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Cubic Γ-n normed linear spaces

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Abstract

This paper is aimed to propose the notion of cubic Γ-n-normed linear spaces based on the theory of cubic n-normed linear space, fuzzy n-normed linear space, interval valued fuzzy n-normed linear space and cubic sets. The concept of convergence and Cauchy sequences in cubic Γ-n-normed linear space are introduced and we provide some results on it. Also, this paper introduces the notion of completeness in cubic Γ-n-normed linear space.

Keywords

Cubic Γ-n-normed linear space, cubic n-normed linear space, interval valued fuzzy n-normed linear space, cubic sets.

AMS Subject Classification

46-XX, 46S40, 03E72.

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Contents

1. Introduction

A significant theory on 2-normed space was initially introduced by Gahler [2]. Consequently, Misak [9], Kim and Cho, Malceski [8], Hendra Gunwan and Mashadi [3] took an effort in developing this theory to a great extent. Zadeh [17] in 1965, first introduced the notion of fuzzy sets. This introduction laid foundation for the development of various structures in mathematics. This theory has a wide range of applications in several branches of mathematics such as logic set theory, group theory, real analysis, measure theory, topology etc. Fuzzy groups, fuzzy rings, fuzzy semigroup, fuzzy topology, fuzzy norm and so on are few interesting topics emerged after the development of fuzzy sets. Fuzzy concepts also play a vital role in image processing, Pattern recognition, medical diagnosis, neural network theory on so on. Later on, the notion of interval-valued fuzzy sets was introduced by Zadeh [18] in 1975, as an extension of fuzzy sets, that is, fuzzy sets with interval valued membership functions. Katsaras and Liu [7] introduced the concepts of fuzzy vector and fuzzy topological vector spaces. In studying fuzzy topological vector spaces, Katsaras in 1984 [6], first introduced the notion of fuzzy norm on a linear space. In [10] Vijayabalaji introduced the notion of fuzzy n-normed linear space as a generalisation of n-normed space by Gunwan and Mashadi. The concept of intutionistic n-normed linear space, interval valued fuzzy linear space and interval valued fuzzy n-normed linear space are discussed in [14], [13]. Jun et al.[4] have introduced a noticeable theory of cubic sets which comprises of interval-valued fuzzy set and a fuzzy set. A detailed theory of cubic linear space can be found in [16], [15]. The concept of Γ-ring was introduced by Nobusawa [11] more general than a ring. Barnes [1] gave the definition of Γ-ring as a generalisation of a ring and he has developed some other concepts of Γ-rings such as Γ-homomorphism, prime and primary ideals, m-systems etc. The notion of Γ-vector spaces was introduced by Sabur Uddin and Payer Ahamed [12]. Inspired by the above theories Vijayabalaji [5] constructed 2-normed and n-normed left Γ-linear space as a generalisation of n-normed linear space. He also introduced the notion of n-functional in n-normed left Γ-linear space. Inspirited by the above theory we introduce the notion of cubic Γ-n-normed linear space and also define convergent and cauchy sequences in cubic Γ-n-normed linear space.

2. Preliminaries

Definition 2.1. *Let M and Γ be two additive abelian groups. Suppose that there is a mapping from $M \times \Gamma \times M \to M$ (send-*

ing (x,α,y) into $x\alpha y$) such that
(1) $(x+y)\alpha z = x\alpha z + y\alpha z$
(2) $x(\alpha+\beta)z = x\alpha z + x\beta z$
(3) $x\alpha(y+z) = x\alpha y + x\alpha z$
(4) $(x\alpha y)\beta z = x\alpha(y\beta z)$
where $x,y,z\in M$ and $\alpha,\beta\in\Gamma$. Then M is called a Γ-ring.

Definition 2.2. *A subset A of the Γ-ring M is a left (right) ideal of M if A is an additive abelian subgroup of M and $M\Gamma A=\{c\alpha a|c\in M,\alpha\in\Gamma,a\in A\}$ ($A\Gamma M=\{a\alpha c|a\in A,\alpha\in\Gamma,c\in M\}$) is contained in A. If A is both a left and a right ideal of M, then we say that A is an ideal or two sided ideal of M.*

Definition 2.3. *Let M be a Γ-ring. Then M is called a division Γ-ring if it has an identity element and its only non zero ideal is itself.*

Definition 2.4. *Let $(V,+)$ be an abelian group. Let Δ be a division Γ-ring with identity 1 and let $\varphi:\Delta\times\Gamma\times V\to V$, where we denote $\varphi(\delta,\gamma,v)$ by $(\delta\gamma v)$. Then V is called a left Γ-vector space over Δ, if for all $\delta_1,\delta_2\in\Delta$, $v_1,v_2\in V$ and $\beta,\gamma\in\Gamma$, the following hold*
(1) $\delta_1\gamma(v_1+v_2) = \delta_1\gamma v_1 + \delta_2\gamma v_2$
(2) $(\delta_1+\delta_2)\gamma v_1 = \delta_1\gamma v_1+\delta_2\gamma v_2$
(3) $(\delta_1\beta\delta_2)\gamma v_1 = \delta_1\beta(\delta_2\gamma v_1)$
(4) $1\gamma v_1 = v_1$ *for some $\gamma\in\Gamma$*
We call the elements v of V are vectors and the elements δ of Δ are scalars. We also call $\delta\gamma v$ the scalar multiple of v by δ. Similarly, we can also define right Γ-vector space over Δ.

Definition 2.5. *Let V be a left Γ- linear space over Δ. A real valued function $\|.,.\|:V\times V\to[0,\infty)$ satisfying the following properties.*
(1) $\|\delta_1\gamma v_1,\delta_2\gamma v_2\|=0$ *if and only if v_1 and v_2 are linearly Γ dependent over Δ*
(2) $\|\delta_1\gamma v_1,\delta_2\gamma v_2||=\|\delta_2\gamma v_2,\delta_1\gamma v_1||$
(3) $\|\delta_1\gamma v_1,\alpha\delta_2\gamma v_2\|=|\alpha|\|\delta_1\gamma v_1,\delta_2\gamma v_2\|$ *for any $\alpha\in\Gamma$*
(4) $\|\delta_1\gamma v_1,\delta_2\gamma v_2+\delta_3\gamma v_3\|\le\|\delta_1\gamma v_1,\delta_2\gamma v_2\|+\|\delta_1\gamma v_1,\delta_3\gamma v_3\|$
for all $\delta_1,\delta_2,\delta_3\in\Delta,v_1,v_2,v_3\in V,\gamma\in\Gamma$.
is called 2-norm on left Γ-linear space V and the pair (V, $||.,.||$) is called an 2-normed left Γ-linear space over Δ.

Definition 2.6. *Let V be a left Γ- linear space over Δ. A real valued function on V^n satisfying the following four properties:*
(1) $\|\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n\|=0$ *if any only if v_1,v_2,..., v_n are linearly Γ-dependent over Δ*
(2) $\|\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n\|$ *is invariant under any permutation of v_1, v_2, ..., v_n*
(3) $\|\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\alpha\delta_n\gamma v_n\|=|\alpha|\ \|\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n\|$, *for any $\alpha\in\Gamma$*
(4) $\|\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_{n-1}\gamma v_{n-1},\delta\gamma y+\delta'\gamma z\|\le\|\delta_1\gamma v_1,\dots,\delta_{n-1}\gamma v_{n-1},\delta\gamma y\|+\|\delta_1\gamma v_1,\dots,\delta_{n-1}\gamma v_{n-1},\delta'\gamma z\|$
for all $\delta_1,\delta_2,\dots,\delta_n,\delta_n,\delta,\delta'\in\Delta,v_1,v_2,\dots,v_n,y,z\in V$, $\gamma\in\Gamma$
is called an n-norm on left Γ-linear space V and the pair $(V,\|.,...,.\|)$ is called an n-normed left Γ-linear space over Δ.

Example 2.7. *Let $V=\mathbb{R}$, be a left Γ-linear space over a division $\Gamma-$ ring $\Delta=\mathbb{R}$. Let $\Gamma=\mathbb{Z}$ be an additive abelian group and define $\|.,...,.\|$ on V by*

$$\|\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n\|=\sum_{k_1}\dots\sum_{k_n}|det(\delta_i\gamma v_{k_i})|$$

Then $(V,\|.,...,.\|)$ is called an n-normed left Γ-linear space over Δ.

Definition 2.8. *A sequence $\{\delta_n\gamma v_n\}$ in an n-normed left Γ-linear space $(V,\|.,...,.\|)$ is said to converge to $\delta\gamma v\in V$ if*

$$\lim_{n\to\infty}\|\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_{n-1}\gamma v_{n-1},\delta_n\gamma v_n-\delta\gamma v\|=0.$$

Definition 2.9. *A sequence $\{\delta_n\gamma v_n\}$ in an n-normed left Γ-linear space $(V,\|.,...,.\|)$ is called a cauchy sequence if*

$$\lim_{n,k\to\infty}\|\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_{n-1}\gamma v_{n-1},\delta_n\gamma v_n-\delta_k\gamma v_k\|=0.$$

Definition 2.10. *An n-normed left Γ-linear space is said to be complete if every cauchy sequence in it is convergent.*

Definition 2.11. *An interval number on $[0,1]$, say $\overline{a}$, is a closed sub interval of $[0,1]$, that is $\overline{a}=[a^-,a^+]$, where $0\le a^-\le a^+\le 1$. Let $D[0,1]$ denote the family of all closed sub-intervals of $[0,1]$, that is,*
$D[0,1]=\{\overline{a}=[a^-,a^+]:a^-\le a^+$ *and* $a^-,a^+\in[0,1]\}$.

Definition 2.12. *Let X be a set. A mapping $\overline{A}:\mathrm{X}\to D[0,1]$ is called an interval valued fuzzy set (briefly, an i-v fuzzy set) of X, where $\overline{A}(x)=[A^-(x),A^+(x)]$, for all $x\in\mathrm{X}$, and A^- and A^+ are fuzzy sets in X such that $A^-(x)\le A^+(x)$ for all $x\in\mathrm{X}$.*

Definition 2.13. *Let X be a nonempty set. A cubic set $\mathscr{A}$ in a set X is a structure*
$\mathscr{A}=\{\langle x,\overline{\mu}_A(x),\lambda(x)\rangle:x\in X\}$ which is briefly denoted by $\mathscr{A}$ $=\langle\overline{\mu}_A,\lambda\rangle$ where $\overline{\mu}_A=[\mu_A^-,\mu_A^+]$ is an interval valued fuzzy set (briefly, IVF) in X and $\lambda:X\to[0,1]$ is a fuzzy set in X.

Definition 2.14. *Let V be a linear space over a field F, $(V,\overline{\mu})$ be an interval-valued fuzzy linear space and (V,η) be a fuzzy linear space of V. A cubic set $\mathscr{A}=\langle\overline{\mu},\eta\rangle$ in V is called a cubic linear space of V if it satisfies for all $x,y\in V$ and $\alpha,\beta\in F$:*
(a) $\overline{\mu}(\alpha x*\beta y)\ge min\{\overline{\mu}(x),\overline{\mu}(y)\}$,
(b) $\eta(\alpha x*\beta y)\le max\{\eta(x),\eta(y)\}$.

Definition 2.15. *A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:*
(1) $*$ is commutative and associative.
(2) $*$ is continuous.
(3) $a*1=a$ for all $a\in[0,1]$.
(4) $a*b\le c*d$ whenever $a\le c$ and $b\le d$ and $a,b,c,d\in[0,1]$.

Definition 2.16. *A binary operation $\diamond:[0,1]\times[0,1]\to[0,1]$ is a continuous t-co-norm if $\diamond$ satisfies the following conditions:*
(1) $\diamond$ is commutative and associative.
(2) $\diamond$ is continuous.
(3) $a\diamond 0=a$, for all $a\in[0,1]$.
(4) $a\diamond b\le c\diamond d$ whenever $a\le c$ and $b\le d$ and $a,b,c,d\in[0,1]$.

3. Main Results

Let V be a left Γ- linear space over Δ. Let $N : V^n \times [0,\infty) \to [0,1]$ and $\overline{N} : X^n \times [0,\infty) \to [0,1]$ be a fuzzy set and an interval-valued fuzzy set respectively. A structure $\mathscr{C} = (V,N,\overline{N})$ is a cubic Γ-n-normed linear space (or) briefly cubic Γ-n- NLS if it satisfies the following properties:

(1) $N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n,t) > 0$.

(2) $N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n,t) = 0$ if and only if $v_1,v_2,...,v_n$ are linearly dependent.

(3) $N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n,t)$ is invariant under any permutation of $v_1,v_2,...,v_n$.

(4) $N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,c\delta_n\gamma v_n,t) = N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n,\frac{t}{|c|})$, if $c \neq 0, c \in \Gamma$.

(5) $N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n+\delta_n\gamma v_n^{'},s+t) \le N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n,s) \diamond N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n^{'},t)$.

(6) $N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n,t)$ is left continuous and non-increasing function of $t \in \mathbb{R}$ such that

$$\lim_{t\to\infty} N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n,t) = 0.$$

(7) $\overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n,t) > \overline{0}$.

(8) $\overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n,t) = \overline{1}$ if and only if $v_1,v_2,...,v_n$ are linearly dependent.

(9) $\overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n,t)$ is invariant under any permutation of $v_1,v_2,...,v_n$.

(10) $\overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,c\delta_n\gamma v_n,t) = \overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n,\frac{t}{|c|})$, if $c \neq 0, c \in \Gamma$.

(11) $\overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n+\delta_n\gamma v_n^{'},s+t) \ge \overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n,s) * \overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n^{'},t)$.

(12) $\overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n,t)$ is left continuous and non-decreasing function of $t \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n,t) = \overline{1}.$$

Example

Let $(V,\|\cdot,\dots,\cdot\|)$ be an n-normed left Γ-linear space over Δ. Define $a*b = \min\{a,b\}$ and $a\diamond b = \max\{a,b\}$ for $a,b \in [0,1]$. Also define

$$N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n,t) = \frac{\|\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n\|}{t+\|\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n\|}$$

and

$$\overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n,t) = \frac{t}{t+\|\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n\|}.$$

Then $\mathscr{C} = (V,N,\overline{N})$ is a cubic Γ-n-normed linear space.

Notion of Convergent sequence and Cauchy sequence in a cubic Γ-n-normed linear space

Definition 3.1. *A sequence $\{\delta_n\gamma v_n\}$ in $\mathscr{C} = (V,N,\overline{N})$ a cubic Γ-n-NLS is said to converge to $\delta\gamma v$ if given $r > 0$, $t > 0$, $0 < r < 1$, there exists an integer $n_0 \in \mathrm{N}$ such that $N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v,t) < r$ and $\overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v,t) > 1-r$ for all $n \ge n_0$*

Theorem 3.2. *In a cubic Γ n-NLS $\mathscr{C} = (V,N,\overline{N})$ a sequence $\{\delta_n\gamma v_n\}$ converges to $\delta\gamma v$ if and only if $N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v,t) \to 0$ and $\overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v,t) \to 1$, as $n \to \infty$*

Proof. Fix $t > 0$. Suppose $\{\delta_n\gamma v_n\}$ converges to $\delta\gamma v$. Then for a given r, $0 < r < 1$, there exists an integer $n_0 \in \mathrm{N}$ such that $N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v,t) < r$ and $\overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v,t) > 1-r$.
Thus $N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v,t) < r$ and $1-\overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v,t) < r$ and hence $N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v,t) \to 0$ and
$\overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v,t) \to 1$, as $n \to \infty$.
conversely, if for each $t > 0$, $N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v,t) \to 0$ and $\overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v,t) \to 1$, as $n \to \infty$, then for every r, $0 < r < 1$, there exists an integer n_0 such that $N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v,t) < r$ and $1-\overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v,t) < r$ for all $n \ge n_0$.
Thus $N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v,t) < r$ and $\overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v,t) > 1-r$ for all $n \ge n_0$. Hence $\{\delta_n\gamma v_n\}$ converges to $\delta\gamma v$ in $\mathscr{C} = (V,N,\overline{N})$. □

Definition 3.3. *A sequence $\{\delta_n\gamma v_n\}$ in a cubic Γ-n-NLS $\mathscr{C} = (V,N,\overline{N})$ is said to be cauchy sequence if given $\varepsilon > 0$, with $0 < \varepsilon < 1$, $t > 0$ there exists an integer $n_0 \in \mathrm{N}$ such that $N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta_k\gamma v_k,t) < \varepsilon$ and $\overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta_k\gamma v_k,t) > 1-\varepsilon$ for all $n,k \ge n_0$.*

Theorem 3.4. *In a cubic Γ-n-NLS $\mathscr{C} = (V,N,\overline{N})$ every convergent sequence is a cauchy sequence.*

Proof. Let $\{\delta_n\gamma v_n\}$ be a convergent sequence in $\mathscr{C} = (V,N,\overline{N})$. Suppose $\{\delta_n\gamma v_n\}$ converges to $\delta\gamma v$. Let $t > 0$ and $\varepsilon \in (0,1)$. Choose $r \in (0,1)$ such that $r\diamond r < \varepsilon$ and $(1-r)*(1-r) > 1-\varepsilon$.
Since $\{\delta_n\gamma v_n\}$ converges to $\delta\gamma v$,
we have an integer $n_0 \in \mathrm{N} \ni N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v,\frac{t}{2}) < r$ and $\overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v,\frac{t}{2}) > 1-r$ for all $n \ge n_0$.
Now,

$N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta_k\gamma v_k,t)$
$= N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v + \delta\gamma v - \delta_k\gamma v_k,\frac{t}{2}+\frac{t}{2})$
$\le N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v,\frac{t}{2}) \diamond N(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v,\frac{t}{2})$
$< r\diamond r \quad forall \quad n,k \ge n_0$
$< \varepsilon \quad forall \quad n,k \ge n_0$.

Also,

$\overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta_k\gamma v_k,t)$
$= \overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v + \delta\gamma v - \delta_k\gamma v_k,\frac{t}{2}+\frac{t}{2})$
$\ge \overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v,\frac{t}{2}) * \overline{N}(\delta_1\gamma v_1,\delta_2\gamma v_2,\dots,\delta_n\gamma v_n - \delta\gamma v,\frac{t}{2})$
$> (1-r)*(1-r) \quad forall \quad n,k \ge n_0$

$> 1-\varepsilon \quad forall \quad n,k \geq n_0$
Therefore $\{\delta_n\gamma v_n\}$ is a cauchy sequence in $\mathscr{C} = (V,N,\overline{N})$. □

Definition 3.5. *A cubic Γ-n-NLS $\mathscr{C} = (V,N,\overline{N})$ is said to be complete if every cauchy sequence in it is convergent.*

Remark 3.6. *The following example shows that there may exist cauchy sequence in cubic Γ-n-NLS $\mathscr{C} = (V,N,\overline{N})$ which is not convergent.*

Example 3.7. *Consider a cubic Γ-n-NLS $\mathscr{C} = (V,N,\overline{N})$ as in the previous example*
Let $\{\delta_n\gamma v_n\}$ be a sequence in $\mathscr{C} = (V,N,\overline{N})$ then
(a) $\{\delta_n\gamma v_n\}$ is a cauchy sequence in $(V,\|.,...,.\|)$ if and only if $\{\delta_n\gamma v_n\}$ is a cauchy sequence in $\mathscr{C} = (V,N,\overline{N})$.
(b) $\{\delta_n\gamma v_n\}$ is a convergent sequence in $(V,\|.,...,.\|)$ if and only if $\{\delta_n\gamma v_n\}$ is convergent in $\mathscr{C} = (V,N,\overline{N})$.

Proof. (a) $\{\delta_n\gamma v_n\}$ is a cauchy sequence in $(V,\|.,...,.\|)$
$\Leftrightarrow \lim_{n,k\to\infty} \|\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta_k\gamma v_k\| = 0$
$\Leftrightarrow \lim_{n,k\to\infty} N(\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta_k\gamma v_k)$
$= \frac{\|\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta_k\gamma v_k\|}{t+\|\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta_k\gamma v_k\|} = 0$ and
$\lim_{n,k\to\infty} \overline{N}(\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta_k\gamma v_k)$
$= \frac{t}{t+\|\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta_k\gamma v_k\|} = 1$
$\Leftrightarrow N(\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta_k\gamma v_k, t) \to 0$ and
$\overline{N}(\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_{n-1}\gamma v_{n-1}, \delta_n\gamma v_n - \delta_k\gamma v_k, t) \to 1$ as $n,k \to \infty$
$\Leftrightarrow N(\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta_k\gamma v_k, t) < r$ and
$\overline{N}(\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta_k\gamma v_k, t) > 1-r,\ r \in (0,1)$,
$n,k \geq n_0$
$\Leftrightarrow \{\delta_n\gamma v_n\}$ is a cauchy sequence in $\mathscr{C}$
(b) $\{\delta_n\gamma v_n\}$ is a convergent sequence in $(V,\|.,...,.\|)$
$\Leftrightarrow \lim_{n\to\infty} \|\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta\gamma v\| = 0$
$\Leftrightarrow \lim_{n,k\to\infty} N(\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta\gamma v)$
$= \frac{\|\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta\gamma v\|}{t+\|\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta\gamma v\|} = 0$ *and*
$\lim_{n,k\to\infty} \overline{N}(\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta\gamma v)$
$= \frac{t}{t+\|\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta\gamma v\|} = 1$
$\Leftrightarrow N(\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta\gamma v, t) \to 0$ *and*
$\overline{N}(\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta\gamma v, t) \to 1$ *as* $n \to \infty$
$\Leftrightarrow N(\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta\gamma v, t) < r$ and
$\overline{N}(\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta\gamma v, t) > 1-r,\ r \in (0,1)$,
$n \geq n_0$
$\Leftrightarrow \{\delta_n\gamma v_n\}$ is a convergent sequence in $\mathscr{C}$
Thus if there exists an n-normed left Γ-linear space $(V,\|.,...,.\|)$ which is not complete, then the cubic Γ-n norm induced by such a crisp n-norm $\|.,...,.\|$ on an incomplete n-normed left Γ linear space V is an incomplete cubic Γ-n normed linear space. □

Theorem 3.8. *A cubic Γ-n-NLS $\mathscr{C} = (V,N,\overline{N})$ in which every cauchy sequence has a convergent subsequence is complete.*

Proof. Let $\{\delta_n\gamma v_n\}$ be a cauchy sequence in $\mathscr{C} = (V,N,\overline{N})$ and $\{\delta_n\gamma v_{n_k}\}$ be a subsequence of $\{\delta_n\gamma v_n\}$ that converges to $\delta\gamma v$. We need to prove that $\{\delta_n\gamma v_n\}$ converges to $\delta\gamma v$. Let $t > 0$ and $\varepsilon \in (0,1)$. Choose $r \in (0,1)$ suchthat $r \diamond r < \varepsilon$ and $(1-r) * (1-r) > 1-\varepsilon$.
Given that $\{\delta_n\gamma v_n\}$ is a cauchy sequence,
there exists an integer $n_0 \in \mathrm{N} \ni N(\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta_k\gamma v_k, \frac{t}{2}) < r$ and $\overline{N}(\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta_k\gamma v_k, \frac{t}{2}) > 1-r$ for all $n,k \geq n_0$.
Also since $\{\delta_n\gamma v_{n_k}\}$ converges to $\delta\gamma v$, there is a positive $i_k > n_0 \ni N(\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_{i_k} - \delta\gamma v, \frac{t}{2}) < r$ and $\overline{N}(\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_{i_k}\text{-}\delta\gamma v, \frac{t}{2}) > 1-r$
Now,
$N(\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta\gamma v, t)$
$= N(\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta_n\gamma v_{i_k} + \delta_n\gamma v_{i_k} - \delta\gamma v, \frac{t}{2} + \frac{t}{2})$
$\leq N(\delta_1\gamma v_1, \ldots, \delta_n\gamma v_n - \delta_n\gamma v_{i_k}, \frac{t}{2}) \diamond N(\delta_1\gamma v_1, \ldots, \delta_n\gamma v_{i_k} - \delta\gamma v, \frac{t}{2})$
$< r \diamond r$
$< \varepsilon$.
Also
$\overline{N}(\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta\gamma v, t)$
$= \overline{N}(\delta_1\gamma v_1, \delta_2\gamma v_2, \ldots, \delta_n\gamma v_n - \delta_n\gamma v_{i_k} + \delta_n\gamma v_{i_k} - \delta\gamma v \frac{t}{2} + \frac{t}{2})$
$\geq \overline{N}(\delta_1\gamma v_1, \ldots, \delta_n\gamma v_n - \delta_n\gamma v_{i_k}, \frac{t}{2}) * \overline{N}(\delta_1\gamma v_1, \ldots, \delta_n\gamma v_{i_k} - \delta\gamma v, \frac{t}{2})$
$> (1-r) * (1-r)$
$> 1-\varepsilon$.
Therefore $\{\delta_n\gamma v_n\}$ converges to $\delta\gamma v$ in $\mathscr{C} = (V,N,\overline{N})$ and hence it is complete. □

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