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Analysis of an $M^{[X]}/G_1(a,b),G_2(a,b)/1$ unreliable G-queue with optional re-service, Bernoulli vacation, delay time to two phase of repair

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Abstract

In this paper, we consider the queueing system where the batch of customers arrive at the system according to the compound Poisson process and two types of service, each of which has an optional reservice is provided to the server under Bernoulli vacation. After completion of each type of service, the customer may go for reservice of the same type of service without joining the tail of the queue or they may depart the system. An unpredictable breakdown may occur at any moment during the functioning of the server with any type of service or re-service and at that situation, the service channel will breakoff for a short period of time. A breakdown in a busy server is represented by the arrival of a negative customer which consequently leads to the loss of the customer who is in service. Delay time is referred to as the waiting time of the server for the two phase of repair to start. By considering elapsed service time as the supplementary variable, the PGF of the number of customers in the queue at a random epoch is derived and this PGF is further used to establish explicitly some of the following performance measures namely various states of the system, the mean queue length, and the mean waiting time in the queue. At last, some particular cases are discussed and the numerical illustrations are provided.

Keywords

Two types of service, Re-service, Bernoulli vacation, G-queue, Delay time to repair, Two phase of repair time.

AMS Subject Classification (2010)

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Contents

1. Introduction

A considerable amount of work has been done on the modelling and analysis for the queueing system using the supplementary variable technique where the service is rendered in bulk. Most of the queueing models assume that customers are served singly which is a contradiction to some of the real-life situations where the service is provided in bulk. Bulk service queue was first dealt by Bailey [2]. The "General Bulk Service Rule " (GBSR) was proposed by Neuts [13] in which service initiates only when a certain number of customers in the queue are available. A detailed survey on bulk queueing models can be seen in the studies of Chaudhury and Templeton [3]. Lee et al. [11] discussed the decompositions of the batch service queue with server vacations. Recently, Haridass and Arumuganathan [8] studied a batch arrival general bulk service queueing system by considering the supplementary variable as the remaining service time.

In many real-life situations, the concept of reservice may be easily seen. For example, in bank counters, supermarket, doctor clinics etc. Recently, Rajadurai et al. [14] analyzed the queueing system with optional re-service under modified vacation policy. Most recently, Choudhury and Chandi Ram Kalita [6] studied the queueing model with two types of service and optional repeated service. For this model, they derived the joint distribution of state of the server and queue size by considering both elapsed and remaining service time.

In a vacation queueing system, the term vacation is referred to as the period of time during which the server is unavailable due to many reasons like being checked for maintenance, scanning for new work or simply taking tea break. Bernoulli schedule vacation means that, with probability θ, the server may go for a vacation after the completion of service. Otherwise, with probability $1-\theta$, he may continue to stay in the system and this vacation policy is considered in this paper. A queueing model with a modified Bernoulli schedule vacation was briefly investigated by Choudhury and Madan [4] under N-policy. Queueing model with single working vacation and working interruption was examined by Gao and Liu [7] under Bernoulli schedule. Ayyappan and Shymala [1] have discussed about the concept of Bernoulli schedule vacation and random setup time.

G-queues are the queues with negative customers and this type of negative customers will remove and destroy a positive customer in service and consequently the positive customers loss his service and leave the system. G-queue with server breakdown, working vacation and vacation interruption has been analyzed by Zhang and Liu [16]. A non-Markovian retrial queue with negative customers under Bernoulli schedule vacation was considered by Wu and Lian [15].

G-queues with an unreliable server has also found applications in communication networks. In these models, if a negative customer arrives at a queue, a customer or a batch of customers in service may be removed which causes server failure. Madan and Ebrahim Malalla [12] discussed the two-phase repair with a delay in a bulk input single server queue. A queueing system with an unreliable server, randomized vacation policy and delayed repair has been analyzed by Ke and Huang [10] whereas the batch arrival unreliable server queue under randomised vacation policy has been discussed Ke et al. [9] and Choudhury and Deka [5].

The outline of the remaining sections is as follows. In section 2, we give the description of the present model. In section 3, we present the definitions, Kolmogorov forward equations and the transient solution of our model. In section 4 and 5, we finding the probability generating function of the stationary queue length at the random epoch and the system stability condition respectively. Some performance measures in the various states of the system, the mean queue size are derived in section 6. Some important particular cases are given in section 7. Computational results and graphs are presented in section 8. At last, summary of the work is presented in section 9.

2. Model Description

In this paper, the authors' best of our knowledge, no investigation published in the queueing literature with combination of batch arrival, bulk service and two types of service and re-service under Bernoulli schedule, G-queue (negative arrival), delay time to repair, two phase of repair. Customers arrive at the system in batches of variable size in a compound Poisson process. Let $\Lambda^+ c_i dt$ ($i \geq 1$) be the first order probability that a batch of i customers arrive at the system during a short interval of time $(t,t+dt]$, where $0 \leq c_i \leq 1$ and $\sum_{i=1}^{\infty} c_i = 1$ and $\Lambda^+ > 0$ is the mean arrival rate of batches. The server serves the customer under 'GBSR' rule. we consider a queueing system with two types of service where each type consists of an optional re-service. We presumed that the probability of providing First Type of Service (FTS) is p_1 and Second Type of Service (STS) is p_2 $(p_1+p_2=1)$. The server may repeat type i^{th} service to a batch of customers for whom the i^{th} type service is just completed, with probability π_i (i=1,2). If not, the batch of customers may leave the system with probability $(1-\pi_i)$. In addition, we assume that either service may be repeated only once. The server may opt to go for a vacation with probability θ or proceed to serve the next batch, if exist, with probability $(1-\theta)$ immediately after the completion of both type of service and re-service. Otherwise, the server remains idle in the system until a customer arrives. The negative customers arrive from outside the system according to a Poisson arrival rate λ^-. Negative customers cannot accumulate in a queue and do not receive service, will remove the positive customers being in service from the system. The server breakdown may be caused by such type of negative customers and for a short duration of time, the service channel may fail. As soon as the server gets fail, it takes delay time to start two phases of repair. The server will treat as good as new just after the completion of two phase of repair.

The service time, re-service time, vacation time, delay time to repair and two phase of repair time follow general distribution and notations used for the Cumulative Distribution Function(CDF), the probability density functions(pdf) are given in Table 1.

Table 1. Some notations for distribution function

Time	CDF	Hazard rate	pdf
i^{th} type of service $1 \le i \le 2$	$U_i(u)$	$\mu_i(u)$	$\mathrm{u}_i(h) = \mu_i(h)e^{-\int_0^h \mu_i(u)du}$
i^{th} type of re-service $1 \le i \le 2$	$U_i(u)$	$\mu_i(u)$	$\mathrm{u}_i(h) = \mu_i(h)e^{-\int_0^h \mu_i(u)du}$
Bernoulli vacation	$V(u)$	$\gamma(u)$	$\mathrm{v}(r) = \gamma(r)e^{-\int_0^r \gamma(u)du}$
Delay time to repair	$D(u)$	$\xi(u)$	$\mathrm{d}(g) = \xi(g)e^{-\int_0^g \xi(u)du}$
First phase of repair	$R_1(u)$	$\beta_1(u)$	$\mathrm{r}_1(w) = \beta_1(w)e^{-\int_0^w \beta_1(u)du}$
Second phase of repair	$R_2(u)$	$\beta_2(u)$	$\mathrm{r}_2(w) = \beta_2(w)e^{-\int_0^w \beta_2(u)du}$

3. Equations Governing the Systems

In this section, we have defined the system state equations for its stationary queue size distribution, by treating elapsed service time, elapsed re-service time, elapsed vacation time, elapsed delay time and the elapsed two phase of repair time, as the supplementary variables. Then these equations are solved and the PGFs of the stationary queue size distribution is derived.

Denote

$\mathscr{N}(t)$ - the queue size (including one batch of customers being served, if any) at time t.
$U_1^0(t)$ - the elapsed first type of service/re-service time at time t.
$U_2^0(t)$ - the elapsed second type of service/re-service time at time t.
$V^0(t)$ - the elapsed vacation time at time t.
$D^0(t)$ - the elapsed delay time to repair at time t.
$R_1^0(t)$ - the elapsed first phase of repair time at time t.
$R_2^0(t)$ - the elapsed second phase of repair time at time t.

Further, we introduce the following random variable:

$$\mathscr{Y}(t) = \begin{cases} 0, & \text{if the server is idle at time } t. \\ 1, & \text{if the server is busy at first type service at time } t. \\ 2, & \text{if the server is busy at second type service at time } t. \\ 3, & \text{if the server is busy at first type re-service at time } t. \\ 4, & \text{if the server is busy at second type re-service at time } t. \\ 5, & \text{if the server is on Bernoulli vacation period at time } t. \\ 6, & \text{if the server is under delay time to repair at time } t. \\ 7, & \text{if the server is under first phase of repair at time } t. \\ 8, & \text{if the server is under second phase of repair at time } t. \end{cases}$$

Thus the supplementary variable $U_1^0(t)$, $U_2^0(t)$, $V^0(t)$, $D^0(t)$, $R_1^0(t)$ and $R_2^0(t)$ for $i = 1,2$ are introduced in order to obtain a bivariate Markov process $\{\mathscr{N}(t), \mathscr{Y}(t)\}$ and define the following probabilities as:

$$\mathscr{Q}_r(t)dt = P\{\mathscr{N}(t) = r, \mathscr{Y}(t) = 0\}, \text{for } t \ge 0, \text{and } 0 \le r \le a-1$$

$$\mathscr{H}_{1,n}(t,u)du = P\{\mathscr{N}(t) = n, \mathscr{Y}(t) = 1; u \le U_1^0(t) \le u+du\}, \text{ for } t \ge 0, u \ge 0 \text{ and } n \ge 0$$

$$\mathscr{H}_{2,n}(t,u)du = P\{\mathscr{N}(t) = n, \mathscr{Y}(t) = 2; u \le U_2^0(t) \le u+du\}, \text{ for } t \ge 0, u \ge 0 \text{ and } n \ge 0$$

$$\mathscr{A}_{1,n}(t,u)du = P\{\mathscr{N}(t) = n, \mathscr{Y}(t) = 3; u \le U_1^0(t) \le u+du\}, \text{ for } t \ge 0, u \ge 0 \text{ and } n \ge 0$$

$$\mathscr{A}_{2,n}(t,u)du = P\{\mathscr{N}(t) = n, \mathscr{Y}(t) = 4; u \le U_2^0(t) \le u+du\}, \text{ for } t \ge 0, u \ge 0 \text{ and } n \ge 0$$

$$\mathscr{V}_n(t,u)du = P\{\mathscr{N}(t) = n, \mathscr{Y}(t) = 5; u \le V^0(t) \le u+du\}, \text{ for } t \ge 0, u \ge 0 \text{ and } n \ge 0$$

$$\mathscr{D}_n(t,u)du = P\{\mathscr{N}(t) = n, \mathscr{Y}(t) = 6; u \le D^0(t) \le u+du\}, \text{ for } t \ge 0, u \ge 0 \text{ and } n \ge 0$$

$$\mathscr{R}_{1,n}(t,u)du = P\{\mathscr{N}(t) = n, \mathscr{Y}(t) = 7; u \le R_1^0(t) \le u+du\}, \text{ for } t \ge 0, u \ge 0 \text{ and } n \ge 0$$

$$\mathscr{R}_{2,n}(t,u)du = P\{\mathscr{N}(t) = n, \mathscr{Y}(t) = 8; u \le R_2^0(t) \le u+du\}, \text{ for } t \ge 0, u \ge 0 \text{ and } n \ge 0$$

The Kolmogorov forward equations to govern the model; where sub index $i = 1,2$ denotes the FTS and STS respectively can be formulated as follows:

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial u}+(\Lambda^{+}+\Lambda^{-}+\mu_i(u))\right)\mathscr{H}_{i,n}(t,u)= \Lambda^{+}(1-\delta_{n,0})\sum_{k=1}^{n}c_k\mathscr{H}_{i,n-k}(t,u),\ n\geq 0 \quad (3.1)$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial u}+(\Lambda^{+}+\Lambda^{-}+\mu_1(u))\right)\mathscr{A}_{1,n}(t,u)= \Lambda^{+}(1-\delta_{n,0})\sum_{k=1}^{n}c_k\mathscr{A}_{1,n-k}(t,u),\ n\geq 0 \quad (3.2)$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial u}+(\Lambda^{+}+\Lambda^{-}+\mu_2(u))\right)\mathscr{A}_{2,n}(t,u)= \Lambda^{+}(1-\delta_{n,0})\sum_{k=1}^{n}c_k\mathscr{A}_{2,n-k}(t,u),\ n\geq 0 \quad (3.3)$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial u}+(\Lambda^{+}+\gamma(u))\right)\mathscr{V}_{n}(t,u)= \Lambda^{+}(1-\delta_{n,0})\sum_{k=1}^{n}c_k\mathscr{V}_{n-k}(t,u),\ n\geq 0 \quad (3.4)$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial u}+(\Lambda^{+}+\xi(u))\right)\mathscr{D}_{n}(t,u)= \Lambda^{+}(1-\delta_{n,0})\sum_{k=1}^{n}c_k\mathscr{D}_{n-k}(t,u),\ n\geq 0 \quad (3.5)$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial u}+(\Lambda^{+}+\beta_1(u))\right)\mathscr{R}_{1,n}(t,u)= \Lambda^{+}(1-\delta_{n,0})\sum_{k=1}^{n}c_k\mathscr{R}_{1,n-k}(t,u),\ n\geq 0 \quad (3.6)$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial u}+(\Lambda^{+}+\beta_2(u))\right)\mathscr{R}_{2,n}(t,u)= \Lambda^{+}(1-\delta_{n,0})\sum_{k=1}^{n}c_k\mathscr{R}_{2,n-k}(t,u),\ n\geq 0 \quad (3.7)$$

$$\begin{aligned}\frac{d}{dt}\mathscr{Q}_r(t)=&-\Lambda^{+}\mathscr{Q}_r(t)+\Lambda^{+}(1-\delta_{r,0})\sum_{k=1}^{r}c_k\mathscr{Q}_{r-k}(t)\\&+(1-\theta)\Big[(1-\pi_1)\int_0^\infty\mathscr{H}_{1,r}(t,u)\mu_1(u)du\\&+(1-\pi_2)\int_0^\infty\mathscr{H}_{2,r}(t,u)\mu_2(u)du\\&+\int_0^\infty\mathscr{A}_{1,r}(t,u)\mu_1(u)du\\&+\int_0^\infty\mathscr{A}_{2,r}(t,u)\mu_2(u)du\Big]\\&+\int_0^\infty\mathscr{V}_r(t,u)\gamma(u)du\\&+\int_0^\infty\mathscr{R}_{2,r}(t,u)\beta_2(u)du,\\&\qquad 0\leq r\leq a-1. \quad (3.8)\end{aligned}$$

where $\delta_{i,j}$ denotes Kronecker's delta.

These set of equations are to be solved under the following boundary conditions at $u=0$:

$$\begin{aligned}\mathscr{H}_{i,0}(t,0)=p_i\Big[&\Lambda^{+}\sum_{r=a}^{b}\sum_{k=0}^{a-1}c_{r-k}\mathscr{Q}_k(t)\\&+(1-\theta)\Big[(1-\pi_1)\sum_{r=a}^{b}\int_0^\infty\mathscr{H}_{1,r}(t,u)\mu_1(u)du\\&+(1-\pi_2)\sum_{r=a}^{b}\int_0^\infty\mathscr{H}_{2,r}(t,u)\mu_2(u)du\\&+\sum_{r=a}^{b}\int_0^\infty\mathscr{A}_{1,r}(t,u)\mu_1(u)du\\&+\sum_{r=a}^{b}\int_0^\infty\mathscr{A}_{2,r}(t,u)\mu_2(u)du\Big]\\&+\sum_{r=a}^{b}\int_0^\infty\mathscr{V}_r(t,u)\gamma(u)du\\&+\sum_{r=a}^{b}\int_0^\infty\mathscr{R}_{2,r}(t,u)\beta_2(u)du\Big],i=1,2 \quad (3.9)\end{aligned}$$

$$\begin{aligned}\mathscr{H}_{i,n}(t,0)=p_i\Big[&\Lambda^{+}\sum_{k=0}^{a-1}c_{b+n-k}\mathscr{Q}_k(t)\\&+(1-\theta)\Big[(1-\pi_1)\int_0^\infty\mathscr{H}_{1,n+b}(t,u)\mu_1(u)du\\&+(1-\pi_2)\int_0^\infty\mathscr{H}_{2,n+b}(t,u)\mu_2(u)du\\&+\int_0^\infty\mathscr{A}_{1,n+b}(t,u)\mu_1(u)du\\&+\int_0^\infty\mathscr{A}_{2,n+b}(t,u)\mu_2(u)du\Big]\\&+\int_0^\infty\mathscr{V}_{n+b}(t,u)\gamma(u)du\\&+\int_0^\infty\mathscr{R}_{2,n+b}(t,u)\beta_2(u)du\Big],i=1,2 \quad (3.10)\end{aligned}$$

$$\mathscr{A}_{1,n}(t,0)=\pi_1\int_0^\infty\mathscr{H}_{1,n}(t,u)\mu_1(u)du,\ n\geq 0 \quad (3.11)$$

$$\mathscr{A}_{2,n}(t,0)=\pi_2\int_0^\infty\mathscr{H}_{2,n}(t,u)\mu_2(u)du,\ n\geq 0 \quad (3.12)$$

$$\begin{aligned}\mathscr{V}_n(t,0)=\theta\Big[&(1-\pi_1)\int_0^\infty\mathscr{H}_{1,n}(t,u)\mu_1(u)du\\&+(1-\pi_2)\int_0^\infty\mathscr{H}_{2,n}(t,u)\mu_2(u)du\\&+\int_0^\infty\mathscr{A}_{1,n}(t,u)\mu_1(u)du\\&+\int_0^\infty\mathscr{A}_{2,n}(t,u)\mu_2(u)du\Big],\ n\geq 0 \quad (3.13)\end{aligned}$$

$$\begin{aligned}\mathscr{D}_n(t,0)=&\Lambda^{-}\int_0^\infty\mathscr{H}_{1,n}(t,u)du+\Lambda^{-}\int_0^\infty\mathscr{H}_{2,n}(t,u)du\\&+\Lambda^{-}\int_0^\infty\mathscr{A}_{1,n}(t,u)du\\&+\Lambda^{-}\int_0^\infty\mathscr{A}_{2,n}(t,u)du,\ n\geq 0 \quad (3.14)\end{aligned}$$

$$\mathscr{R}_{1,n}(t,0)=\int_0^\infty \mathscr{D}_n(t,u)\xi(u)du,\ n\geq 0 \tag{3.15}$$

$$\mathscr{R}_{2,n}(t,0)=\int_0^\infty \mathscr{R}_{1,n}(t,u)\beta_1(u)du,\ n\geq 0. \tag{3.16}$$

Further, it is assume that initially there are no adequate number of customers in the system and the server is idle. So the initial conditions are

$$\begin{aligned}&\mathscr{Q}_0(0)=1,\mathscr{Q}_r(0)=0\ \text{ for }\ 1\leq r\leq a-1,\\&\mathscr{H}_{i,n}(0)=\mathscr{A}_{i,n}(0)=\mathscr{R}_{1,n}(0)=\mathscr{R}_{2,n}(0)\\&=\mathscr{D}_n(0)=\mathscr{V}_n(0)=0 \text{ for } n\geq 0,\ i=1,2.\end{aligned} \tag{3.17}$$

Here, we use the probability generating functions to simplify equations (3.1) to (3.16)

$$\begin{aligned}&\mathscr{B}_i(t,u,w)=\sum_{n=0}^{\infty}\mathscr{B}_{i,n}(t,u)w^n\\&\mathscr{B}_i(t,w)=\sum_{n=0}^{\infty}\mathscr{B}_{i,n}(t)w^n;\mathscr{C}(w)=\sum_{n=1}^{\infty}c_nw^n;\\&\mathscr{G}(t,u,w)=\sum_{n=0}^{\infty}\mathscr{G}(t,u)w^n\\&\mathscr{G}(t,w)=\sum_{n=0}^{\infty}\mathscr{G}(t)w^n;\mathscr{Q}(w)=\sum_{r=0}^{a-1}\mathscr{Q}_rw^r;|w|=1\end{aligned} \tag{3.18}$$

where $\mathscr{B}=\mathscr{H},\mathscr{A},\mathscr{R}$; $\mathscr{G}$=$\mathscr{D},\mathscr{V}$: i=1,2.
Taking the Laplace transform of equations (3.1) to (3.16) and using (3.18), we get

$$\begin{aligned}&\left(\frac{\partial}{\partial u}+(s+\Lambda^++\Lambda^-+\mu_i(u))\right)\bar{\mathscr{H}}_{i,n}(s,u)=\\&\Lambda^+(1-\delta_{n,0})\sum_{k=1}^{n}c_k\bar{\mathscr{H}}_{i,n-k}(s,u),\ n\geq 0,i=1,2\end{aligned} \tag{3.19}$$

$$\begin{aligned}&\left(\frac{\partial}{\partial u}+(s+\Lambda^++\Lambda^-+\mu_1(u))\right)\bar{\mathscr{A}}_{1,n}(s,u)=\\&\Lambda^+(1-\delta_{n,0})\sum_{k=1}^{n}c_k\bar{\mathscr{A}}_{1,n-k}(s,u),\ n\geq 0\end{aligned} \tag{3.20}$$

$$\begin{aligned}&\left(\frac{\partial}{\partial u}+(s+\Lambda^++\Lambda^-+\mu_2(u))\right)\bar{\mathscr{A}}_{2,n}(s,u)=\\&\Lambda^+(1-\delta_{n,0})\sum_{k=1}^{n}c_k\bar{\mathscr{A}}_{2,n-k}(s,u),\ n\geq 0\end{aligned} \tag{3.21}$$

$$\begin{aligned}&\left(\frac{\partial}{\partial u}+(s+\Lambda^++\gamma(u))\right)\bar{\mathscr{V}}_n(s,u)=\\&\Lambda^+(1-\delta_{n,0})\sum_{k=1}^{n}c_k\bar{\mathscr{V}}_{n-k}(s,u),\ n\geq 0\end{aligned} \tag{3.22}$$

$$\begin{aligned}&\left(\frac{\partial}{\partial u}+(s+\Lambda^++\xi(u))\right)\bar{\mathscr{D}}_n(s,u)=\\&\Lambda^+(1-\delta_{n,0})\sum_{k=1}^{n}c_k\bar{\mathscr{D}}_{n-k}(s,u),\ n\geq 0\end{aligned} \tag{3.23}$$

$$\begin{aligned}&\left(\frac{\partial}{\partial u}+(s+\Lambda^++\beta_1(u))\right)\bar{\mathscr{R}}_{1,n}(s,u)=\\&\Lambda^+(1-\delta_{n,0})\sum_{k=1}^{n}c_k\bar{\mathscr{R}}_{1,n-k}(s,u),\ n\geq 0\end{aligned} \tag{3.24}$$

$$\begin{aligned}&\left(\frac{\partial}{\partial u}+(s+\Lambda^++\beta_2(u))\right)\bar{\mathscr{R}}_{2,n}(s,u)=\\&\Lambda^+(1-\delta_{n,0})\sum_{k=1}^{n}c_k\bar{\mathscr{R}}_{2,n-k}(s,u),\ n\geq 0\end{aligned} \tag{3.25}$$

$$\begin{aligned}(s+\Lambda^+)\bar{\mathscr{Q}}_0(s)=1+(1-\theta)&\\&\left[(1-\pi_1)\int_0^\infty\bar{\mathscr{H}}_{1,0}(s,u)\mu_1(u)du\right.\\&+(1-\pi_2)\int_0^\infty\bar{\mathscr{H}}_{2,0}(s,u)\mu_2(u)du\\&+\int_0^\infty\bar{\mathscr{A}}_{1,0}(s,u)\mu_1(u)du\\&\left.+\int_0^\infty\bar{\mathscr{A}}_{2,0}(s,u)\mu_2(u)du\right]\\&+\int_0^\infty\bar{\mathscr{V}}_0(s,u)\gamma(u)du\\&+\int_0^\infty\bar{\mathscr{R}}_{2,0}(s,u)\beta_2(u)du\end{aligned} \tag{3.26}$$

$$\begin{aligned}(s+\Lambda^+)\bar{\mathscr{Q}}_r(s)=\Lambda^+\sum_{k=1}^{r}c_k\bar{\mathscr{Q}}_{r-k}(s)+(1-\theta)&\\&\left[(1-\pi_1)\int_0^\infty\bar{\mathscr{H}}_{1,r}(s,u)\mu_1(u)du\right.\\&+(1-\pi_2)\int_0^\infty\bar{\mathscr{H}}_{2,r}(s,u)\mu_2(u)du\\&+\int_0^\infty\bar{\mathscr{A}}_{1,r}(s,u)\mu_1(u)du\\&\left.+\int_0^\infty\bar{\mathscr{A}}_{2,r}(s,u)\mu_2(u)du\right]\\&+\int_0^\infty\bar{\mathscr{V}}_r(s,u)\gamma(u)du\\&+\int_0^\infty\bar{\mathscr{R}}_{2,r}(s,u)\beta_2(u)du,\\&1\leq r\leq a-1\end{aligned} \tag{3.27}$$

$$\begin{aligned}\bar{\mathscr{H}}_{i,0}(s,0)=p_i\Bigg[\Lambda^+\sum_{r=a}^{b}\sum_{k=0}^{a-1}c_{r-k}\bar{\mathscr{Q}}_k(s)+(1-\theta)&\\&\left[(1-\pi_1)\sum_{r=a}^{b}\int_0^\infty\bar{\mathscr{H}}_{1,r}(s,u)\mu_1(u)du\right.\\&+(1-\pi_2)\sum_{r=a}^{b}\int_0^\infty\bar{\mathscr{H}}_{2,r}(s,u)\mu_2(u)du\\&+\sum_{r=a}^{b}\int_0^\infty\bar{\mathscr{A}}_{1,r}(s,u)\mu_1(u)du\\&\left.+\sum_{r=a}^{b}\int_0^\infty\bar{\mathscr{A}}_{2,r}(s,u)\mu_2(u)du\right]\\&+\sum_{r=a}^{b}\int_0^\infty\bar{\mathscr{V}}_r(s,u)\gamma(u)du\end{aligned}$$

$$+\sum_{r=a}^{b}\int_0^\infty \bar{\mathscr{R}}_{2,r}(s,u)\beta_2(u)du\Bigg], i=1,2 \quad (3.28)$$

$$\begin{aligned}\bar{\mathscr{H}}_{i,n}(s,0) = p_i\Bigg[&\Lambda^+\sum_{k=0}^{a-1} c_{b+n-k}\bar{\mathscr{Q}}_k(s)+(1-\theta)\\ &\Bigg[(1-\pi_1)\int_0^\infty \bar{\mathscr{H}}_{1,n+b}(s,u)\mu_1(u)du\\ &+(1-\pi_2)\int_0^\infty \bar{\mathscr{H}}_{2,n+b}(s,u)\mu_2(u)du\\ &+\int_0^\infty \bar{\mathscr{A}}_{1,n+b}(s,u)\mu_1(u)du\\ &+\int_0^\infty \bar{\mathscr{A}}_{2,n+b}(s,u)\mu_2(u)du\Bigg]\\ &+\int_0^\infty \bar{\mathscr{V}}_{n+b}(s,u)\gamma(u)du\\ &+\int_0^\infty \bar{\mathscr{R}}_{2,n+b}(s,u)\beta_2(u)du\Bigg],\\ &i=1,2\; n\geq 0 \end{aligned} \quad (3.29)$$

$$\bar{\mathscr{A}}_{1,n}(s,0)=\pi_1\int_0^\infty \bar{\mathscr{H}}_{1,n}(s,u)\mu_1(u)du,\; n\geq 0 \quad (3.30)$$

$$\bar{\mathscr{A}}_{2,n}(s,0)=\pi_2\int_0^\infty \bar{\mathscr{H}}_{2,n}(s,u)\mu_2(u)du,\; n\geq 0 \quad (3.31)$$

$$\begin{aligned}\bar{\mathscr{V}}_n(s,0)=\theta\Bigg[&(1-\pi_1)\int_0^\infty \bar{\mathscr{H}}_{1,n}(s,u)\mu_1(u)du\\ &+(1-\pi_2)\int_0^\infty \bar{\mathscr{H}}_{2,n}(s,u)\mu_2(u)du\\ &+\int_0^\infty \bar{\mathscr{A}}_{1,n}(s,u)\mu_1(u)du\\ &+\int_0^\infty \bar{\mathscr{A}}_{2,n}(s,u)\mu_2(u)du\Bigg],\\ &n\geq 0\end{aligned} \quad (3.32)$$

$$\bar{\mathscr{R}}_{1,n}(s,0)=\int_0^\infty \bar{\mathscr{D}}_n(s,u)\xi(u)du,\; n\geq 0 \quad (3.33)$$

$$\begin{aligned}\bar{\mathscr{D}}_n(s,0)=\Lambda^-&\int_0^\infty \bar{\mathscr{H}}_{1,n}(s,u)du\\ &+\Lambda^-\int_0^\infty \bar{\mathscr{H}}_{2,n}(s,u)du\\ &+\Lambda^-\int_0^\infty \bar{\mathscr{A}}_{1,n}(s,u)du\\ &+\Lambda^-\int_0^\infty \bar{\mathscr{A}}_{2,n}(s,u)du,\; n\geq 0\end{aligned} \quad (3.34)$$

$$\bar{\mathscr{R}}_{2,n}(s,0)=\int_0^\infty \bar{\mathscr{R}}_{1,n}(s,u)\beta_1(u)du,\; n\geq 0. \quad (3.35)$$

By multiplying equations (3.19) to (3.25) by the appropriate power of w^n and sum accordingly, and use the equation (3.18), we get

$$\left(\frac{\partial}{\partial u}+(s+\Lambda^+(1-\mathscr{C}(w))+\Lambda^-+\mu_i(u))\right)\bar{\mathscr{H}}_i(s,u,w) = 0, i=1,2 \quad (3.36)$$

$$\left(\frac{\partial}{\partial u}+(s+\Lambda^+(1-\mathscr{C}(w))+\Lambda^-+\mu_1(u))\right)\bar{\mathscr{A}}_1(s,u,w) = 0 \quad (3.37)$$

$$\left(\frac{\partial}{\partial u}+(s+\Lambda^+(1-\mathscr{C}(w))+\Lambda^-+\mu_2(u))\right)\bar{\mathscr{A}}_2(s,u,w) = 0 \quad (3.38)$$

$$\left(\frac{\partial}{\partial u}+(s+\Lambda^+(1-\mathscr{C}(w))+\gamma(u))\right)\bar{\mathscr{V}}(s,u,w) = 0 \quad (3.39)$$

$$\left(\frac{\partial}{\partial u}+(s+\Lambda^+(1-\mathscr{C}(w))+\xi(u))\right)\bar{\mathscr{D}}(s,u,w) = 0 \quad (3.40)$$

$$\left(\frac{\partial}{\partial u}+(s+\Lambda^+(1-\mathscr{C}(w))+\beta_1(u))\right)\bar{\mathscr{R}}_1(s,u,w) = 0 \quad (3.41)$$

$$\left(\frac{\partial}{\partial u}+(s+\Lambda^+(1-\mathscr{C}(w))+\beta_2(u))\right)\bar{\mathscr{R}}_2(s,u,w) = 0 \quad (3.42)$$

Multiplying two sides of equation (3.29) by the appropriate power of w^n and sum accordingly, and use the equation (3.28), we get

$$\begin{aligned}w^b\bar{\mathscr{H}}_i(s,0,w)=p_i\Bigg[&\Lambda^+\sum_{r=0}^{a-1}\sum_{n=1}^{b-r-1}c_n\bar{\mathscr{Q}}_r(s)(w^b-w^{n+r})\\ &-w^b\sum_{r=0}^{a-1}(s+\Lambda^+)\bar{\mathscr{Q}}_r(s)+w^b\\ &+\Lambda^+\sum_{r=0}^{a-1}\mathscr{C}(w)\bar{\mathscr{Q}}_r(s)w^r+(1-\theta)\\ &\Bigg[(1-\pi_1)\int_0^\infty \bar{\mathscr{H}}_1(s,u,w)\mu_1(u)du\\ &+(1-\pi_2)\int_0^\infty \bar{\mathscr{H}}_2(s,u,w)\mu_2(u)du\\ &+\int_0^\infty \bar{\mathscr{A}}_1(s,u,w)\mu_1(u)du\\ &+\int_0^\infty \bar{\mathscr{A}}_2(s,u,w)\mu_2(u)du\Bigg]\\ &+\int_0^\infty \bar{\mathscr{V}}(s,u,w)\gamma(u)du\\ &+\int_0^\infty \bar{\mathscr{R}}_2(s,u,w)\beta_2(u)du\\ &+(1-\theta)\sum_{r=0}^{b-1}(w^b-w^r)\\ &\Bigg[(1-\pi_1)\int_0^\infty \bar{\mathscr{H}}_{1,r}(s,u)\mu_1(u)du\\ &+(1-\pi_2)\int_0^\infty \bar{\mathscr{H}}_{2,r}(s,u)\mu_2(u)du\\ &+\int_0^\infty \bar{\mathscr{A}}_{1,r}(s,u)\mu_1(u)du\end{aligned}$$

$$
\begin{aligned}
&+\int_0^\infty \bar{\mathscr{A}}_{2,r}(s,u)\mu_2(u)du\Big]\\
&+\sum_{r=0}^{b-1}(w^b-w^r)\Big[\int_0^\infty \bar{\mathscr{V}}_r(s,u)\gamma(u)du\\
&+\int_0^\infty \bar{\mathscr{R}}_{2,r}(s,u)\beta_2(u)du\Big]\Big],\\
&\quad i=1,2.
\end{aligned} \tag{3.43}
$$

Similarly from equations (3.30) to (3.35), we get

$$\bar{\mathscr{A}}_1(s,0,w)=\pi_1\bar{\mathscr{H}}_1(s,0,w)\bar{U}_1(\Psi(s,w)), \tag{3.44}$$

$$\bar{\mathscr{A}}_2(s,0,w)=\pi_2\bar{\mathscr{H}}_2(s,0,w)\bar{U}_2(\Psi(s,w)), \tag{3.45}$$

$$
\begin{aligned}
\bar{\mathscr{V}}(s,0,w)=\theta\Big[&(1-\pi_1)\bar{\mathscr{H}}_1(s,0,w)\bar{U}_1(\Psi(s,w))\\
&+(1-\pi_2)\bar{\mathscr{H}}_2(s,0,w)\bar{U}_2(\Psi(s,w))\\
&+\pi_1\bar{\mathscr{H}}_1(s,0,w)(\bar{U}_1(\Psi(s,w)))^2\\
&+\pi_2\bar{\mathscr{H}}_2(s,0,w)(\bar{U}_2(\Psi(s,w)))^2\Big]
\end{aligned} \tag{3.46}
$$

$$
\begin{aligned}
\bar{\mathscr{D}}(s,0,w)=\Lambda^-\bar{\mathscr{H}}_1(s,0,w)&\left[\frac{1-\bar{U}_1(\Psi(s,w))}{\Psi(s,w)}\right]\\
&[1+\pi_1\bar{U}_1(\Psi(s,w))]\\
&+\Lambda^-\bar{\mathscr{H}}_2(s,0,w)\\
&\left[\frac{1-\bar{U}_2(\Psi(s,w))}{\Psi(s,w)}\right]\\
&[1+\pi_2\bar{U}_2(\Psi(s,w))]
\end{aligned} \tag{3.47}
$$

$$
\begin{aligned}
\bar{\mathscr{R}}_1(s,0,w)=\Lambda^-\bar{\mathscr{H}}_1(s,0,w)&\bar{D}(\Phi(s,w))\\
&\left[\frac{1-\bar{U}_1(\Psi(w,s))}{\Psi(w,s)}\right]\\
&[1+\pi_1\bar{U}_1(\Psi(w,s))]\\
&[+\Lambda^-\bar{\mathscr{H}}_2(0,w,s)\bar{D}(\Phi(w,s))\\
&\left[\frac{1-\bar{U}_2(\Psi(w,s))}{\Psi(w,s)}\right]\\
&[1+\pi_2\bar{U}_2(\Psi(w,s))]
\end{aligned} \tag{3.48}
$$

$$
\begin{aligned}
\bar{\mathscr{R}}_2(s,0,w)=\Lambda^-\bar{\mathscr{H}}_1(s,0,w)&\bar{D}(\Phi(s,w))\\
&\bar{R}_1(\Phi(s,w))\left[\frac{1-\bar{U}_1(\Psi(s,w))}{\Psi(s,w)}\right]\\
&[1+\pi_1\bar{U}_1(\Psi(s,w))]+\Lambda^-\bar{\mathscr{H}}_2(s,0,w)\\
&\bar{D}(\Phi(s,w))\bar{R}_1(\Phi(s,w))\\
&[1+\pi_2\bar{U}_2(\Psi(s,w))]\\
&\left[\frac{1-\bar{U}_2(\Psi(s,w))}{\Psi(s,w)}\right].
\end{aligned} \tag{3.49}
$$

Solving the partial differential equations (3.36) to (3.42), it follows that

$$\bar{\mathscr{H}}_i(s,u,w)=\bar{\mathscr{H}}_i(s,0,w)e^{-\Psi(s,w)u-\int_0^u \mu_i(t)dt} \tag{3.50}$$

$$\bar{\mathscr{A}}_1(s,u,w)=\bar{\mathscr{A}}_1(s,0,w)e^{-\Psi(s,w)u-\int_0^u \mu_1(t)dt} \tag{3.51}$$

$$\bar{\mathscr{A}}_2(s,u,w)=\bar{\mathscr{A}}_2(s,0,w)e^{-\Psi(s,w)u-\int_0^u \mu_2(t)dt} \tag{3.52}$$

$$\bar{\mathscr{V}}(s,u,w)=\bar{\mathscr{V}}(s,0,w)e^{-\Phi(s,w)u-\int_0^u \gamma(t)dt} \tag{3.53}$$

$$\bar{\mathscr{D}}(s,u,w)=\bar{\mathscr{D}}(s,0,w)e^{-\Phi(s,w)u-\int_0^u \xi(t)dt} \tag{3.54}$$

$$\bar{\mathscr{R}}_1(s,u,w)=\bar{\mathscr{R}}_1(s,0,w)e^{-\Phi(s,w)u-\int_0^u \beta_1(t)dt} \tag{3.55}$$

$$\bar{\mathscr{R}}_2(s,u,w)=\bar{\mathscr{R}}_2(s,0,w)e^{-\Phi(s,w)u-\int_0^u \beta_2(t)dt}. \tag{3.56}$$

Now multiplying both sides of equations (3.50) to (3.56) by $\mu_i(u)$, $\mu_1(u)$, $\mu_2(u)$, $\gamma(u)$, $\xi(u)$, $\beta_1(u)$ and $\beta_2(u)$ respectively, and integrating, we obtain for i=1,2

$$\int_0^\infty \bar{\mathscr{H}}_i(s,u,w)\mu_i(u)du=\bar{\mathscr{H}}_i(s,0,w)\bar{U}_i(\Psi(s,w)),\quad i=1,2 \tag{3.57}$$

$$\int_0^\infty \bar{\mathscr{A}}_1(s,u,w)\mu_1(u)du=\bar{\mathscr{A}}_1(s,0,w)\bar{U}_1(\Psi(s,w)) \tag{3.58}$$

$$\int_0^\infty \bar{\mathscr{A}}_2(s,u,w)\mu_2(u)du=\bar{\mathscr{A}}_2(s,0,w)\bar{U}_2(\Psi(s,w)) \tag{3.59}$$

$$\int_0^\infty \bar{\mathscr{V}}(s,u,w)\gamma(u)du=\bar{\mathscr{V}}(s,0,w)\bar{V}(\Phi(s,w)) \tag{3.60}$$

$$\int_0^\infty \bar{\mathscr{D}}(s,u,w)\xi(u)du=\bar{\mathscr{D}}(s,0,w)\bar{D}(\Phi(s,w)) \tag{3.61}$$

$$\int_0^\infty \bar{\mathscr{R}}_1(s,u,w)\beta_1(u)du=\bar{\mathscr{R}}_1(s,0,w)\bar{R}_1(\Phi(s,w)) \tag{3.62}$$

$$\int_0^\infty \bar{\mathscr{R}}_2(s,u,w)\beta_2(u)du=\bar{\mathscr{R}}_2(s,0,w)\bar{R}_2(\Phi(s,w)) \tag{3.63}$$

Again integrating equations (3.50) to (3.56) by parts with respect to u and using the equation (3.44) to (3.49), we get

$$\bar{\mathscr{H}}_1(s,w)=\bar{\mathscr{H}}_1(s,0,w)\left[\frac{1-\bar{U}_1(\Psi(s,w))}{\Psi(s,w)}\right], \tag{3.64}$$

$$\bar{\mathscr{H}}_2(s,w)=\bar{\mathscr{H}}_2(s,0,w)\left[\frac{1-\bar{U}_2(\Psi(s,w))}{\Psi(s,w)}\right], \tag{3.65}$$

$$
\begin{aligned}
\bar{\mathscr{A}}_1(s,w)=\pi_1\bar{\mathscr{H}}_1(s,0,w)&\bar{U}_1(\Psi(s,w))\\
&\left[\frac{1-\bar{U}_1(\Psi(s,w))}{\Psi(s,w)}\right],
\end{aligned} \tag{3.66}
$$

$$\bar{\mathscr{A}}_2(s,w) = \pi_2\bar{\mathscr{H}}_2(s,0,w)\bar{U}_2(\Psi(s,w)) \left[\frac{1-\bar{U}_2(\Psi(s,w))}{\Psi(w,s)}\right], \tag{3.67}$$

$$\begin{aligned}\bar{\mathscr{V}}(s,w) = \theta\big[&(1-\pi_1)\bar{\mathscr{H}}_1(s,0,w)\bar{U}_1(\Psi(s,w))\\ &+(1-\pi_2)\bar{\mathscr{H}}_2(s,0,w)\bar{U}_2(\Psi(s,w))\\ &+\pi_1\bar{\mathscr{H}}_1(s,0,w)(\bar{U}_1(\Psi(s,w)))^2\\ &+\pi_2\bar{\mathscr{H}}_2(s,0,w)(\bar{U}_2(\Psi(s,w)))^2\big]\\ &\left[\frac{1-\bar{V}(\Phi(s,w))}{\Phi(s,w)}\right],\end{aligned} \tag{3.68}$$

$$\begin{aligned}\bar{\mathscr{D}}(s,w) = \Lambda^-\Bigg[&\bar{\mathscr{H}}_1(s,0,w)\left[\frac{1-\bar{U}_1(\Psi(s,w))}{\Psi(s,w)}\right]\\ &[1+\pi_1\bar{U}_1(\Psi_1(s,w))]+\bar{\mathscr{H}}_2(s,0,w)\\ &\left[\frac{1-\bar{U}_2(\Psi(s,w))}{\Psi(s,w)}\right]\\ &[1+\pi_2\bar{U}_2(\Psi(s,w))]\Bigg]\\ &\left[\frac{1-\bar{D}(\Phi(s,w))}{\Phi(s,w)}\right]\end{aligned} \tag{3.69}$$

$$\begin{aligned}\bar{\mathscr{R}}_1(s,w) = \Lambda^-\Bigg[&\bar{\mathscr{H}}_1(s,0,w)\bar{D}(\Phi(s,w))\\ &\left[\frac{1-\bar{U}_1(\Psi(s,w))}{\Psi(s,w)}\right]\\ &[1+\pi_1\bar{U}_1(\Psi(s,w))]\\ &+\bar{\mathscr{H}}_2(s,0,w)\bar{D}(\Phi(s,w))\\ &\left[\frac{1-\bar{U}_2(\Psi(s,w))}{\Psi(s,w)}\right]\\ &[1+\pi_2\bar{U}_2(\Psi(s,w))]\Bigg]\\ &\left[\frac{1-\bar{R}_1(\Phi(s,w))}{\Phi(s,w)}\right]\end{aligned} \tag{3.70}$$

$$\begin{aligned}\bar{\mathscr{R}}_2(s,w) = \Lambda^-\Bigg[&\bar{\mathscr{H}}_1(s,0,w)\bar{D}(\Phi(s,w))\\ &\bar{R}_1(\Phi(s,w))\left[\frac{1-\bar{U}_1(\Psi(s,w))}{\Psi(s,w)}\right]\\ &[1+\pi_1\bar{U}_1(\Psi(s,w))]+\bar{\mathscr{H}}_2(s,0,w)\\ &\bar{D}(\Phi(s,w))\bar{R}_1(\Phi(s,w))\\ &\left[\frac{1-\bar{U}_2(\Psi(s,w))}{\Psi(s,w)}\right]\\ &[1+\pi_2\bar{U}_2(\Psi(s,w))]\Bigg]\\ &\left[\frac{1-\bar{R}_2(\Phi(s,w))}{\Phi(s,w)}\right].\end{aligned} \tag{3.71}$$

Inserting the equations (3.57), (3.58), (3.59) and (3.63) into the equation (3.43), we get for $i=1,2$

$$\bar{\mathscr{H}}_i(s,0,w) = \frac{p_i\left(\begin{aligned}&\Lambda^+\sum_{r=0}^{a-1}\sum_{n=1}^{b-r-1}c_n\bar{\mathscr{Q}}_r(s)(w^b-w^{n+r})\\ &+\Lambda^+\sum_{r=0}^{a-1}\mathscr{C}(w)\bar{\mathscr{Q}}_r(s)w^r-w^b(s+\Lambda^+)\\ &\sum_{r=0}^{a-1}\bar{\mathscr{Q}}_r(s)+w^b+\sum_{r=0}^{b-1}(w^b-w^r)(1-\theta)\\ &\Big[(1-\pi_1)\int_0^\infty\bar{\mathscr{H}}_{1,r}(s,u)\mu_1(u)du\\ &+(1-\pi_2)\int_0^\infty\bar{\mathscr{H}}_{2,r}(s,u)\mu_2(u)du\\ &+\int_0^\infty\bar{\mathscr{A}}_{1,r}(s,u)\mu_1(u)du\\ &+\int_0^\infty\bar{\mathscr{A}}_{2,r}(s,u)\mu_2(u)du\Big]\\ &+\sum_{r=0}^{b-1}(w^b-w^r)\Big[\int_0^\infty\bar{\mathscr{V}}_r(s,u)\gamma(u)du\\ &+\int_0^\infty\bar{\mathscr{R}}_{2,r}(s,u)\beta_2(u)du\Big]\end{aligned}\right)}{Dr(s,w)} \tag{3.72}$$

where

$$\begin{aligned}Dr(s,w) = \Psi(s,w)w^b-\big[&\Psi(s,w)[(1-\theta)+\theta\bar{V}(\Phi(s,w))]\\ &\big[p_1(1-\pi_1)\bar{U}_1(\Psi(s,w))+p_1\pi_1(\bar{U}_1(\Psi(s,w)))^2\\ &+p_2(1-\pi_2)\bar{U}_2(\Psi(s,w))+p_2\pi_2(\bar{U}_2(\Psi(s,w)))^2\big]\\ &+\Lambda^-\bar{D}(\Phi(s,w))\bar{R}_1(\Phi(s,w))\bar{R}_2(\Phi(s,w))\\ &\big[p_1[1-\bar{U}_1(\Psi(s,w))][1+\pi_1\bar{U}_1(\Psi(s,w))]\\ &+p_2[1-\bar{U}_2(\Psi(s,w))][1+\pi_2\bar{U}_2(\Psi(s,w))]\big]\big]\end{aligned}$$

$$\begin{aligned}\Psi(s,w) &= s+\Lambda^-+\Lambda^+(1-\mathscr{C}(w))\\ \Phi(s,w) &= s+\Lambda^+(1-C(w))\end{aligned}$$

substituting the equation (3.72) into the equations (3.64) to (3.71) and taking the inverse Laplace transform of these equations, we get the probability generating fuctions of various states of the system under transient state.

4. The steady state results

The steady state results can be obtained by applying the well-known Tauberian theorem, that is,

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t). \tag{4.1}$$

The PGF of the server's state queue size distribution under the steady state condition are given by

$$\mathscr{H}_1(w) = \mathscr{H}_1(0,w)\left[\frac{1-\bar{U}_1(\Psi(w))}{\Psi(w)}\right], \tag{4.2}$$

$$\mathscr{H}_2(w) = \mathscr{H}_2(0,w)\left[\frac{1-\bar{U}_2(\Psi(w))}{\Psi(w)}\right], \tag{4.3}$$

$$\mathscr{A}_1(w) = \pi_1\mathscr{H}_1(0,w)\bar{U}_1(\Psi(w))\left[\frac{1-\bar{U}_1(\Psi(w))}{\Psi(w)}\right], \tag{4.4}$$

$$\mathscr{A}_2(w) = \pi_2\mathscr{H}_2(0,w)\bar{U}_2(\Psi(w))\left[\frac{1-\bar{U}_2(\Psi(w))}{\Psi(w)}\right], \tag{4.5}$$

$$\mathscr{V}(w) = \theta\Big[(1-\pi_1)\mathscr{H}_1(0,w)\bar{U}_1(\Psi(w)) + (1-\pi_2)H_2(0,w)\bar{U}_2(\Psi(w)) + \pi_1\mathscr{H}_1(0,w)(\bar{U}_1(\Psi(w)))^2 + \pi_2\mathscr{H}_2(0,w)(\bar{U}_2(\Psi(w)))^2\Big]\left[\frac{1-\bar{V}(\Phi(w))}{\Phi(w)}\right], \tag{4.6}$$

$$\mathscr{D}(w) = \Lambda^-\Big[\mathscr{H}_1(0,w)\left[\frac{1-\bar{U}_1(\Psi(w))}{\Psi(w)}\right][1+\pi_1\bar{U}_1(\Psi(w))] + \mathscr{H}_2(0,w)\left[\frac{1-\bar{U}_2(\Psi(w))}{\Psi(w)}\right],[1+\pi_2\bar{U}_2(\Psi(w))]\Big]\left[\frac{1-\bar{D}(\Phi(w))}{\Phi(w)}\right] \tag{4.7}$$

$$\mathscr{R}_1(w) = \Lambda^-\Big[\mathscr{H}_1(0,w)\bar{D}(\Phi(w))\left[\frac{1-\bar{U}_1(\Psi(w))}{\Psi(w)}\right][1+\pi_1\bar{U}_1(\Psi(w))] + \bar{\mathscr{H}}_2(0,w)\bar{D}(\Phi(w))\left[\frac{1-\bar{U}_2(\Psi(w))}{\Psi(w)}\right][1+\pi_2\bar{U}_2(\Psi(w))]\Big]\left[\frac{1-\bar{R}_1(\Phi(w))}{\Phi(w)}\right], \tag{4.8}$$

$$\mathscr{R}_2(w) = \Lambda^-\Big[\mathscr{H}_1(0,w)\bar{D}(\Phi(w))\bar{R}_1(\Phi(w))\left[\frac{1-\bar{U}_1(\Psi(w))}{\Psi_1(w)}\right][1+\pi_1\bar{U}_1(\Psi(w))] + \mathscr{H}_2(0,w)\bar{D}(\Phi(w))\bar{R}_1(\Phi(w))\left[\frac{1-\bar{U}_2(\Psi(w))}{\Psi(w)}\right][1+\pi_2\bar{U}_2(\Psi(w))]\Big]\left[\frac{1-\bar{R}_2(\Phi(w))}{\Phi(w)}\right], \tag{4.9}$$

where

$$\mathscr{H}_i(0,w) = \frac{p_i\left(\begin{array}{l}\Lambda^+\sum_{r=0}^{a-1}c_n\mathscr{Q}_r\sum_{n=1}^{b-r-1}(w^b-w^{n+r}) \\ +\Lambda^+\sum_{r=0}^{a-1}(\mathscr{C}(w)w^r-w^b) \\ +\sum_{r=0}^{b-1}(w^b-w^r)(1-\theta) \\ \Big[(1-\pi_1)\int_0^\infty \mathscr{H}_{1,r}(u)\mu_1(u)du \\ +(1-\pi_2)\int_0^\infty \mathscr{H}_{2,r}(u)\mu_2(u)du \\ +\int_0^\infty \mathscr{A}_{1,r}(u)\mu_1(u)du \\ +\int_0^\infty \mathscr{A}_{2,r}(u)\mu_2(u)du\Big] \\ +\sum_{r=0}^{b-1}(w^b-w^r)\Big[\int_0^\infty \mathscr{V}_r(u)\gamma(u)du \\ +\int_0^\infty \mathscr{R}_{2,r}(u)\beta_2(u)du\Big]\end{array}\right)}{Dr(w)}, i=1,2 \tag{4.10}$$

$$\begin{aligned}Dr(w) = {}&\Psi(w)w^b - \Big[\Psi(w)[(1-\theta)+\theta\bar{V}(\Phi(w))] \\ &[p_1(1-\pi_1)\bar{U}_1(\Psi(w)) + p_1\pi_1(\bar{U}_1(\Psi(w)))^2 \\ &+ p_2(1-\pi_2)\bar{U}_2(\Psi(w)) + p_2\pi_2(\bar{U}_2(\Psi(w)))^2] \\ &+ \Lambda^-\bar{D}(\Phi(w))\bar{R}_1(\Phi(w))\bar{R}_2(\Phi(w)) \\ &[p_1[1-\bar{U}_1(\Psi(w))][1+\pi_1\bar{U}_1(\Psi(w))] \\ &+ p_2[1-\bar{U}_2(\Psi(w))][1+\pi_2\bar{U}_2(\Psi(w))]]\Big]\end{aligned}$$

$$\Psi(w) = \Lambda^- + \Lambda^+(1-\mathscr{C}(w))$$
$$\Phi(w) = \Lambda^+(1-\mathscr{C}(w)).$$

4.1 Queue size distribution at a random epoch

By adding (4.2) to (4.9) with idle term, we get the PGF of the queue size distribution at a random epoch.

$$P(w) = \mathscr{H}_1(w) + \mathscr{H}_2(w) + \mathscr{A}_1(w) + \mathscr{A}_2(w) + \mathscr{V}(w)\ \mathscr{D}(w) + \mathscr{R}_1(w) + \mathscr{R}_2(w) + \mathscr{Q}(w)$$

$$P(w)=\frac{\left(\begin{array}{l}\Big[\Lambda^{+}\sum_{r=0}^{a-1}\sum_{n=1}^{b-r-1}c_n\mathscr{Q}_r(w^b-w^{n+r})\\ +\sum_{r=0}^{b-1}(w^b-w^r)\mathscr{W}_r+\Lambda^{+}\sum_{r=0}^{a-1}\mathscr{Q}_r\\ (\mathscr{C}(w)w^r-w^b)\Big]\times\big[p_1\Phi(w)(1-\bar{U}_1(\Psi(w)))\\ +p_2\Phi(w)(1-\bar{U}_2(\Psi(w)))+\pi_1p_1\Phi(w)\\ \bar{U}_1(\Psi(w))(1-\bar{U}_1(\Psi_1(w)))\\ +\pi_2p_2\Phi(w)\bar{U}_2(\Psi(w))(1-\bar{U}_2(\Psi(w)))\\ +\theta(1-\pi_1)p_1\Psi(w)\bar{U}_1(\Psi(w))[1-\bar{V}(\Phi(w))]\\ +\theta(1-\pi_2)p_2\Psi(w)\bar{U}_2(\Psi(w))[1-\bar{V}(\Phi(w))]\\ +\theta\pi_1p_1\Psi(w)(\bar{U}_1(\Psi(w)))^2[1-\bar{V}(\Phi(w))]\\ +\theta\pi_2p_2\Psi(w)(\bar{U}_2(\Psi(w)))^2[1-\bar{V}(\Phi(w))]\\ +\Lambda^{-}p_1(1+\pi_1\bar{U}_1(\Psi(w)))(1-\bar{U}_1(\Psi(w)))\\ (1-\bar{D}(\Phi(w)))+\Lambda^{-}p_2(1+\pi_2\bar{U}_2(\Psi(w)))\\ (1-\bar{U}_2(\Psi(w)))(1-\bar{D}(\Phi(w)))+\Lambda^{-}p_1\\ \bar{D}(\Phi(w))(1+\pi_1\bar{U}_1(\Psi(w)))(1-\bar{U}_1(\Psi(w)))\\ (1-\bar{R}_1(\Phi(w)))+\Lambda^{-}p_2\bar{D}(\Phi(w))\\ (1+\pi_2\bar{U}_2(\Psi(w)))(1-\bar{U}_2(\Psi(w)))\\ (1-\bar{R}_1(\Phi(w)))+\Lambda^{-}p_1\bar{D}(\Phi(w))\bar{R}_1(\Phi(w))\\ (1+\pi_1\bar{U}_1(\Psi(w)))(1-\bar{U}_1(\Psi(w)))\\ (1-\bar{R}_2(\Phi(w)))+\Lambda^{-}p_2\bar{D}(\Phi(w))\bar{R}_1(\Phi(w))\\ (1+\pi_2\bar{U}_2(\Psi(w)))(1-\bar{U}_2(\Psi(w)))\\ +(1-\bar{R}_2(\Phi(w)))\big]Dr(w)\mathscr{Q}(w)\end{array}\right)}{[Dr(w)\Phi(w)]}. \tag{4.11}$$

where

$$\Psi(w)=\Lambda^{-}+\Lambda^{+}(1-\mathscr{C}(w));\Phi(w)=\Lambda^{+}(1-\mathscr{C}(w))$$

$$\mathscr{H}_r=(1-\pi_1)\int_0^\infty\mathscr{H}_{1,r}(u)\mu_1(u)du+(1-\pi_2)\int_0^\infty\mathscr{H}_{2,r}(u)\mu_2(u)du$$

$$\mathscr{A}_r=\int_0^\infty\mathscr{A}_{1,r}(u)\mu_1(u)du+\int_0^\infty\mathscr{A}_{2,r}(u)\mu_2(u)du$$

$$\mathscr{W}_r=(1-\theta)\mathscr{H}_r+(1-\theta)\mathscr{A}_r+\int_0^\infty\mathscr{V}_r(u)\gamma(u)du+\int_0^\infty\mathscr{R}_{2,r}(u)\beta_2(u)du$$

5. Stability condition

The condition P(1)=1 should be satisfied by the probability generating function. To satisfy this condition, apply L'Hospital's rule and equating the expression to 1, we get

$$X_1\times\Big[p_1(1-\bar{U}_1(\Lambda^{-}))(1+\pi_1\bar{U}_1(\Lambda^{-}))+p_2(1-\bar{U}_2(\Lambda^{-}))(1+\pi_2\bar{U}_2(\Lambda^{-}))+\Lambda^{-}\theta E(V)M_1+(\Lambda^{-}E(D)+\Lambda^{-}E(R_1)+\Lambda^{-}E(R_2))M_4\Big]+C_1\times\sum_{r=0}^{a-1}\mathscr{Q}_r=C_1 \tag{5.1}$$

Next, the unknown probabilities, $\mathscr{B}_r,\ r=0,1,2,...,b-1$ are calculated and related to the idle-server probabilities, $\mathscr{Q}_r,\ r=0,1,2,...,a-1$. The LHS of the above expression must be positive. Thus, the required condition $P(1)=1$ is satisfied if

$$\Big[\Psi(w)w^b-\big[\Psi(w)[(1-\theta)+\theta\bar{V}(\Phi(w))]\big[p_1(1-\pi_1)\bar{U}_1(\Psi(w))+p_1\pi_1(\bar{U}_1(\Psi(w)))^2+p_2(1-\pi_2)\bar{U}_2(\Psi(w))+p_2\pi_2(\bar{U}_2(\Psi(w)))^2\big]+\Lambda^{-}\bar{D}(\Phi(w))\bar{R}_1(\Phi(w))\bar{R}_2(\Phi(w))\big[p_1[1-\bar{U}_1(\Psi(w))][1+\pi_1\bar{U}_1(\Psi(w))]+p_2[1-\bar{U}_2(\Psi(w))][1+\pi_2\bar{U}_2(\Psi(w))]\big]\big]\Big]>0.$$

$$\text{If }\rho=\frac{\begin{array}{l}\big[\Lambda^{+}E(X)(\theta E(V)M_1-M_2+p_1\bar{U}_1^{'}(\Lambda^{-})\\ (1-\pi_1+2\pi_1\bar{U}_1(\Lambda^{-}))\\ +p_2\bar{U}_2^{'}(\Lambda^{-})(1-\pi_2+2\pi_2\bar{U}_2(\Lambda^{-}))\\ +(E(D)+E(R_1)+E(R_2))M_4)\big]\end{array}}{b} \tag{5.2}$$

then the condition to be satisfied by the model under consideration for the existence of steady state is $\rho<1$. There are b+a unknowns in equation (4.11). Using the following result, we can express $\mathscr{B}_r$ in terms of $\mathscr{Q}_r$ in such a way that numerator have only 'b' constants. Now, equation(4.11) gives the PGF of the number of customers involving 'b' unknowns. By Rouche's theorem, the expression Dr(w) has $b-1$ zeros inside and one on the unit circle $|w|=1$. The numerator of equation (4.11) must vanish at these points, since P(w) is analytic within and on the unit circle and as a result we get 'b' equations in 'b' unknowns . These equations can be solved by any appropriate numerical technique.

5.1 Result: Let $\mathscr{W}_r$ can be expressed in terms of $\mathscr{Q}_r$ as

$$\sum_{r=0}^{a-1}\mathscr{W}_r=\Lambda^{+}\sum_{r=0}^{a-1}\mathscr{Q}_r-\Lambda^{+}\sum_{r=0}^{a-1}\mathscr{Q}_r\sum_{k=1}^{a-r-1}c_k$$

where, $\mathscr{W}_r$ is the probabilities of the 'r' customers in the queue during idle period and X_1, M_1, M_2, M_4 C_1, $E(X)$ are given in Section 6.

6. Performance measures

In this section, we derive system state probabilities and the mean number of customers in the queue (L_q) and the mean waiting time in the queue (W_q).

6.1 System state probabilities

Differentiating (4.2) to (4.9) and applying L'Hospital's rule whenever necessary, we get the following results

Let $\mathscr{H}_q(1)$, $\mathscr{V}_q(1)$, $\mathscr{D}_q(1)$, $\mathscr{R}_q(1)$ be the probabilities that the server is in a busy, Bernoulli vacation, delay time to repair and repair state respectively. We can give that

$$\mathscr{H}_q(1)=\mathscr{H}_1(1)+\mathscr{H}_2(1)+\mathscr{A}_1(1)+\mathscr{A}_2(1)=\frac{X_1M_4}{C_1}$$
$$\mathscr{V}_q(1)=\frac{X_1\Lambda^-\theta E(V)M_1}{C_1}$$
$$\mathscr{D}_q(1)=\frac{X_1\Lambda^-E(D)M_4}{C_1}$$
$$\mathscr{R}_q(1)=\mathscr{R}_1(1)+\mathscr{R}_2(1)=\frac{X_1\Lambda^-(E(R_1)+E(R_2))M_4}{C_1}$$

6.2 Mean queue size

1. Differentiating (4.11) and using L'Hospital's rule, we can obtain the mean number of customers in the queue (L_q) as follows:

$$L_q=\lim_{w\to 1}\frac{d}{dw}P(w)=\frac{N'''(1)D''(1)-D'''(1)N''(1)}{3(D'')^2}$$

where

$$\begin{aligned}
D'' &= -2\Lambda^+E(X)C_1\\
D''' &= 3\big[(-\Lambda^+E(X))\big(-\Lambda^+E(X(X-1))\\
&\quad -2\Lambda^+E(X)b+\Lambda^-b(b-1)\\
&\quad -\big[(-\Lambda^+E(X(X-1))\\
&\quad -2(\Lambda^+E(X))^2\theta E(V)+\Lambda^-\theta S_1)M_1\\
&\quad +[2(\Lambda^+E(X))^2-\Lambda^-(\Lambda^+E(X))^2\theta E(V)\\
&\quad -\Lambda^-\Lambda^+E(X(X-1))]M_2\\
&\quad +\Lambda^-(\Lambda^+E(X))^2M_3+\Lambda^-\Lambda^+E(X(X-1))\\
&\quad [p_1\bar{U}_1'(\Lambda^-)+p_2\bar{U}_2'(\Lambda^-)+M_4(E(D)\\
&\quad +E(R_1)+E(R_2))]+\Lambda^-(\Lambda^+E(X))^2\\
&\quad [-p_1\bar{U}_1''(\Lambda^-)[1-\pi_1+2\pi_1\bar{U}_1(\Lambda^-)]\\
&\quad -p_2\bar{U}_2''(\Lambda^-)[1-\pi_2+2\pi_2\bar{U}_2(\Lambda^-)]\\
&\quad -2p_1\pi_1(\bar{U}_1'(\Lambda^-))^2-2p_2\pi_2(\bar{U}_2'(\Lambda^-))^2\\
&\quad +2(E(D)+E(R_1)+E(R_2))[p_1\bar{U}_1'(\Lambda^-)\\
&\quad [1-\pi_1+2\pi_1\bar{U}_1(\Lambda^-)]\\
&\quad +p_2\bar{U}_2'(\Lambda^-)[1-\pi_2+2\pi_2\bar{U}_2(\Lambda^-)]]\\
&\quad +M_4M_5]\big]\big)-\Lambda^+E(X(X-1))C_1\big]\\
N'' &= -2\Lambda^+E(X)\big(X_1[p_1(1-\bar{U}_1(\Lambda^-))(1+\pi_1\bar{U}_1(\Lambda^-))\\
&\quad +p_2(1-\bar{U}_2(\Lambda^-))(1+\pi_2\bar{U}_2(\Lambda^-))\\
&\quad +\Lambda^-\theta E(V)M_1+(\Lambda^-E(D)+\Lambda^-E(R_1)\\
&\quad +\Lambda^-E(R_2))M_4]+C_1\sum_{r=0}^{a-1}\mathscr{Q}_r\big)\\
N''' &= 3\big[-X_2\Lambda^+E(X)[M_4+\Lambda^-\theta E(V)M_1\\
&\quad +(\Lambda^-E(D)+\Lambda^-E(R_1)+\Lambda^-E(R_2))M_4]\\
&\quad +X_1\big(2(\Lambda^+E(X))^2[-p_1\bar{U}_1'(\Lambda^-)\\
&\quad -p_2\bar{U}_2'(\Lambda^-)+\pi_1p_1[-\bar{U}_1(\Lambda^-)\bar{U}_1'(\Lambda^-)\\
&\quad +(1-\bar{U}_1(\Lambda^-))\bar{U}_1'(\Lambda^-)+2\theta\Lambda^-E(V)\\
&\quad \bar{U}_1'(\Lambda^-)\bar{U}_1(\Lambda^-)+\theta(\bar{U}_1(\Lambda^-))^2E(V)\\
&\quad +\Lambda^-\bar{U}_1'(\Lambda^-)(1-\bar{U}_1(\Lambda^-))E(D)\\
&\quad +\Lambda^-E(R_1)\bar{U}_1'(\Lambda^-)(1-\bar{U}_1(\Lambda^-))\\
&\quad +\Lambda^-E(R_2)\bar{U}_1'(\Lambda^-)(1-\bar{U}_1(\Lambda^-))]\\
&\quad +\pi_2p_2[-\bar{U}_2(\Lambda^-)\bar{U}_2'(\Lambda^-)+(1-\bar{U}_2(\Lambda^-))\\
&\quad \bar{U}_2'(\Lambda^-)+2\theta\Lambda^-E(V)\bar{U}_2'(\Lambda^-)\bar{U}_2(\Lambda^-)\\
&\quad +\theta(\bar{U}_2(\Lambda^-))^2E(V)+\Lambda^-\bar{U}_2'(\Lambda^-)\\
&\quad (1-\bar{U}_2(\Lambda^-))E(D)+\Lambda^-E(R_1)\bar{U}_2'(\Lambda^-)\\
&\quad (1-\bar{U}_2(\Lambda^-))+\Lambda^-E(R_2)\bar{U}_2'(\Lambda^-)\\
&\quad (1-\bar{U}_2(\Lambda^-))]+\theta(1-\pi_1)p_1E(V)A_1\\
&\quad +\theta(1-\pi_2)p_2E(V)A_2-\Lambda^-p_1\\
&\quad (1+\pi_1\bar{U}_1'(\Lambda^-))\bar{U}_1'(\Lambda^-)E(D)\\
&\quad -\Lambda^-p_2(1+\pi_2\bar{U}_2'(\Lambda^-))\bar{U}_2'(\Lambda^-)E(D)\\
&\quad -\Lambda^-p_1(1+\pi_1\bar{U}_1(\Lambda^-))E(R_1)A_3\\
&\quad -\Lambda^-p_2(1+\pi_2\bar{U}_2(\Lambda^-))E(R_1)A_4\\
&\quad -\Lambda^-p_1(1+\pi_1\bar{U}_1(\Lambda^-))E(R_2)A_5\\
&\quad -\Lambda^-p_2(1+\pi_2\bar{U}_2(\Lambda^-))E(R_2)A_6]\\
&\quad -\Lambda^-E(X(X-1))[p_1(1-\bar{U}_1(\Lambda^-))\\
&\quad +p_2(1-\bar{U}_2(\Lambda^-))+\pi_1p_1\bar{U}_1(\Lambda^-)\\
&\quad (1-\bar{U}_1(\Lambda^-))+\pi_2p_2\bar{U}_2(\Lambda^-)\\
&\quad (1-\bar{U}_2(\Lambda^-))]-\Lambda^-S_1[\theta(1-\pi_1)p_1\\
&\quad \bar{U}_1(\Lambda^-)+\theta(1-\pi_2)p_2\bar{U}_2(\Lambda^-)\\
&\quad +\theta\pi_1p_1(\bar{U}_1(\Lambda^-))^2+\theta\pi_2p_2(\bar{U}_2(\Lambda^-))^2]\\
&\quad -\Lambda^-M_4(S_2+S_3+S_4)\big)-\Lambda^+E(X)\sum_{r=0}^{a-1}\mathscr{Q}_r\\
&\quad \big(-\Lambda^+E(X(X-1))-2\Lambda^+E(X)b\\
&\quad +\Lambda^-b(b-1)-\big[(-\Lambda^+E(X(X-1))\\
&\quad -2(\Lambda^+E(X))^2\theta E(V)+\Lambda^-\theta S_1)M_1\\
&\quad +[2(\Lambda^+E(X))^2-\Lambda^-(\Lambda^+E(X))^2\theta E(V)\\
&\quad -\Lambda^-\Lambda^+E(X(X-1))]M_2\\
&\quad +\Lambda^-(\Lambda^+E(X))^2M_3\\
&\quad +\Lambda^-\Lambda^+E(X(X-1))[p_1\bar{U}_1'(\Lambda^-)\\
&\quad +p_2\bar{U}_2'(\Lambda^-)+M_4(E(D)+E(R_1)\\
&\quad +E(R_2))]+\Lambda^-(\Lambda^+E(X))^2
\end{aligned}$$

$$[-p_1\bar{U}_1''(\Lambda^-)[1-\pi_1+2\pi_1\bar{U}_1'(\Lambda^-)]$$
$$-p_2\bar{U}_2''(\Lambda^-)[1-\pi_2+2\pi_2\bar{U}_2'(\Lambda^-)]$$
$$-2p_1\pi_1(\bar{U}_1'(\Lambda^-))^2-2p_2\pi_2(\bar{U}_2'(\Lambda^-))^2$$
$$+2(E(D)+E(R_1)+E(R_2))[p_1\bar{U}_1'(\Lambda^-)$$
$$[1-\pi_1+2\pi_1\bar{U}_1(\Lambda^-)]$$
$$+p_2\bar{U}_2'(\Lambda^-)[1-\pi_2+2\pi_2\bar{U}_2(\Lambda^-)]]$$
$$+M_4M_5]]\big)-[\Lambda^+E(X(X-1))\sum_{r=0}^{a-1}\mathscr{Q}_r$$
$$+2\Lambda^+E(X)\sum_{r=0}^{a-1}r\mathscr{Q}_r]C_1]$$

$$X_1=\Lambda^+\sum_{r=0}^{a-1}\sum_{n=1}^{b-r-1}c_n\mathscr{Q}_r(b-n-r)$$
$$+\Lambda^+\sum_{r=0}^{a-1}\mathscr{Q}_r(E(X)+r-b)+\sum_{r=0}^{b-1}(b-r)\mathscr{W}_r$$

$$X_2=\Lambda^+\sum_{r=0}^{a-1}\sum_{n=1}^{b-r-1}c_n\mathscr{Q}_r(b(b-1)$$
$$-(n+r)(n+r-1))+\Lambda^+\sum_{r=0}^{a-1}\mathscr{Q}_r$$
$$(E(X(X-1))+2E(X)r+r(r-1)$$
$$-b(b-1))+\sum_{r=0}^{b-1}(b(b-1)-r(r-1))\mathscr{W}_r$$

$$C_1=-\Lambda^+E(X)+\Lambda^-b-[\Lambda^-\Lambda^+E(X)[\theta E(V)M_1$$
$$-M_2+p_1\bar{U}_1'(\Lambda^-)[1-\pi_1+2\pi_1\bar{U}_1(\Lambda^-)]$$
$$+p_2\bar{U}_2'(\Lambda^-)[1-\pi_2+2\pi_2\bar{U}_2(\Lambda^-)]$$
$$+(E(D)+E(R_1)+E(R_2))M_4]$$
$$-\Lambda^+E(X)M_1]$$

$$M_1=p_1(1-\pi_1)\bar{U}_1(\Lambda^-)+p_1\pi_1(\bar{U}_1(\Lambda^-))^2$$
$$+p_2(1-\pi_2)\bar{U}_2(\Lambda^-)+p_2\pi_2(\bar{U}_2(\Lambda^-))^2$$

$$M_2=p_1(1-\pi_1)\bar{U}_1'(\Lambda^-)+p_2(1-\pi_2)\bar{U}_2'(\Lambda^-)$$
$$+2p_1\pi_1\bar{U}_1'(\Lambda^-)\bar{U}_1(\Lambda^-)$$
$$+2p_2\pi_2\bar{U}_2'(\Lambda^-)\bar{U}_2(\Lambda^-)$$

$$M_3=p_1(1-\pi_1)\bar{U}_1''(\Lambda^-)+p_2(1-\pi_2)\bar{U}_2''(\Lambda^-)$$
$$+2p_1\pi_1\bar{U}_1''(\Lambda^-)\bar{U}_1(\Lambda^-)$$
$$+2p_2\pi_2\bar{U}_2''(\Lambda^-)\bar{U}_2(\Lambda^-)$$
$$+2p_1\pi_1(\bar{U}_1'(\Lambda^-))^2$$
$$+2p_2\pi_2(\bar{U}_2'(\Lambda^-))^2$$

$$M_4=p_1(1+\pi_1\bar{U}_1(\Lambda^-))(1-\bar{U}_1(\Lambda^-))$$
$$+p_2(1+\pi_2\bar{U}_2(\Lambda^-))(1-\bar{U}_2(\Lambda^-))$$

$$M_5=2E(D)E(R_1)+2E(D)E(R_2)$$
$$+2E(R_2)E(R_1)+E(D^2)+E(R_1^2)+E(R_2^2)$$

$$A_1=\bar{U}_1(\Lambda^-)+\Lambda^-\bar{U}_1'(\Lambda^-)$$
$$A_2=\bar{U}_2(\Lambda^-)+\Lambda^-\bar{U}_2'(\Lambda^-)$$
$$A_3=\bar{U}_1'(\Lambda^-)+(1-\bar{U}_1(\Lambda^-))E(D)$$
$$A_4=\bar{U}_2'(\Lambda^-)+(1-\bar{U}_2(\Lambda^-))E(D)$$
$$A_5=\bar{U}_1'(\Lambda^-)+(1-\bar{U}_1'(\Lambda^-))(E(D)+E(R_1))$$
$$E(X)=C'(1)$$
$$A_6=\bar{U}_2'(\Lambda^-)+(1-\bar{U}_2'(\Lambda^-))(E(D)+E(R_1))$$
$$E(X(X-1))=C''(1)$$
$$S_1=\Lambda^+E(X(X-1))E(V)+(\Lambda^+E(X))^2E(V^2)$$
$$S_2=\Lambda^+E(X(X-1))E(D)+(\Lambda^+E(X))^2E(D^2)$$
$$S_3=\Lambda^+E(X(X-1))E(R_1)+(\Lambda^+E(X))^2E(R_1^2)$$
$$S_4=\Lambda^+E(X(X-1))E(R_2)+(\Lambda^+E(X))^2E(R_2^2)$$

2. Mean waiting time in the queue is obtained by using Little's formula

$$W_q=\frac{L_q}{\Lambda^+E(X)}$$

7. Particular cases

Case 1: If batch arrival, single service ($a=b=1$), no Bernoulli vacation ($\theta=0$) and no negative arrival is considered then (4.11) reduces to

$$P(w)=\frac{\Big[p_1[1-\bar{U}_1(\Psi(w))][1+\pi_1\bar{U}_1(\Psi(w))]+p_2[1-\bar{U}_2(\Psi(w))][1+\pi_2\bar{U}_2(\Psi(w))]\Big]Q}{\big[p_1(1-\pi_1)\bar{U}_1(\Psi(w))+p_1\pi_1(\bar{U}_1(\Psi(w)))^2+p_2(1-\pi_2)\bar{U}_2(\Psi(w))+p_2\pi_2(\bar{U}_2(\Psi(w)))^2\big]-w}$$

where $\Psi(w)=\Lambda^+(1-\mathscr{C}(w))$

$$\mathscr{Q}=1-\rho,\ \rho=\Lambda^+E(X)(p_1E(U_1)(1+\pi_1)+p_2E(U_2)(1+\pi_2))$$

These expressions are exactly matched with the results by Madan et. al (2004).

Case 2: If batch arrival, single service ($a=b=1$), no re-service for two types of service, no Bernoulli vacation ($\theta=0$) and no negative arrival is considered then (4.11) reduces to

$$P(w)=\frac{\Big[p_1[1-\bar{U}_1(\psi(w))]+p_2[1-\bar{U}_2(\Psi(w))]\Big]Q}{[p_1\bar{U}_1(\Psi(w))+p_2\bar{U}_2(\Psi(w))]-w}$$

where $\Psi(w)=\Lambda^+(1-\mathscr{C}(w));\ \ \mathscr{Q}=1-\rho$

$$\rho=\Lambda^+E(X)(p_1E(U_1)+p_2E(U_2))$$

These expressions agree with the results by Baruah et.al (2014).

8. Numerical results

In this section, we present some numerical results and graphs using MATLAB that provide insight into the system behavior.

1. The distribution of arriving batches is assumed to be geometric with mean 2.

2. Service times, Reservice times, vacation times, delay times and two phase of repair times are exponentially and Erlangianly distributed.

3. The arbitrary values to the parameters are so chosen such that they satisfy the stability condition.

Table 2 and 3 shows that when Type 1 service rate (μ_1) increases, then the utilization factor (ρ) decreases, the mean queue size (Lq) decreases and the mean waiting time in the queue (W_q) are also decreases for the values of $a=2$, $b=5$, $\theta=0.3$, $\Lambda^+=1$, $\Lambda^-=1.1$, $\mu_2=14$, $\gamma=7$, $\beta_1=3$, $\beta_2=2.5$, $\xi=1.20$, $\pi_1=0.3$, $\pi_2=0.2$, $p_1=0.2$, $p_2=0.8$. $\xi=1.20$, $\pi_1=0.3$, $\pi_2=0.2$, $p_1=0.2$, $p_2=0.8$.
Table 4 and 5 shows that when vacation rate (γ) increases, then the utilization factor (ρ) decreases, the mean queue size (Lq) decreases and the mean waiting time in the queue (W_q) are also decreases for the values of $a=2$, $b=5$, $\theta=0.3$, $\Lambda^+=1$, $\Lambda^-=1.1$, $\mu_1=17$, $\mu_2=14$, $\beta_1=3$, $\beta_2=2.5$, $\xi=1.20$, $\pi_1=0.3$, $\pi_2=0.2$, $p_1=0.2$, $p_2=0.8$.

Table 2. The impact of service rate (μ_1) on ρ, L_q, W_q

	Exponential		
μ_1	ρ	L_q	W_q
8	0.0195	10.1755	5.0877
9	0.0191	9.9879	4.9940
10	0.0188	9.8355	4.9177
11	0.0185	9.7091	4.8545
12	0.0183	9.6026	4.8013
13	0.0182	9.5116	4.7558
14	0.0181	9.4331	4.7165
15	0.0180	9.3645	4.6823
16	0.0179	9.3041	4.6521
17	0.0178	9.2506	4.6253

Table 3. The impact of service rate (μ_1) on ρ, L_q, W_q

	Erlang-2 stage		
μ_1	ρ	L_q	W_q
8	0.0182	10.6451	5.3226
9	0.0179	10.4415	5.2207
10	0.0176	10.2764	5.1382
11	0.0174	10.1398	5.0699
12	0.0173	10.0250	5.0125
13	0.0172	9.9270	4.9635
14	0.0171	9.8426	4.9213
15	0.0170	9.7689	4.8844
16	0.0169	9.7041	4.8521
17	0.0168	9.6467	4.8234

Table 4. The impact of vacation rate (γ) on ρ, L_q, W_q

	Exponential		
γ	ρ	L_q	W_q
5	0.0185	8.9519	4.4759
6	0.0149	8.7697	4.3849
7	0.0122	8.6442	4.3221
8	0.0103	8.5526	4.2763
9	0.0088	8.4829	4.2415
10	0.0075	8.4281	4.2140
11	0.0065	8.3838	4.1919
12	0.0057	8.3474	4.1737
13	0.0050	8.3169	4.1585
14	0.0044	8.2910	4.1455

Table 5. The impact of vacation rate (γ) on ρ, L_q, W_q

	Erlang-2 stage		
γ	ρ	L_q	W_q
5	0.0174	9.6865	4.8433
6	0.0138	9.4978	4.7489
7	0.0112	9.3687	4.6844
8	0.0092	9.2750	4.6375
9	0.0077	9.2040	4.6020
10	0.0065	9.1484	4.5742
11	0.0055	9.1036	4.5518
12	0.0047	9.0669	4.5334
13	0.0040	9.0361	4.5181
14	0.0033	9.0100	4.5050

For the effect of the parameters, μ_1, γ on the system performance measures, two dimensional graphs are drawn in Figures 1 and 2. Fig.1 and Fig.2 shows respectively that as the values of first type service rate (μ_1) and vacation rate (γ) increases individually, then the utilization factor (ρ), the mean queue size (L_q) and the mean waiting time in the queue (W_q) decreases.

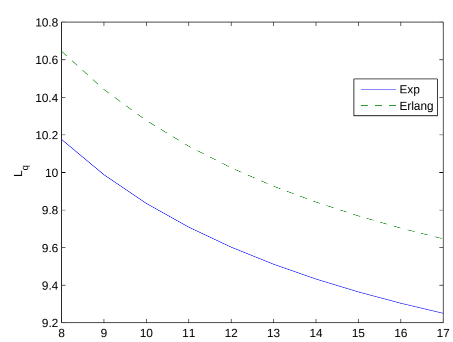

Figure 1. L_q versus μ_1

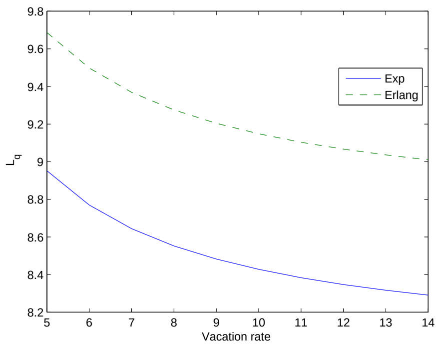

Figure 2. L_q versus γ

9. Conclusion and further work

In this paper, we have studied an $M^{[X]}/G_1(a,b),G_2(a,b)/1$ unreliable G-queue with optional re-service, Bernoulli vacation, delay time to two phase of repair. Where the server provides two types of service and each type consist of an optional re-service. We derive the probability generating function of the number of customers in the queue at a random epoch in transient and steady state conditions. The performance measures of the system state probabilities, the mean queue size and the mean waiting time in the queue are determined under steady state conditions. Some particular cases are discussed. The results are validated with the support of numerical illustrations. To this end, we can extend this model to optional re-service G-queue with working vacations and vacation interruption under Bernoulli schedule.

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