \geqslant t], \min\limits_{\alpha\leqslant\beta\leqslant1}[f(\tilde{x}^U_\alpha - t)/\tilde{x}^U_\alpha \geqslant t]\right\}}{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1}[\bar{F}(\tilde{x}^L_\alpha - t)/\tilde{x}^L_\alpha \geqslant t], \min\limits_{\alpha\leqslant\beta\leqslant1}[\bar{F}(\tilde{x}^U_\alpha - t)/\tilde{x}^U_\alpha \geqslant t]\right\}} \geqslant \frac{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^L_\beta\right), \min\limits_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^U_\beta\right)\right\}}{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{y}^L_\beta\right), \min\limits_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^U_\beta\right)\right\}}$$

and

$$\frac{\max\left\{ \max\limits_{\alpha\leqslant\beta\leqslant1}[f(\tilde{x}^L_\alpha - t)/\tilde{x}^L_\alpha \geqslant t], \max\limits_{\alpha\leqslant\beta\leqslant1}[f(\tilde{x}^U_\alpha - t)/\tilde{x}^U_\alpha \geqslant t]\right\}}{\max\left\{ \max\limits_{\alpha\leqslant\beta\leqslant1}[\bar{F}(\tilde{x}^L_\alpha - t)/\tilde{x}^L_\alpha \geqslant t], \max\limits_{\alpha\leqslant\beta\leqslant1}[\bar{F}(\tilde{x}^U_\alpha - t)/\tilde{x}^U_\alpha \geqslant t]\right\}} \geqslant \frac{\max\left\{ \max\limits_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^L_\beta\right), \max\limits_{\alpha\leqslant\beta\leqslant1} f\left(\tilde{x}^U_\beta\right)\right\}}{\max\left\{ \max\limits_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^L_\beta\right), \max\limits_{\alpha\leqslant\beta\leqslant1} \bar{F}\left(\tilde{x}^U_\beta\right)\right\}}$$

By defining

$$\mathfrak{K}_\alpha(ix) = \frac{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1}[f(\tilde{x}^L_\alpha - t)/\tilde{x}^L_\alpha \geqslant t], \min\limits_{\alpha\leqslant\beta\leqslant1}[f(\tilde{x}^U_\alpha - t)/\tilde{x}^U_\alpha \geqslant t]\right\}}{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1}[\bar{F}(\tilde{x}^L_\alpha - t)/\tilde{x}^L_\alpha \geqslant t], \min\limits_{\alpha\leqslant\beta\leqslant1}[\bar{F}(\tilde{x}^U_\alpha - t)/\tilde{x}^U_\alpha \geqslant t]\right\}}, \quad i = 1$$

and

$$\mathfrak{K}_\alpha(ix) = \frac{\max\left\{ \max\limits_{\alpha\leqslant\beta\leqslant1}[f(\tilde{x}^L_\alpha - t)/\tilde{x}^L_\alpha \geqslant t], \max\limits_{\alpha\leqslant\beta\leqslant1}[f(\tilde{x}^U_\alpha - t)/\tilde{x}^U_\alpha \geqslant t]\right\}}{\max\left\{ \max\limits_{\alpha\leqslant\beta\leqslant1}[\bar{F}(\tilde{x}^L_\alpha - t)/\tilde{x}^L_\alpha \geqslant t], \max\limits_{\alpha\leqslant\beta\leqslant1}[\bar{F}(\tilde{x}^U_\alpha - t)/\tilde{x}^U_\alpha \geqslant t]\right\}}, \quad i = 2L_\alpha(xt) = I_{0,t}(\tilde{x}); \tilde{x}, t > 0$$

.

Then by definition $M_\alpha(it) = \int_0^\infty [\mathfrak{K}_\alpha(i,x)L_\alpha(xt)]dx$

Thus we conclude that,

$$\frac{\int_0^t \min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1} F(\tilde{x})^L_\beta, \min\limits_{\alpha\leqslant\beta\leqslant1} F(\tilde{x})^L_\beta\right\}dx}{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1} F(\tilde{t})^L_\beta, \min\limits_{\alpha\leqslant\beta\leqslant1} F(\tilde{t})^L_\beta\right\}} \leqslant \frac{\int_0^t \min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1} G(\tilde{y})^U_\beta, \min\limits_{\alpha\leqslant\beta\leqslant1} G(\tilde{y})^U_\beta\right\}dx}{\min\left\{ \min\limits_{\alpha\leqslant\beta\leqslant1} G(\tilde{t})^U_\beta, \min\limits_{\alpha\leqslant\beta\leqslant1} G(\tilde{t})^U_\beta\right\}}$$

and

$$\frac{\int_0^t \max\left\{\max\limits_{\alpha\leqslant\beta\leqslant 1} F(\tilde{x})_\beta^L, \max\limits_{\alpha\leqslant\beta\leqslant 1} F(\tilde{x})_\beta^L\right\} dx}{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant 1} F(\tilde{t})_\beta^L, \max\limits_{\alpha\leqslant\beta\leqslant 1} F(\tilde{t})_\beta^L\right\}} \leqslant \frac{\int_0^t \max\left\{\max\limits_{\alpha\leqslant\beta\leqslant 1} G(\tilde{y})_\beta^U, \max\limits_{\alpha\leqslant\beta\leqslant 1} G(\tilde{y})_\beta^U\right\} dx}{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant 1} G(\tilde{t})_\beta^U, \max\limits_{\alpha\leqslant\beta\leqslant 1} G(\tilde{t})_\beta^U\right\}}$$

is decreasing in $x > 0$. Now using previous lemma proof is complete.
Other parts are similarly to proved. □

Theorem 5.4. *Let X and Y are two fuzzy random variables have a common fuzzy mean past life $F(\tilde{x})$ and $G(\tilde{y})$ respectively. Then,*
(a) If,

$$\frac{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant 1}[f(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t], \min\limits_{\alpha\leqslant\beta\leqslant 1}[f(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t]\right\}}{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant 1}[\bar{F}(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t], \min\limits_{\alpha\leqslant\beta\leqslant 1}[\bar{F}(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t]\right\}} \geqslant \frac{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant 1} f\left(\tilde{x}_\beta^L\right), \min\limits_{\alpha\leqslant\beta\leqslant 1} f\left(\tilde{x}_\beta^U\right)\right\}}{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant 1} \bar{F}\left(\tilde{y}_\beta^L\right), \min\limits_{\alpha\leqslant\beta\leqslant 1} \bar{F}\left(\tilde{x}_\beta^U\right)\right\}}$$

and

$$\frac{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant 1}[f(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t], \max\limits_{\alpha\leqslant\beta\leqslant 1}[f(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t]\right\}}{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant 1}[\bar{F}(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t], \max\limits_{\alpha\leqslant\beta\leqslant 1}[\bar{F}(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t]\right\}} \geqslant \frac{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant 1} f\left(\tilde{x}_\beta^L\right), \max\limits_{\alpha\leqslant\beta\leqslant 1} f\left(\tilde{x}_\beta^U\right)\right\}}{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant 1} \bar{F}\left(\tilde{x}_\beta^L\right), \max\limits_{\alpha\leqslant\beta\leqslant 1} \bar{F}\left(\tilde{x}_\beta^U\right)\right\}}$$

is increasing in $t > 0$, then $X \leqslant_{FRH1} Y$ if and only if $X \leqslant_{FMIT4} Y$.
(b) If,

$$\frac{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant 1}[f(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t], \min\limits_{\alpha\leqslant\beta\leqslant 1}[f(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t]\right\}}{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant 1}[\bar{F}(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t], \min\limits_{\alpha\leqslant\beta\leqslant 1}[\bar{F}(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t]\right\}} \geqslant \frac{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant 1} f\left(\tilde{x}_\beta^L\right), \min\limits_{\alpha\leqslant\beta\leqslant 1} f\left(\tilde{x}_\beta^L\right)\right\}}{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant 1} \bar{F}\left(\tilde{y}_\beta^L\right), \min\limits_{\alpha\leqslant\beta\leqslant 1} \bar{F}\left(\tilde{x}_\beta^L\right)\right\}}$$

and

$$\frac{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant 1}[f(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t], \max\limits_{\alpha\leqslant\beta\leqslant 1}[f(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t]\right\}}{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant 1}[\bar{F}(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t], \max\limits_{\alpha\leqslant\beta\leqslant 1}[\bar{F}(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geqslant t]\right\}} \geqslant \frac{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant 1} f\left(\tilde{x}_\beta^L\right), \max\limits_{\alpha\leqslant\beta\leqslant 1} f\left(\tilde{x}_\beta^L\right)\right\}}{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant 1} \bar{F}\left(\tilde{x}_\beta^L\right), \max\limits_{\alpha\leqslant\beta\leqslant 1} \bar{F}\left(\tilde{x}_\beta^L\right)\right\}}$$

is increasing in $t > 0$, then $X \leqslant_{FRH2} Y$ if and only if $X \leqslant_{FMIT2} Y$.
(C) If,

$$\frac{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant 1}[f(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t], \min\limits_{\alpha\leqslant\beta\leqslant 1}[f(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t]\right\}}{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant 1}[\bar{F}(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t], \min\limits_{\alpha\leqslant\beta\leqslant 1}[\bar{F}(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t]\right\}} \geqslant \frac{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant 1} f\left(\tilde{x}_\beta^U\right), \min\limits_{\alpha\leqslant\beta\leqslant 1} f\left(\tilde{x}_\beta^U\right)\right\}}{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant 1} \bar{F}\left(\tilde{y}_\beta^U\right), \min\limits_{\alpha\leqslant\beta\leqslant 1} \bar{F}\left(\tilde{x}_\beta^U\right)\right\}}$$

and

$$\frac{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant 1}[f(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t], \max\limits_{\alpha\leqslant\beta\leqslant 1}[f(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t]\right\}}{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant 1}[\bar{F}(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t], \max\limits_{\alpha\leqslant\beta\leqslant 1}[\bar{F}(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geqslant t]\right\}} \geqslant \frac{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant 1} f\left(\tilde{x}_\beta^U\right), \min\limits_{\alpha\leqslant\beta\leqslant 1} f\left(\tilde{x}_\beta^U\right)\right\}}{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant 1} \bar{F}\left(\tilde{y}_\beta^U\right), \min\limits_{\alpha\leqslant\beta\leqslant 1} \bar{F}\left(\tilde{x}_\beta^U\right)\right\}}$$

is increasing in $t > 0$, then $X \leqslant_{FRH3} Y$ if and only if $X \leqslant_{FMIT3} Y$.

(d) If,

$$\frac{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant1}[f(\tilde{x}_\alpha^L-t)/\tilde{x}_\alpha^L\geqslant t],\ \min\limits_{\alpha\leqslant\beta\leqslant1}[f(\tilde{x}_\alpha^L-t)/\tilde{x}_\alpha^L\geqslant t]\right\}}{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant1}[\bar{F}(\tilde{x}_\alpha^L-t)/\tilde{x}_\alpha^L\geqslant t],\ \min\limits_{\alpha\leqslant\beta\leqslant1}[\bar{F}(\tilde{x}_\alpha^L-t)/\tilde{x}_\alpha^L\geqslant t]\right\}}\geqslant\frac{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant1}f\left(\tilde{x}_\beta^U\right),\ \min\limits_{\alpha\leqslant\beta\leqslant1}f\left(\tilde{x}_\beta^U\right)\right\}}{\min\left\{\min\limits_{\alpha\leqslant\beta\leqslant1}\bar{F}\left(\tilde{y}_\beta^U\right),\ \min\limits_{\alpha\leqslant\beta\leqslant1}\bar{F}\left(\tilde{x}_\beta^U\right)\right\}}$$

and

$$\frac{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant1}[f(\tilde{x}_\alpha^L-t)/\tilde{x}_\alpha^L\geqslant t],\ \max\limits_{\alpha\leqslant\beta\leqslant1}[f(\tilde{x}_\alpha^L-t)/\tilde{x}_\alpha^L\geqslant t]\right\}}{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant1}[\bar{F}(\tilde{x}_\alpha^L-t)/\tilde{x}_\alpha^L\geqslant t],\ \max\limits_{\alpha\leqslant\beta\leqslant1}[\bar{F}(\tilde{x}_\alpha^L-t)/\tilde{x}_\alpha^L\geqslant t]\right\}}\geqslant\frac{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant1}f\left(\tilde{x}_\beta^U\right),\ \max\limits_{\alpha\leqslant\beta\leqslant1}f\left(\tilde{x}_\beta^U\right)\right\}}{\max\left\{\max\limits_{\alpha\leqslant\beta\leqslant1}\bar{F}\left(\tilde{x}_\beta^U\right),\ \max\limits_{\alpha\leqslant\beta\leqslant1}\bar{F}\left(\tilde{x}_\beta^U\right)\right\}}$$

is increasing in $t>0$, then $X\leqslant_{FRH4}Y$ if and only if $X\leqslant_{FMIT1}Y$.

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