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Observations on $x^2 + y^2 + 2(x+y) + 2 = 10z^2$

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Abstract
The quadratic equation with three unknowns given by $x^2 + y^2 + 2(x+y) + 2 = 10z^2$ is analysed for its non-zero distinct integer solutions. Given a solution, formula for generating sequence of solutions is obtained.

Keywords
Second degree equation, three unknowns, lattice points.

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1. Introduction

It is well-known that there are various choices of quadratic equations with three unknowns to obtain lattice points satisfying them [1,18]. Particularly in [2-17], different types of problems are presented. This paper deals with a different quadratic equation with three unknowns given by $x^2 + y^2 + 2(x+y) + 2 = 10z^2$to obtain a sequence of integral solutions. Further, a general formula for generating sequence of solutions based on the given solution is illustrated.

2. Method of analysis

The quadratic diophantine equation with three unknowns under consideration is

$$x^2 + y^2 + 2(x+y) + 2 = 10z^2 \tag{2.1}$$

Assuming

$$x = u + v\ ,\ y = u - v\ ,\ u \neq v \neq 0 \tag{2.2}$$

in (2.1), it gives

$$(u+1)^2 + v^2 = 5z^2 \tag{2.3}$$

Solving (2.3) through various approaches and employing (2.2), different sets of integer solutions to (2.1) are obtained. The above process is illustrated below:

Method: 1

Write 5 as

$$5 = (2+i)(2-i) \tag{2.4}$$

Let

$$z = a^2 + b^2\ ,\ a, b \neq 0 \tag{2.5}$$

Using (2.4) and (2.5) in (2.3) and applying factorization, consider

$$u + 1 + iv = (2+i)(a+ib)^2$$

Equating the real and imaginary parts, one gets

$$u = 2(a^2 - b^2 - ab) - 1$$
$$v = a^2 - b^2 + 4ab$$

In view of (2.2), the values of x and y are given by

$$\begin{cases} x = 3a^2 - 3b^2 + 2ab - 1 \\ y = a^2 - b^2 - 6ab - 1 \end{cases} \tag{2.6}$$

Thus, (2.5) and (2.6) represent the non-zero distinct integer solutions to (2.1).

Note: 1

Observe that (2.5) is also written as

$$5=(1+2i)(1-2i) \tag{2.7}$$

Following the analysis as presented above, the corresponding values of x and y are given by

$$x=3a^2-3b^2-2ab-1$$
$$y=-a^2+b^2-6ab-1$$

The above values of x and y along with (2.5) represent the integer solutions to (2.1).

Method: 2

One may write (2.3) as

$$(u+1)^2+v^2=5z^2*1 \tag{2.8}$$

Write 1 as

$$1=\frac{(p^2-q^2+i2pq)(p^2-q^2-i2pq)}{(p^2+q^2)^2}\ ,\ p>q>0 \tag{2.9}$$

Substituting (2.4), (2.5) and (2.9) in (2.8) and using factorization, define

$$u+1+iv=\frac{(p^2-q^2+i2pq)(2+i)(a+ib)^2}{(p^2+q^2)}$$

from which, one obtains

$$u+1=\frac{1}{(p^2+q^2)}[2(p^2-q^2)(a^2-b^2-ab) - 2pq(a^2-b^2+4ab)]$$

$$v=\frac{1}{(p^2+q^2)}[(p^2-q^2)(a^2-b^2+4ab)+4pq(a^2-b^2-ab)]$$

Employing (2.2) , note that

$$\begin{cases} x=\frac{1}{(p^2+q^2)}[(p^2-q^2)(3a^2-3b^2+2ab) \\ +2pq(a^2-b^2-6ab)]-1 \\ \\ y=\frac{1}{(p^2+q^2)}[(p^2-q^2)(a^2-b^2-6ab) \\ -2pq(3a^2-3b^2+2ab)]-1 \end{cases} \tag{2.10}$$

Hence, (2.10) and (2.5) represent integer solutions to (2.1) for suitable choices of a and b.

Note: 2

In (2.8), one may consider (2.7) for (2.5) and proceeding similarly, another choice for x and y is found.

3. Formula for generating sequence of solutions

Let (x_0,y_0,z_0) satisfy (2.1). The solution may be in real integers or in Gausssian integers or in irrational numbers. Let (x_1,y_1,z_1) be the second solution of (2.1), where

$$x_1=x_0+2h, y_1=y_0+2h, z_1=h-z_0 \tag{3.1}$$

in which h is an unknown to be determined.
Substitution of (3.1) in (2.1) gives

$$h=2x_0+2y_0+10z_0+4 \tag{3.2}$$

Using (3.2) in (3.1) , the second solution (x_1,y_1,z_1) of (2.1) is expressed in the matrix form as

$$(x_1,y_1,z_1)^t=M(x_0,y_0,z_0)^t$$

where t is the transpose and

$$M=\begin{pmatrix} 5 & 4 & 20 & 8 \\ 4 & 5 & 20 & 8 \\ 2 & 2 & 9 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Following the above procedure, the general solution $(x_{n+1},y_{n+1},z_{n+1})$ of (2.1) is written in the matrix form as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \\ 1 \end{pmatrix}=\begin{pmatrix} \frac{Y_n+1}{2} & \frac{Y_n-1}{2} & 10X_n & Y_n-1 \\ \frac{Y_n-1}{2} & \frac{Y_n+1}{2} & 10X_n & Y_n-1 \\ X_n & X_n & Y_n & 2X_n \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{pmatrix}$$

$n=0,1,2,3,....$ where (X_n,Y_n) is the general solution of the Pellian equation $Y^2=20X^2+1$.

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References

[1] L.E.Dickson, History of Theory of Numbers, Vol.II, *Chelsea Publishing Company, New York*, 1952.

[2] M.A. Gopalan and G. Srividhya, Observations on $y^2=2x^2+z^2$, *Archimedes J.Math.*, 2(1)(2012), 7–15.

[3] M.A. Gopalan and G. Sangeetha, Observations on $y^2=3x^2-2z^2$, *Antarctica J. Math.*, 9(4)(2012), 359–362.

[4] M.A. Gopalan and R. Vijayalakshmi, On the ternary quadratic equation $x^2=(\alpha^2-1)(y^2-z^2)$, $\alpha>1$, *Bessel J. Mat.h*, 2(2)(2012), 147–151.

[5] Manju Somanath, G. Sangeetha and M.A. Gopalan, On the homogeneous ternary quadratic Diophantine equation $x^2+(2k+1)y^2=(k+1)^2z^2$, *Bessel J. Math.*, 2(2)(2012), 107–110.

[6] Manju Somanath, G. Sangeetha and M.A. Gopalan, Observations on the ternary quadratic equation $y^2=3x^2+z^2$, *Bessel J. Math.,* 2(2)(2012), 101–105.

[7] M.A. Gopalan, S. Vidhyalakshmi and C. Nithya, Integral points on the ternary quadratic Diophantine equation $3x^2+5y^2=128z^2$,*Bulletin of Mathematics and Statistics Research*, 2(1)(2014), 25–31.

[8] K. Meena, S. Vidhyalakshmi, M.A. Gopalan and S. Aarthy Thangam, Integer solutions on the homogeneous cone $4x^2+3y^2=28z^2$, *Bulletin of Mathematics and Statistics Research*, 2(1)(2012), 47–53.

[9] M.A. Gopalan, S. Vidhyalakshmi and J. Umarani, On the ternary quadratic diophantine equation $6(x^2+y^2)-8xy=21z^2$, *Sch. J. Eng. Tech.,* 2(2A)(2014), 108–112.

[10] K. Meena, S. Vidhyalakshmi, M.A. Gopalan and S. Aarthy Thangam, On homogeneous ternary quadratic diophantine equation $2(x^2+y^2)-3xy=16z^2$, *International Journal of Engineering, Science and Mathematics*, 3(2)(2014), 93–99.

[11] K. Meena, S. Vidhyalakshmi, S. Divya and M.A. Gopalan, Integral points on the cone $Z^2=41X^2+Y^2$, *Sch. J. Eng. Tech.,* 2(2B)(2014), 301–304.

[12] M.A. Gopalan, S. Vidhyalakshmi, S. Devibala and J. Umavathy, On the ternary quadratic diophantine equation $3(x^2+y^2)-5xy=60z^2$, *International Journal of Applied Research*, 1(5)(2015), 234–238.

[13] S. Vidhyalakshmi, T. Geetha and R. Sridevi, On ternary quadratic diophantine equation $2x^2-7y^2=25z^2$, *International Journal of Applied Research*, 1(4)(2015), 111–114.

[14] M.A. Gopalan, S. Vidhyalakshmi, U.K. Rajalakshmi, On ternary quadratic diophantine equation $5(x^2+y^2)-6xy=196z^2$, *IJRDO- Journal of Mathematics,* 3(5)(2017), 1–10.

[15] S. Vidhyalakshmi, M.A. Gopalan and S. Aarthy Thangam, Observation on the elliptic paraboloid $x^2+y^2=19z$, *Asian Journal of Applied Science and Technology,* 1(9)(2017), 37–39.

[16] S. Vidhyalakshmi and S. Thenmozhi, On the ternary quadratic diophantine equation $3(X^2+Y^2)-5XY=75Z^2$, *Journal of Mathematics and Informatics,* 10(2017), 11–19.

[17] S. Vidhyalakshmi and A. Priya, On the non-homogeneous ternary quadratic diophantine equation $2(x^2+y^2)-3xy+(x+y)+1=z^2$, *Journal of Mathematics and Informatics*, 10(2017), 49–55.

[18] L.J. Mordell, *Diophantine Equations*, Academic press, New York, 1969.

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