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Hopf real hypersurface of a complex space form

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Abstract
In this paper we study Hopf real hypersurface of complex space form. We give a characterization of projective and hyperbolic complex space form based on curvature conditions of real hypersurface of complex space form.

Keywords
Complex space form, Real hypersurface, Hopf hypersurface, structure Jacobi operator.

AMS Subject Classification
53D10, 53D15.

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Contents

1. Introduction

An n- dimensional Kaehler manifold (M_n, J, g) of constant holomorphic sectional curvature c is called a complex space form and it is denoted by $M_n(c)$, where J is a complex structure of $M_n(c)$. Let M be a real hypersurface of $M_n(c)$. If $A\xi = \lambda\xi$ is satisfied, then the structure vector field ξ of M is a principal vector and M is then called a Hopf hypersurface of $M_n(c)$, where A denotes the shape operator of M and $\lambda = g(A\xi, \xi)$. Complex space form is a complex Euclidean space E^n, a complex projective space $P_n(c)$ or a complex hyperbolic space $H_n(c)$ if $c = 0$, $c > 0$ or $c < 0$. Real hypersurfaces of complex space forms were studied by many differential geometers such as Okumura [10], Ki and Suh [4], Montiel and Romero ([7], [6]), Maeda [5] and Niebergall-Ryan [8]. Particularly first time Takagi ([12], [13]) classified all homogeneous real hypersurfaces in $P_n(c)$ into six model spaces A_1, A_2, B, C, D and E. Later on Cecil and Ryan [2] studied real hypersurfaces in $P_n(c)$ on which $J\xi$ is principal. Afterwards Berndt [1] characterised all homogeneous real hypersurfaces in $H_n(c)$ with ξ as principal vector into four model spaces A_0, A_1, A_2, and B. The class of Riemannian manifolds satisfying the condition $\nabla R = 0$ is a natural generalization of the class of manifolds of constant curvature, where ∇ is the Levi-Civita connection on Riemannian manifold and R is the corresponding curvature tensor. A Riemannian manifold is called semi-symmetric if $R \cdot R = 0$. A general study of semi-symmetric Riemannian manifolds was made by Szabo [11] and Tripathi [14]. The semi-symmetric hypersurfaces of Euclidean spaces were classified by Nomizu [9]. A Riemannian manifold is said to be Ricci-semi-symmetric [3] if the condition $R \cdot S = 0$ is satisfied, where S is the Ricci tensor. The aim of the present paper is to classify complex space forms when Hopf hypersurface of complex space form satisfying certain symmetry and semi-symmetry conditions. The scope of studying Hopf hypersurfaces is that, computations are more tractable because of the structure vector field being a principal vector. The paper is organised as follows: In section 2, we give a brief introduction of real hypersurfaces of complex space forms. In section 3, we give a characterization of projective , hyperbolic and Euclidean complex space forms based on the semi-symmetry, symmetry and Ricci recurrency conditions on Hopf hypersurface of the complex space form.

2. Preliminaries

Let $M_n(c)$ be an n-dimensional complex space form and M be a connected real hypersurface of a complex space form. Then M has an almost contact structure (ϕ, ξ, η, g) induced from the Kahler structure (J, g) of $M^n(c)$. In an almost contact metric manifold (M, ϕ, ξ, η, g), the structure tensors satisfy the following:

$$\phi^2 X = -X + \eta(X)\xi,\ g(\phi X, \phi Y) = -g(X,Y),\ \phi\xi = 0,$$
$$\eta(\phi X) = 0,\ \eta(\xi) = 1.$$

(2.1)

From the parallelism of J, we get

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX,Y)\xi, \tag{2.2}$$

$$\nabla_X \xi = \phi AX. \tag{2.3}$$

Let R be the curvature tensor of M. Then we have following equations of Gauss and Codazzi:

$$\begin{aligned} R(X,Y)Z = &\frac{c}{4}\big[g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X \\ &- g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\big] \\ &+ g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{2.4}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\big[\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi\big]. \tag{2.5}$$

From the Gauss equation (2.4), we have the following:

$$R(X,Y)\xi = \frac{c}{4}\big[\eta(Y)X - \eta(X)Y\big] + \lambda\big[\eta(Y)AX - \eta(X)AY\big], \tag{2.6}$$

$$S(Y,Z) = \frac{c}{4}\big[(2n+1)g(Y,Z)\big] + hg(AY,Z) - g(AY,AZ), \tag{2.7}$$

$$QY = \frac{c}{4}\big[(2n+1)Y\big] + hAY - A^2Y, \tag{2.8}$$

where Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$. From (2.7), the scalar curvature r of M is given by

$$r = \frac{c}{4}\big[(2n)^2 - 1\big] + h^2 - 2n + 1. \tag{2.9}$$

From the Gauss equation (2.4), the structure Jacobi operator R_ξ is given by

$$R_\xi X = R(X,\xi)\xi = \frac{c}{4}\big[X - \eta(X)\xi\big] + \eta(A\xi)AX - \eta(AX)A\xi. \tag{2.10}$$

A Riemannian manifold is called locally symmetric if its curvature tensor R is parallel i.e., $\nabla R = 0$, where ∇ denotes the Levi-civita connection. As a proper generalization of locally symmetric manifold the notion of semi-symmetric manifold [11] was defined by

$$(R(U,W)\cdot R)(X,Y)Z = 0. \tag{2.11}$$

A 3-dimensional Riemannian manifold in which R satisfies the condition

$$R(\xi,X)\cdot S = 0 \tag{2.12}$$

is called ξ-Ricci semi-symmetric manifold.
The Ricci tensor of the Riemannian manifold is said to be cyclic-parallel if it satisfies

$$\sum \nabla S(X,Y,Z) = (\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0, \tag{2.13}$$

for any vector fields X,Y and Z, where $\sum$ denotes the cyclic sum.

Definition 2.1. *The hypersurface M is totally η- umbilical if $AX = \alpha X + \beta\eta(X)\xi$, where α and β are constants.*

From this definition, we have

$$(\nabla_X A)Z = \beta(\nabla_X \eta)Z\xi + \beta\eta(Z)\nabla_X \xi. \tag{2.14}$$

3. Hopf hypersurface of a complex space form

Let M be a Hopf hypersurface of a complex space form $M^n(c)$ and $\{\ e_i\ \}$ be an orthonormal basis of the tangent space at each point of the manifold. Then $A\xi = \lambda\xi$, $\lambda = g(A\xi,\xi)$. Replacing Y by AY in (2.7), we get

$$S(AY,Z) = \frac{c}{4}\big[(2n+1)g(AY,Z)\big] + hg(A^2Y,Z) - g(A^2Y,AZ). \tag{3.1}$$

We now prove the following:

Theorem 3.1. *Let M be a Hopf hypersurface of a complex space form $M^n(c)$ satisfying $R\cdot R = 0$. Then $M^n(c)$ is a complex hyperbolic space.*

Proof. Suppose (2.11) holds in M. Then

$$\begin{aligned} &R(U,\xi)R(X,Y)Z - R(R(U,\xi)X,Y)Z \\ &- R(X,R(U,\xi)Y)Z - R(X,Y)R(U,\xi)Z = 0. \end{aligned} \tag{3.2}$$

If $A\xi = \lambda\xi$, taking $Y = Z = \xi$ in (3.2) and using (2.4) in (3.2), we get

$$\begin{aligned} &\left[\frac{c^2}{8} + c\lambda^2 + \lambda^4\right]\eta(X)\eta(U) + \left[\frac{c\lambda}{4} + 2\lambda^3\right]\eta(X)\eta(AU) \\ &- \frac{c\lambda}{4}\eta(U)\eta(AX) - \lambda^2\eta(AU)\eta(AX) = 0. \end{aligned} \tag{3.3}$$

Taking $X = U = \xi$ in(3.3), we have

$$16\lambda^4 + 8c\lambda^2 + c^2 = 0. \tag{3.4}$$

From (3.4), we obtain

$$c = -4\lambda^2.$$

□

Theorem 3.2. *Let M be a ξ-Ricci semi symmetric Hopf hypersurface of a complex space form $M_n(c)$ with $traceA = \lambda$, then complex space form $M_n(c)$ is a projective space.*

Proof. Suppose $R(\xi,Y)\cdot S = 0$ holds in M. Then from equation (2.12), we have

$$S(R(\xi,X)Y,Z)+S(Y,R(\xi,X)Z)=0. \tag{3.5}$$

Taking $Z=\xi$ in (3.5) and using (2.4), we get

$$\begin{aligned}&\frac{c}{4}S(X,Y)+\lambda S(AX,Y)=\big[\frac{(2n+1)c}{4}+\lambda h\big]\\&\big[\frac{c}{4}g(X,Y)+\lambda g(AX,Y)+\lambda^2\eta(X)\eta(Y)\big]\\&-\lambda^2\big[\frac{c}{4}g(X,Y)+\lambda g(AX,Y)-\eta(A^2X)\eta(Y)\big]\\&-\big[\frac{nc\lambda}{4}+\lambda^2h\big]\eta(AX)\eta(Y)-\big[\frac{c\lambda}{4}+\lambda^3\big]\eta(X)\eta(AY).\end{aligned} \tag{3.6}$$

Putting $X=Y=\xi$ in(3.6) and using (2.7), we have

$$16\lambda^4-16\lambda^3h-2\lambda^2nc-2\lambda hc+\lambda hc^2=0. \tag{3.7}$$

For $h=\lambda$, from (3.7), we obtain

$$c=2(n+1).$$

□

Theorem 3.3. *Let M be a totally η- umbilical real hypersurface of a non flat complex space form $M_n(c)$. Then Ricci tensor in M is cyclic-parallel if and only if trA is a constant.*

Proof. Differentiating the equation(2.7) covariantly with respect to X, we have

$$\begin{aligned}(\nabla_XS)(Y,Z)=&\frac{(2n+1)c}{4}(\nabla_Xg)(Y,Z)+(\nabla_Xh)g(AY,Z)\\&+hg((\nabla_XA)Y,Z)-g((\nabla_XA)Y,AZ)\\&-g(AY,(\nabla_XA)Z).\end{aligned} \tag{3.8}$$

Writing two more equations by cyclic permutation of X,Y,Z in the above equation and adding, we get from(2.13) the following.

$$\begin{aligned}&(\nabla_XS)(Y,Z)+(\nabla_YS)(Z,X)+(\nabla_ZS)(X,Y)=\\&(\nabla_Xh)g(AY,Z)+(\nabla_Yh)g(AZ,X)+(\nabla_Zh)g(AX,Y)\\&+h\big[g((\nabla_XA)Y,Z)+g((\nabla_YA)Z,X)+g((\nabla_ZA)X,Y)\big]\\&-g((\nabla_XA)Y,AZ)-g(AY,(\nabla_XA)Z)-g((\nabla_YA)Z,AX)\\&-g(AZ,(\nabla_YA)X)-g((\nabla_ZA)X,AY)-g(AX,(\nabla_ZA)Y).\end{aligned} \tag{3.9}$$

Suppose the Ricci tensor in M is cyclic parallel. Then LHS and hence RHS of (3.9) is zero. Substituting (2.3), (2.4) in (3.9), letting $Y=Z=\{e_i\}$, and summing over i, $1\le i\le n$, we get

$$(Xh)h+2(AX)(h)=0, \tag{3.10}$$

$$h(Xh)+2\alpha(Xh)+2\beta\eta(X)(\xi h)=0. \tag{3.11}$$

Replacing X by ξ in above equation, we get either $h=-2(\alpha+\beta)$ or $(\xi h)=0$. If $(\xi h)=0$ in (3.11) then we get $Xh=0$, provided $h\neq-2\lambda$. In either case h is a constant.
Conversely, suppose $h=-2(\alpha+\beta)$ or any other constant. Then from (3.9), we get

$$\begin{aligned}&(\nabla_XS)(Y,Z)+(\nabla_YS)(Z,X)+(\nabla_ZS)(X,Y)=\\&h\Big[g((\nabla_XA)Y,Z)+g((\nabla_YA)Z,X)+g((\nabla_ZA)X,Y)\Big]\\&-g((\nabla_XA)Y,AZ)-g(AY,(\nabla_XA)Z)-g((\nabla_YA)Z,AX)\\&-g(AZ,(\nabla_YA)X)-g((\nabla_ZA)X,AY)-g(AX,(\nabla_ZA)Y.\end{aligned} \tag{3.12}$$

Using (2.4) in (3.12), we obtain

$$(\nabla_XS)(Y,Z)+(\nabla_YS)(Z,X)+(\nabla_ZS)(X,Y)=0. \tag{3.13}$$

□

Definition 3.4. *A complex space form is said to be a ϕ-recurrent manifold if there exists a non-zero 1-form α such that*

$$\phi^2((\nabla_XR)(Y,Z)W)=\alpha(X)R(Y,Z)W, \qquad X,Y,Z,W\in TM. \tag{3.14}$$

Theorem 3.5. *Let M be a totally η-umbilical ϕ-recurrent hypersurface of a complex space form $M^n(c)$ with $h=trA$. If $A\xi=\lambda\xi$ then $M^n(c)$ is*
(i) Euclidean $(c=0)$ if $h=\lambda$,
(ii) projective $(c>0)$ if $h<\lambda$,
(iii) hyperbolic$(c<0)$ if $h>\lambda$.

Proof. Using (2.1) in (3.14), we have

$$-((\nabla_XR)(Y,Z)W)+\eta((\nabla_XR)(Y,Z)W)\xi=\alpha(X)R(Y,Z)W. \tag{3.15}$$

Using (2.2), (2.3) and (2.4) in (3.15), we get

$$\begin{aligned}
&\frac{c}{4}\big[-g(AX,W)\eta(Z)\phi Y+g(AX,Z)\eta(W)\phi Y\\
&-g(\phi Z,W)\eta(Y)AX+g(\phi Z,W)g(AX,Y)\xi\\
&+g(AX,W)\eta(Y)\phi Z-g(AX,Y)\eta(W)\phi Z\\
&+g(\phi Y,W)\eta(Z)AX-g(\phi Y,W)g(AX,Z)\xi\\
&+2g(AX,Z)\eta(Y)\phi W-2g(AX,Y)\eta(Z)\phi W\\
&+g(\phi Y,Z)\eta(W)AX-g(\phi Y,Z)g(AX,W)\xi\\
&+g(\phi Z,W)\eta(Y)\eta(AX)\xi-g(\phi Z,W)g(AX,Y)\xi\\
&-g(\phi Y,W)\eta(Z)\eta(AX)\xi+g(\phi Y,W)g(AX,Z)\xi\\
&-g(\phi Y,Z)\eta(W)\eta(AX)\xi+g(\phi Y,Z)g(AX,W)\xi\big]\\
&-\beta g(\phi AX,Z)\eta(W)AY-\beta g(\phi AX,W)\eta(Z)AY\\
&-\beta g(AZ,W)g(\phi AX,Y)-\beta g(AZ,W)\eta(Y)\phi AX\\
&+\beta g(\phi AX,Y)\eta(W)AZ+\beta g(\phi AX,W)\eta(Y)AZ\\
&+\beta g(AY,W)g(\phi AX,Z)\xi+\beta g(AY,W)\eta(Z)\phi AX\\
&+\beta g(\phi AX,Z)\eta(W)\eta(AY)+\beta g(\phi AX,W)\eta(Z)\\
&\eta(AY)\xi+\beta g(AZ,W)g(\phi AX,Y)\xi-\beta g(\phi AX,Y)\\
&\eta(W)\eta(AZ)\xi-\beta g(\phi AX,W)\eta(Y)\eta(AZ)\xi\\
&-\beta g(AY,W)g(\phi AX,Z)\xi=\alpha(X)R(Y,Z)W.
\end{aligned}\tag{3.16}$$

Letting $W=Z=\xi$ in (3.16) and using (2.4), we get

$$\alpha(X)\Big[\frac{c}{4}Y+\lambda AY-(\frac{c}{4}+\lambda^2)\eta(Y)\xi\Big]=0. \tag{3.17}$$

If $\alpha(X)\neq 0$, then contracting (3.17) with respect to U letting $Y=U=\{e_i\}$, and summing over i, $1\le i\le n$, we obtain

$$c=\frac{2\lambda(\lambda-h)}{n-1}.$$

□

Theorem 3.6. *Let M be a real hypersurface of a complex space form $M_n(c)$, where the structure Jacobi operator R_ξ is of Lie-Codazzi type. Then $M_n(c)$ is a*
(i) complex Euclidean space if $\lambda=1$,
(ii) complex projective space if $\lambda>1$,
(iii)complex hyperbolic space if $0<\lambda<1$.

Proof. Suppose that

$$\begin{aligned}
&(L_XR_\xi)(Y)-(L_YR_\xi)(X)=(\xi\lambda)AY+\lambda\nabla_\xi AY\\
&-\eta(\nabla_\xi AY)\lambda\xi-\lambda\nabla_{AY}\xi-2\lambda\nabla_\xi Y+2\eta(A\nabla_\xi Y)\lambda\xi\\
&+2\lambda A\phi AY-2\eta(A\phi AY)\lambda\xi+\frac{c}{4}\phi AY-\frac{c}{4}\nabla_\xi Y.
\end{aligned}\tag{3.18}$$

Contracting $(L_XR_\xi)(Y)-(L_YR_\xi)(X)=0$ with ξ, we have

$$\big[2\lambda^2-2\lambda-\frac{c}{4}\big]\eta(\nabla_\xi Y)=0. \tag{3.19}$$

Thus, we obtain

$$c=8\lambda(\lambda-1).$$

□

4. Conclusion

When a Hopf real hypersurface M^{n-1} of a complex space form $M^n(c)$ is semi-symmetric, then $c<0$, but when it is Ricci-symmetric with $traceA=\lambda$ then $c>0$. When M^{n-1} is totally η- umbilical, Ricci tensor is cyclic paralle. Furhter it is shown that $M^n(c)$ is projective, Euclidean and hyperbolic according as $traceA=\lambda$ (or) $traceA<\lambda$ (or) $traceA>\lambda$.

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