MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **10(01)**(2022), 63–78.
http://doi.org/10.26637/mjm1001/006

Existence results for $(p_1, ..., p_n)$-biharmonic systems under Navier boundary conditions

Têlé Jonas Doumate[*,1,2], **Lawouè Robert Toyou**[2,3] **and Liamidi A. Leadi**[1,2]

[1] *Faculté des Sciences et Techniques, Département de Mathématiques, Université d'Abomey-Calavi, Bénin.*
[2] *Institut de Mathématiques et de Sciences Physiques, Université d'Abomey-Calavi, Bénin.*
[3] *Haute École de Commerce et de Management, Bénin.*

***R**eceived 19 September 2021; **A**ccepted 29 December 2021*

Abstract. We are concerned with the following $(p_1, \ldots, p_n)$-biharmonic system

$$\begin{cases} \Delta^2_{p_i} u_i - m(x)|u_i|^{\alpha_i - 1} u_i \prod\limits_{j=1, j\neq i}^{n} |u_j|^{\alpha_j+1} = \lambda m_i(x)|u_i|^{p_i-2}u_i, & \text{in } \Omega \\ u_i = \Delta u_i = 0, \text{ for } 1 \leq i \leq n, & \text{on } \partial\Omega. \end{cases}$$

The authors study the existence of weak solutions for the problem above via mountain pass theorem and establish semitrivial principal and strictly principal eigenvalues, positivity and simplicity results.

AMS Subject Classifications: 35D30, 35J35, 35J58, 35P30.

Keywords: Nonlinear eigenvalue problems, Variational methods, $(p_1, \ldots, p_n)$-biharmonic systems, Boundary value problems.

Contents

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a non-empty bounded domain with smooth boundary $\partial\Omega$, $n \geq 1$ be an integer, α_i and p_i (with $i \in \{1, 2, \ldots, n\}$) be real constants such that $\alpha_i \geq 0$, $p_i > 1$ and $\sum\limits_{i=1}^{n} \frac{\alpha_i+1}{p_i} = 1$.

The aim of this work is to study the following interesting problem

[*]**Corresponding author.** Email address: **jonas.doumate@fast.uac.bj/doumatt@yahoo.fr** (Têlé Jonas Doumate), robert.toyou@imsp-uac.org (Lawouè Robert Toyou) and leadiare@imsp-uac.org (Liamidi A. Leadi)

$$(Q):\begin{cases}\Delta_{p_1}^2 u_1 - m(x)|u_1|^{\alpha_1-1}u_1\prod\limits_{i=2}^{n}|u_i|^{\alpha_i+1} = \lambda m_1(x)|u_1|^{p_1-2}u_1, & \text{in }\Omega\\ \dots\\ \Delta_{p_i}^2 u_i - m(x)|u_i|^{\alpha_i-1}u_i\prod\limits_{j=1,j\neq i}^{n}|u_j|^{\alpha_j+1} = \lambda m_i(x)|u_i|^{p_i-2}u_i, & \text{in }\Omega\\ \dots\\ \Delta_{p_n}^2 u_n - m(x)|u_n|^{\alpha_n-1}u_n\prod\limits_{i=1}^{n-1}|u_i|^{\alpha_i+1} = \lambda m_n(x)|u_n|^{p_n-2}u_n, & \text{in }\Omega\\ u_i=\Delta u_i = 0,\ \text{for } 1\leq i\leq n, & \text{on }\partial\Omega\end{cases}$$

where $\Delta_{p_i}^2 u_i = \Delta(|\Delta u_i|^{p_i-2}\Delta u_i)$ is the p_i-biharmonic operator and λ is a real parameter. Here, the coefficients m_i, (with $i = 1, 2, \dots, n$), $m \in L^\infty(\Omega)$ are assumed to be nonnegatives in Ω. Throughout this paper, we let W denote the Cartesian product of n Sobolev spaces $\left(W^{2,p_i}(\Omega)\cap W_0^{1,p_i}(\Omega)\right)$ for $i = 1, \dots, n$, i.e.,

$$W = \left(W^{2,p_1}(\Omega)\cap W_0^{1,p_1}(\Omega)\right)\times\left(W^{2,p_2}(\Omega)\cap W_0^{1,p_2}(\Omega)\right)\times\cdots\times\left(W^{2,p_n}(\Omega)\cap W_0^{1,p_n}(\Omega)\right)$$

endowed with the norm

$$\|(u_1,u_2,\dots,u_n)\| = \sum_{i=1}^{n}\|\Delta u_i\|_{p_i} \tag{1.1}$$

where $\|.\|_p$ stands for the Lebesgue norm in L^p for all $p\in(1,\infty]$. We say that $((u_1,\dots,u_n),\lambda)\in W\times\mathbb{R}$ is a (weak) solution to the problem (Q) if

$$\int_\Omega |\Delta u_i|^{p_i-2}\Delta u_i\Delta\varphi_i dx - \int_\Omega m\prod_{j=1,j\neq i}^{n}|u_j|^{\alpha_j+1}|u_i|^{\alpha_i-1}u_i\varphi_i dx = \lambda\int_\Omega m_i|u|^{p_i-2}u_i\varphi_i dx, \tag{1.2}$$

for $1\leq i\leq n$ and for all $(\varphi_1,\dots,\varphi_n)\in W$.

The study of nonlinear eigenvalue problems involving fourth-order differential equations has aroused a great interest in the scientific world and many applications have been made, including the study of deflections of elastic beams on nonlinear elastic foundations (see [2, 6, 19]), deformations of a rigid body and especially the study of traveling waves in suspension bridges (see, for instance, [14]) . A remarkable work of M. Talbi and N. Tsouli [19] has focused on the scalar version of (Q) with $m\equiv 0$, which reads

$$(P_{a,p,\rho}):\begin{cases}\Delta(\rho|\Delta u|^{p-2}\Delta u) = \lambda a(x)|u|^{p-2}u & \text{in}\quad\Omega\\ u=\Delta u = 0 & \text{on}\quad\partial\Omega\end{cases} \tag{1.3}$$

where $p\in(1,\infty)$, $\rho\in C(\bar\Omega)$ such that $\rho>0$ and $a\in L^\infty(\Omega)$. They proved that $(P_{a,p,\rho})$ possesses at least one non-decreasing sequence of eigenvalues and studied $(P_{a,p,\rho})$ in the particular one dimensional case. The authors, in the same reference gave the first eigenvalue $\lambda_{1,p,\rho}(a)$ and showed that if $a\geq 0$ almost everywhere in Ω and $a\in C(\bar\Omega)$ then $\lambda_{1,p,\rho}(a)$ is simple, isolated and principal i.e. the associated eigenfunction, denoted by $\varphi_{p,\rho,a}$ is positive on Ω with

$$\lambda_{1,p,\rho}(a) = \inf_{u\in\mathcal{A}}\int_\Omega \rho|\Delta u|^p dx \tag{1.4}$$

where

$$\mathcal{A} = \left\{u\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega) : \int_\Omega a|u|^p dx = 1\right\}. \tag{1.5}$$

By using a transformation of $(P_{a,p,\rho})$ to a known Poisson problem when $\rho \equiv 1$ and $a \equiv 1$, the authors in [9] proved the existence of a principal positive simple eigenvalue which is isolated and a description of all eigenvalues and associated eigenfunctions was given as well. The Dirichlet boundary conditions case was analyzed in [10] where it is shown that the spectrum contains at least one non-decreasing sequence of positive eigenvalues. On the other hand, J. Benedikt [4] gave the spectrum of the p-biharmonic operator with Dirichlet and Neumann boundary conditions in the special case $N = 1$, $\rho \equiv 1$ and $a \equiv 1$. They system (Q) in the absence of weights has drawn attention in [12] where the authors used the generalized three critical points of Ricceri, namely, three critical points theorem of Averna and Bonanno to prove the existence of at least three weak solutions for (Q) in case no weight is considered.

It is important to note that (u_1, λ) is solution of problem $(P_{m_1,p_1,1})$ if and only if $[(u_1, 0, \dots, 0), \lambda]$ is solution of (Q). This kind of solution is called "semitrivial" solution for (Q). Consequently, there are n "semitrivial" solutions of the problem (Q) that is $[(u_1, 0, \dots, 0), \lambda]$ with (u_1, λ) solution of problem $(P_{m_1,p_1,1})$, $[(0, \dots, u_i, \dots, 0), \lambda]$ with (u_i, λ) solution of problem $(P_{m_i,p_i,1})$ for $2 \leq i \leq n-1$ and $[(0, 0, \dots, u_n), \lambda]$ with (u_n, λ) solution of problem $(P_{m_n,p_n,1})$. In particular, when $\lambda = \lambda_{1,p_1,1}(m_1)$ (resp. $\lambda = \lambda_{1,p_i,1}(m_i)$) then $[(\varphi_{p_1,1,m_1}, 0), \lambda]$ (resp. $[(0, \varphi_{p_i,1,m_i}), \lambda]$ for $2 \leq i \leq n$) is called "semitrivial" solutions of the problem (Q) and $\lambda_{1,p_1,1}(m_1)$ (resp. $\lambda_{1,p_i,1}(m_i)$) is called "semitrivial" principal eigenvalue of (Q).

Recently, in a very interesting paper, L. A. Leadi and R. L. Toyou [17] studied the simplicity and the existence of the first strictly principal eigenvalue or semitrivial principal eigenvalue of problem (Q) in the particular case of $n = 2$. Motivated by their results we consider in this work the problem (Q), which generalizes the one in [17], and we intend to extend their findings in this general and challenging form of (Q). For this, we shall recall a bit of notations and basic results. The Sobolev space W endowed with the norm defined in (1.1) is a Banach and reflexive space [13, 18] and the weak convergence in W is denoted by $\rightharpoonup$. The positive and negative parts of a function w are denoted by $w^+ = \max\{w, 0\}$ and $w^- = \max\{-w, 0\}$. Equalities (and inequalities) between two functions must be understood almost everywhere (a.e.). Notice that, for all $f \in L^p(\Omega)$, the Poisson equation associated with the Dirichlet problem

$$\begin{cases} -\Delta u = f(x) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

is uniquely solvable in $X_p = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see for example [11]). We denote by Λ the inverse operator of $-\Delta : X_p \longmapsto L^p(\Omega)$. and the following lemma gives us some properties of the operator Λ:

Lemma 1.1. *[9, 19].*

1. *(Continuity) There exists a constant $c_p > 0$ such that*

$$\|\Lambda f\|_{W^{2,p}} \leq c_p \|f\|_p$$

holds for all $p \in (1, \infty)$ and $f \in L^p(\Omega)$.

2. *(Continuity) Given $k \in \mathbb{N}^*$, there exists a constant $c_{p,k} > 0$ such that*

$$\|\Lambda f\|_{W^{k+2,p}} \leq c_{p,k} \|f\|_{W^{k,p}}$$

holds for all $p \in (1, \infty)$ and $f \in W^{k,p}(\Omega)$.

3. *(Symmetry) The identity*

$$\int_\Omega \Lambda u.v dx = \int_\Omega u.\Lambda v dx$$

holds for $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$ with $p' = \frac{p}{p-1}$ and $p \in (1, \infty)$.

4. *(Regularity) Given $f \in L^{\infty}(\Omega)$, we have $\Lambda f \in C^{1,\nu}(\bar{\Omega})$ for all $\nu \in (0,1)$. Moreover, there exists $c_\nu > 0$ such that*
$$\|\Lambda f\|_{C^{1,\nu}(\Omega)} \leq c_\nu \|f\|_\infty.$$
5. *(Regularity and Hopf-type maximum principle) Let $f \in C(\bar{\Omega})$ and $f \geq 0$ then $w = \Lambda f \in C^{1,\nu}(\bar{\Omega})$, for all $\nu \in (0,1)$ and w satisfies: $w > 0$ in Ω, $\frac{\partial w}{\partial n} < 0$ on $\partial\Omega$.*
6. *(Order preserving property) Given f, $g \in L^p(\Omega)$ if $f \leq g$ in Ω, then $\Lambda f < \Lambda g$ in Ω.*

The rest of the paper is organized as follows. The next section sets the functional framework, a review of tools and established results that help in our concern and constructs an eigencurve associated to the system (Q) as well as some well-known properties on obtained eigencurve. Section 3 is devoted to the study of semitrivial solutions and strictly principal eigenvalues of (Q). We thereby find the lowest eigenvalue of problem (Q) which is proved to be unique, positive, semitrivial principal or strictly principal and simple.

2. An eigenvalue curve associated to problem (Q)

We shall adopt the approach used in a number of papers (see for example [5], [7], [8], [16], [15]) by fixing λ and embed the system (Q) into a new system (Q_λ) in order to derive the existence of solutions for (Q) that is:

$$(Q_\lambda): \begin{cases} \Delta^2_{p_1} u_1 - m(x)|u_1|^{\alpha_1 - 1}u_1 \prod\limits_{i=2}^{n} |u_i|^{\alpha_i+1} - \lambda m_1(x)|u_1|^{p_1-2}u_1 = \mu|u_1|^{p_1-2}u_1, & \text{in } \Omega \\ \cdots \\ \Delta^2_{p_i} u_i - m(x)|u_i|^{\alpha_i - 1}u_i \prod\limits_{j=1, j\neq i}^{n} |u_j|^{\alpha_j+1} - \lambda m_i(x)|u_i|^{p_i-2}u_i = \mu|u_i|^{p_i-2}u_i, & \text{in } \Omega \\ \cdots \\ \Delta^2_{p_n} u_n - m(x)|u_n|^{\alpha_n - 1}u_n \prod\limits_{i=1}^{n-1} |u_i|^{\alpha_i+1} - \lambda m_n(x)|u_n|^{p_n-2}u_n = \mu|u_n|^{p_n-2}u_n, & \text{in } \Omega \\ u_i = \Delta u_i = 0, \text{ for } 1 \leq i \leq n, & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where μ is a new real parameter. For convenience, we now give the following definitions:

Definition 2.1. .

1. *We say that $((u_1, \dots, u_n), \mu)$ is a (weak) solution to problem (Q_λ) if $((u_1, \dots, u_n), \mu) \in W \times \mathbb{R}$ and*
$$\int_\Omega |\Delta u_i|^{p_i-2}\Delta u_i \Delta\varphi_i dx - \int_\Omega m \prod_{j=1, j\neq i}^{n} |u_j|^{\alpha_j+1}|u_i|^{\alpha_i-1}u_i\varphi_i dx - \lambda \int_\Omega m_i|u_i|^{p_i-2}u_i\varphi_i dx = \mu \int_\Omega |u_i|^{p_i-2}u_i\varphi_i dx, \text{ for } 1 \leq i \leq n, \tag{2.2}$$
for all $(\varphi_1, \dots, \varphi_n) \in W$.
2. *We say that $((u_1, \dots, u_n), \lambda)$ is a (weak) solution to problem (Q) if $((u_1, \dots, u_n), \lambda) \in W \times \mathbb{R}$ and*
$$\int_\Omega |\Delta u_i|^{p_i-2}\Delta u_i \Delta\varphi_i dx - \int_\Omega m \prod_{j=1, j\neq i}^{n} |u_j|^{\alpha_j+1}|u_i|^{\alpha_i-1}u_i\varphi_i dx = \lambda \int_\Omega m_i|u|^{p_i-2}u_i\varphi_i dx, \tag{2.3}$$
for $1 \leq i \leq n$ and for all $(\varphi_1, \dots, \varphi_n) \in W$.
3. *If $((u_i, \dots, u_n), \lambda) \in W \times \mathbb{R}$ (resp. $((u_1, \dots, u_n), \mu) \in W \times \mathbb{R}$) is a (weak) solution to problem (Q) (resp. (Q_λ)), then $(u_1, \dots, u_n)$ shall be called an eigenfunction of the problem (Q) (resp. (Q_λ)) associated to the eigenvalue λ (resp. $\mu(\lambda)$).*

4. *Let us agree to say that an eigenvalue of (Q) or (Q_λ) is strictly principal (resp. semitrivial principal) if it is associated to an eigenfunction $(u_1, \ldots, u_n)$ such that $u_i > 0$ or $u_i < 0$, $\forall i \in \{1, \ldots, n\}$ (resp. there exist $\emptyset \neq J_n \subset \{1, \ldots, n\}$ such that $u_k \equiv 0$, $\forall k \in J_n$ and $u_i > 0$ or $u_i < 0$, $\forall i \in \{1, \cdots, n\} \setminus J_n$).*

Based on variational approach, for a fixed real λ, we define the energy functional

$$\begin{aligned} J_\lambda : W &\longrightarrow \mathbb{R} \\ (u_1, \ldots, u_n) &\longmapsto J_\lambda(u_1, \ldots, u_n) = \sum_{i=1}^n \frac{\alpha_i + 1}{p_i} \|\Delta u_i\|_{p_i}^{p_i} - V(u_1, \ldots, u_n) - \lambda M(u_1, \ldots, u_n), \end{aligned}$$

where

$$V(u_1, \ldots, u_n) = \int_\Omega m \prod_{i=1}^n |u_i|^{\alpha_i + 1} dx, \quad \text{and} \quad M(u_1, \ldots, u_n) = \sum_{i=1}^n \frac{\alpha_i + 1}{p_i} M_i(u_i)$$

with

$$M_i(u_i) = \int_\Omega m_i |u_i|^{p_i} dx, \quad \forall (u_1, \ldots, u_n) \in W.$$

Equalities (2.2) are equivalent to

$$\nabla J_\lambda(u_1, \ldots, u_n) = \mu \nabla I(u_1, \ldots, u_n)$$

where

$$I(u_1, \ldots, u_n) = \sum_{i=1}^n \frac{\alpha_i + 1}{p_i} \|u_i\|_{p_i}^{p_i} \quad \forall (u_1, \ldots, u_n) \in W.$$

We now state the main result of this section which generalizes the result of Proposition 2.1 in [17] where (p, q)-biharmonic system case is treated.

Theorem 2.2. *The value*

$$\mu_1(\lambda) := \inf\{J_\lambda(u_1, \ldots, u_n) : (u_1, \ldots, u_n) \in \mathcal{M}\} \tag{2.4}$$

where

$$\mathcal{M} = \{(u_1, \ldots, u_n) \in W : I(u_1, \ldots, u_n) = 1\},$$

is the smallest eigenvalue of (Q_λ).

The proof of Theorem 2.2 relies on the following lemma:

Lemma 2.3. *Let $(\omega_1, \ldots, \omega_n) \in [L^\infty(\Omega)]^n$. If $\omega_1, \ldots, \omega_n > 0$ on Ω then there exist $n + 1$ positive constants $c_1, \ldots, c_{n+1}$ such that*

$$\sum_{i=1}^n \|\Delta u_i\|_{p_i}^{p_i} \leq c_{n+1} J_\lambda(u_1, \ldots, u_n) + \sum_{i=1}^n c_i \int_\Omega \omega_i |u_i|^{p_i} dx \tag{2.5}$$

for every $(u_1, \ldots, u_n) \in W$.

Proof. We borrow ideas from [17]. Indeed, we know that $M_i(u_i) \leq \|m_i\|_\infty \|u_i\|_{p_i}^{p_i}$, for $1 \leq i \leq n$. Since $\sum\limits_{i=1}^n \frac{\alpha_i+1}{p_i} = 1$, it well known by Young inequality that:

$$V(u_1, \ldots, u_n) \leq \|m\|_\infty \int_\Omega \left(\sum_{i=1}^n \frac{\alpha_i + 1}{p_i} |u_i|^{p_i} \right) dx. \tag{2.6}$$

Setting $k_3 = \max\{k_1, k_2\}$ with

$$k_1 = \|m\|_\infty \max\left\{\frac{\alpha_i+1}{p_i},\ \text{for } 1 \le i \le n\right\} \quad \text{and} \quad k_2 = |\lambda| \max\left\{\frac{\alpha_i+1}{p_i}\|m_i\|_\infty,\ \text{for } 1 \le i \le n\right\},$$

one deduces that

$$V(u_1,\dots,u_n) \le k_1 \left(\sum_{i=1}^n \|u_i\|_{p_i}^{p_i}\right) \quad \text{and} \quad |\lambda M(u,v)| \le k_2 \left(\sum_{i=1}^n \|u_i\|_{p_i}^{p_i}\right).$$

On the other hand for $\varepsilon > 0$ there exist $M_{i,\varepsilon} > 0$ for $1 \le i \le n$ such that

$$\|u_i\|_{p_i}^{p_i} \le \varepsilon \|\Delta u_i\|_{p_i}^{p_i} + M_{i,\varepsilon} \int_\Omega \omega_i |u_i|^{p_i} dx.$$

Now, we have

$$\sum_{i=1}^n \left(\frac{\alpha_i+1}{p_i}\|\Delta u_i\|_{p_i}^{p_i}\right) = J_\lambda(u_1,\dots,u_n) - V(u_1,\dots,u_n) + \lambda M(u_1,\dots,u_n).$$

Then,

$$\begin{aligned}\sum_{i=1}^n \left(\frac{\alpha_i+1}{p_i}\|\Delta u_i\|_{p_i}^{p_i}\right) &\le J_\lambda(u_1,\dots,u_n) + 2k_3\left(\sum_{i=1}^n \|u_i\|_{p_i}^{p_i}\right)\\ &\le J_\lambda(u_1,\dots,u_n) + 2\varepsilon k_3\left(\sum_{i=1}^n \|\Delta u_i\|_{p_i}^{p_i}\right) + 2k_3\left(\sum_{i=1}^n M_{i,\varepsilon}\int_\Omega \omega_i|u_i|^{p_i}dx\right).\end{aligned}$$

Let $\varepsilon > 0$ be such that $k_4 = \min\left\{\frac{\alpha_i+1}{p_i} - 2\varepsilon k_3,\ \text{for } 1 \le i \le n\right\} > 0$.

Thus, it reads

$$k_4 \sum_{i=1}^n \left(\|\Delta u_i\|_{p_i}^{p_i}\right) \le J_\lambda(u_1,\dots,u_n) + 2k_3 \sum_{i=1}^n \left(M_{i,\varepsilon}\int_\Omega \omega_i|u_i|^{p_i}dx\right).$$

We deduce

$$\sum_{i=1}^n \left(\|\Delta u_i\|_{p_i}^{p_i}\right) \le \frac{1}{k_4} J_\lambda(u_1,\dots,u_n) + \sum_{i=1}^n \left(\frac{2k_3 M_{i,\varepsilon}}{k_4}\int_\Omega \omega_i|u_i|^{p_i}dx\right)$$

and one can consequently take $c_{n+1} = \dfrac{1}{k_4}$, and $c_i = \dfrac{2k_3 M_{i,\varepsilon}}{k_4}$ for $1 \le i \le n$. This completes the proof. ∎

Proof of Theorem 2.2 . By Lemma 2.3, for $\omega_i \equiv 1$ and $1 \le i \le n$, one can easily show that

$$\begin{aligned}0 \le \sum_{i=1}^n \left(\|\Delta u_i\|_p^p\right) &\le c_{n+1} J_\lambda(u_1,\dots,u_n) + \sum_{i=1}^n \left(c_i \int_\Omega |u_i|^{p_i}dx\right)\\ &\le c_{n+1} J_\lambda(u_1,\dots,u_n) + c_0 \sum_{i=1}^n \left(\frac{\alpha_i+1}{p_i}\int_\Omega |u_i|^{p_i}dx\right)\\ &= c_{n+1} J_\lambda(u_1,\dots,u_n) + c_0, \forall (u_1,\dots,u_n) \in \mathcal{M}\end{aligned}$$

where $c_0 = \max\{\frac{p_i c_i}{\alpha_i+1},\ \text{for } 1 \le i \le n\}$, so that J_λ is bounded below on $\mathcal{M}$. Furthermore any sequence $(u_{1,k},\dots,u_{n,k})$ that minimizes J_λ on $\mathcal{M}$ is bounded in W.

Thus there exists $(u_{1,0}, \dots, u_{n,0}) \in W$ such that, up to a subsequence, $(u_{1,k}, \dots, u_{n,k})$ converges weakly to $(u_{1,0}, \dots, u_{n,0})$ in W and strongly in $\prod_{i=1}^{n} L^{p_i}(\Omega)$. Hence

$$J_\lambda(u_{1,0}, \dots, v_{n,0}) \leq \lim_{k \longrightarrow \infty} J_\lambda(u_{1,k}, \dots, v_{n,k}) = \mu_1(\lambda), \quad (u_{1,0}, \dots, u_{n,0}) \in \mathcal{M}$$

and consequently $J_\lambda(u_{1,0}, \dots, u_{n,0}) = \mu_1(\lambda)$. By the Lagrange multipliers rule, $\mu_1(\lambda)$ is an eigenvalue for (Q_λ) and $(u_{1,0}, \dots, u_{n,0})$ is an associated eigenfunction.

Moreover for any eigenvalue $\mu(\lambda)$ associated to $(u_{\lambda,1}, \dots, u_{\lambda,n}) \in W \setminus \{(0, ..., 0)\}$, one has

$$J_\lambda(u_{\lambda,1}, \dots, u_{\lambda,n}) = \mu(\lambda) I(u_{\lambda,1}, \dots, u_{\lambda,n})$$

with $I(u_{\lambda,1}, \dots, u_{\lambda,n}) > 0$. Consequently

$$\mu_1(\lambda) \leq J_\lambda \left(\frac{u_{\lambda,1}}{I(u_{\lambda,1}, \dots, u_{\lambda,n})^{\frac{1}{p_1}}}, \dots, \frac{u_{\lambda,n}}{I(u_{\lambda,1}, \dots, u_{\lambda,n})^{\frac{1}{p_n}}} \right) = \frac{J_\lambda(u_{\lambda,1}, \dots, u_{\lambda,n})}{I(u_{\lambda,1}, \dots, u_{\lambda,n})} = \mu(\lambda).$$

All in all, we have proved that $\mu_1(\lambda)$ is the smallest eigenvalue of (Q_λ). ■

Remark 2.4. *We can denote by*

$$\mu_0 := \inf \left\{ \sum_{i=1}^{n} \frac{\alpha_i + 1}{p_i} \|\Delta u_i\|_{p_i}^{p_i} : (u_1 ..., u_n) \in \mathcal{M} \right\} \tag{2.7}$$

for $m = m_i \equiv 0$, $\forall i \in \{1, \cdots, n\}$. Since the space $W^{2,p_i}(\Omega) \cap W_0^{1,p_i}(\Omega)$, for $i \in \{1, \cdots, n\}$ does not contain any constant non trivial function, one has $\mu_0 > 0$.

It is straightforward proving the following:

Proposition 2.5. .

1. *μ_1 is concave and differentiable with $\mu_1^{'}(\lambda) = -M(u_{1,0}, ..., u_{n,0})$ where $(u_{1,0}, ..., u_{n,0})$ is some eigenfunction of (Q_λ) associated to $\mu_1(\lambda)$ for all $\lambda \in \mathbb{R}$.*
2. $\lim_{\lambda \to \infty} \mu_1(\lambda) = -\infty$.
3. *μ_1 is strictly decreasing.*

Proof. The proof is partly adapted from analogous technics in literature.

1. The concavity of μ_1 follows from the concavity of the mapping $\lambda \longmapsto J_\lambda(u_1, \dots, u_n)$, for a fixed $(u_1, \dots, u_n) \in W$. In particular μ_1 is continuous. Now let $\lambda_k \longrightarrow \lambda$ and $(u_{1,k}, \dots, u_{n,k})$, $(u_{\lambda,1}, \dots, u_{\lambda,n})$ be the I-normalized eigenfunctions related to $\mu_1(\lambda_k)$ and $\mu_1(\lambda)$ respectively. We apply Lemma 2.3 with $\omega_i \equiv 1$, for $1 \leq i \leq n$, to get

$$\begin{aligned} \sum_{i=1}^{n} (\|\Delta u_{i,k}\|_{p_i}^{p_i}) &\leq c_{n+1} J_\lambda(u_{1,k}, \dots, u_{n,k}) + \sum_{i=1}^{n} (c_i \int_\Omega |u_{i,k}|^{p_i} dx), \\ &\leq c_{n+1} \mu_1(\lambda_k) + \max \left\{ \frac{p_i c_i}{\alpha_i + 1}, \text{ for } 1 \leq i \leq n \right\}. \end{aligned}$$

In addition,

$$\lim_{k \to \infty} c_{n+1} \mu_1(\lambda_k) + \max \left\{ \frac{p_i c_i}{\alpha_i + 1}, \text{ for } 1 \leq i \leq n \right\} = c_{n+1} \mu_1(\lambda) + \max \left\{ \frac{p_i c_i}{\alpha_i + 1}, \text{ for } 1 \leq i \leq n \right\}.$$

So we conclude that $(u_{1,k},\dots,u_{n,k})_k$ is a bounded sequence in W. Hence there exists $(u_{1,0},\dots,u_{n,0})$ such that, up to a subsequence, $(u_{1,k},\dots,u_{n,k}) \rightharpoonup (u_{1,0},\dots,u_{n,0})$ in W, strongly in $\prod_{i=1}^{n} L^{p_i}(\Omega)$. Then $(u_{1,0},\dots,u_{n,0}) \in \mathcal{M}$ and from

$$J_\lambda(u_{1,0},\dots,u_{n,0}) \leq \lim_{k\longrightarrow\infty} J_\lambda(u_{1,k},\dots,u_{n,k}) = \mu_1(\lambda)$$

we infer that $\mu_1(\lambda) = J_\lambda(u_{1,0},\dots,u_{n,0}) = J_\lambda(u_{\lambda,1},\dots,u_{\lambda,n})$ and $(u_{1,0},\dots,u_{n,0})$ is an eigenfunction of (Q_λ) associated to $\mu_1(\lambda)$. Furthermore

$$\begin{cases} \mu_1(\lambda_k) - \mu_1(\lambda) \geq -(\lambda_n - \lambda)M(u_{1,k},\dots,u_{n,k}) \\ \mu_1(\lambda_k) - \mu_1(\lambda) \leq -(\lambda_n - \lambda)M(u_{1,0},\dots,u_{n,0}). \end{cases}$$

Hence

$$\begin{cases} -M(u_{1,k},\dots,u_{n,k}) \leq \dfrac{\mu_1(\lambda_n) - \mu_1(\lambda)}{\lambda_k - \lambda} \leq -M(u_{1,0},\dots,u_{n,0}), \text{ if } \lambda_k > \lambda \\ -M(u_{1,0},\dots,u_{n,0}) \leq \dfrac{\mu_1(\lambda_k) - \mu_1(\lambda)}{\lambda_k - \lambda} \leq -M(u_{1,k},\dots,u_{n,k}), \text{ if } \lambda_k < \lambda. \end{cases}$$

Passing to the limit we get $\mu_1^{'}(\lambda) = -M(u_{1,0},\dots,u_{n,0})$.

2. We know that m_1 is nonnegative, then there exists a function $u_1 \in X_{p_1}$ such that $M_1(u_1) > 0$ and $I(u_1,0,\dots,0) = 1$.
   Then, for all $\lambda \in \mathbb{R}_+^*$, $\mu_1(\lambda) \leq J_\lambda(u_1,0,\dots,0)$. We deduce that

$$\lim_{\lambda\longrightarrow\infty} J_\lambda(u_1,0,\dots,0) = \lim_{\lambda\longrightarrow\infty} E_m(u_1,0,\dots,0) - \lambda M(u_1,0,\dots,0) = -\infty$$

   where

$$E_m(u_1,\dots,u_n) = \sum_{i=1}^{n}\left(\frac{\alpha_i+1}{p_i}\|\Delta u_i\|_{p_i}^{p_i}\right) - \int_\Omega \left(m\prod_{i=1}^{n}|u_i|^{\alpha_i+1}\right)dx.$$

   Thus $\lim_{\lambda\longrightarrow\infty} \mu_1(\lambda) = -\infty$.

3. The result is clear from the fact that $M(u_{\lambda,1},\dots,u_{\lambda,n}) > 0$ for any $\lambda \in \mathbb{R}$. Indeed, if $\lambda_1 < \lambda_2$ then

$$\begin{aligned} \mu_1(\lambda_1) &= E_m(u_{\lambda_1,1},\dots,u_{\lambda_1,n}) - \lambda_1 M(u_{\lambda_1,1},\dots,u_{\lambda_1,n}) \\ &\geq E_m(u_{\lambda_1,1},\dots,u_{\lambda_1,n}) - \lambda_2 M(u_{\lambda_1,1},\dots,u_{\lambda_1,n}) \\ &\geq \mu_1(\lambda_2). \end{aligned}$$

■

3. Existence of solutions for the system (Q)

We address, in this section, the problem (Q) by looking for the zeros of the function $\mu_1(\lambda)$ which by construction solve the problem. Let us make this assumption

$$(H_m) : \|m\|_\infty < \mu_0.$$

We start by proving the following:

Lemma 3.1. *If (H_m) is satisfied, then $\mu_1(0) > 0$ and $\mu_1(\lambda) = 0$ if and only if $\lambda > 0$ is an eigenvalue of (Q).*

Proof. Assume that (H_m) is satisfied. By (2.6), we have $V(u_1, \dots, u_n) \leq \|m\|_\infty I(u_1, \dots, u_n)$, $\forall (u_1, \dots, u_n) \in W$. Then, one has

$$\sum_{i=1}^{n} \left(\frac{\alpha_i + 1}{p_i} \|\Delta u_i\|_{p_i}^{p_i} \right) - \|m\|_\infty I(u_1, \dots, u_n) \leq E_m(u_1, \dots, u_n), \forall (u_1, \dots, u_n) \in W.$$

This implies

$$\mu_0 \leq E_m(u_1, \dots, u_n) + \|m\|_\infty, \quad \forall (u_1, \dots, u_n) \in \mathcal{M},$$
$$\mu_0 - \|m\|_\infty \leq \inf\{E_m(u_1, \dots, u_n), (u_1, \dots, u_n) \in \mathcal{M}\} \leq \mu_1(0).$$

We then conclude that $\mu_1(0) > 0$ and $\mu_1(\lambda) = 0$ if and only if $\lambda > 0$ is an eigenvalue of (Q). ■

From now on, we denote

$$L(\Omega) := \left(\left[\prod_{i=1}^{n} L^{p_i}(\Omega) \right] \setminus \{(0, \dots, 0)\} \right) \times \mathbb{R}, \tag{3.1}$$

$$L_0(\Omega) := \left(\left[\prod_{i=1}^{n} L^{p_i}(\Omega) \right] \setminus \{(0, \dots, 0)\} \right) \times \{0\}. \tag{3.2}$$

We adapt and apply some results proved in [9] and some ideas used in [19] to establish the following.

Remark 3.2. .

1. $\forall u \in X_p$, $\forall v \in L^p(\Omega)$ *(with* $p \in (1, \infty)$*):* $v = -\Delta u \iff u = \Lambda v$.
2. *Let* N_p *(with* $p \in (1, \infty)$*) be the Nemytskii operator defined by*

$$N_p(u)(x) = \begin{cases} |u(x)|^{p-2} u(x) \text{ if } u(x) \neq 0 \\ 0 \qquad\qquad\qquad \text{if } u(x) = 0. \end{cases}$$

   We have

$$\forall v \in L^p(\Omega), \quad \forall w \in L^{p'}(\Omega) : \quad N_p(v) = w \iff v = N_{p'}(w) \tag{3.3}$$

   with $p' = \frac{p}{p-1}$.

3. *If* $(u_1, \dots, u_n)$ *is an eigenfunction of* (Q_λ) *associated with* μ *then* $\varphi_i = -\Delta u_i$*, for* $1 \leq i \leq n$ *satisfy:*

$$N_{p_j}(\varphi_j) = \Lambda \left([\mu(\lambda) + \lambda m_j] N_{p_j}(\Lambda \varphi_j) + m \prod_{i=1, i\neq j}^{n} |\Lambda \varphi_i|^{\alpha_i+1} |\Lambda \varphi_j|^{\alpha_j - 1} \Lambda \varphi_j \right), \text{ for } 1 \leq j \leq n.$$

   Hence:

   (a) $((u_{1,0}, \dots, u_{n,0}), \mu(\lambda))$ *is a solution of* (Q_λ) *if and only if* $((\varphi_{1,0}, \dots, \varphi_{n,0}), \mu(\lambda))$ *is a solution of problem*

$$(Q'_\lambda) : \begin{cases} \text{Find } ((\varphi_1, \dots, \varphi_n), \mu(\lambda)) \in L(\Omega) \text{ such that} \\ N_{p_j}(\varphi_j) = \Lambda \left([\mu(\lambda) + \lambda m_j] N_{p_j}(\Lambda \varphi_j) + m \prod\limits_{i=1, i\neq j}^{n} |\Lambda(\varphi_i)|^{\alpha_i+1} |\Lambda(\varphi_j)|^{\alpha_j - 1} \Lambda(\varphi_j) \right), \\ \text{for } 1 \leq j \leq n, \end{cases}$$

   with $\varphi_{j,0} = -\Delta(u_{j,0})$.

(b) $((\varphi_{1,0},\dots,u_{n,0}),\mu_1(\lambda))\in L_0(\Omega)$ *is a solution of* $(Q^{'}_{\lambda})$ *if and only if* $((\varphi_{1,0},\dots,\varphi_{n,0}),\lambda)\in L(\Omega)$ *is a solution of problem*

$$(Q^{'}):\begin{cases}\textit{Find } ((\varphi_1,\dots,\varphi_n),\lambda)\in L(\Omega) \textit{ such that}\\ N_{p_j}(\varphi_j)=\Lambda\left(\lambda m_j N_{p_j}(\Lambda\varphi_j)+m\prod_{i=1,i\neq j}^{n}|\Lambda(\varphi_i)|^{\alpha_i+1}|\Lambda(\varphi_j)|^{\alpha_j-1}\Lambda(\varphi_j)\right),\\ \textit{for } 1\le j\le n,\end{cases}$$

with $\varphi_{j,0}=-\Delta(u_{j,0})$.

(c)

$$\mu_1(\lambda):=\inf\left\{F_\lambda(\varphi_1,\dots,\varphi_n):(\varphi_1,\dots,\varphi_n)\in\prod_{i=1,i\neq j}^{n}L^{p_i}(\Omega),R(\varphi_1,\dots,\varphi_n)=1\right\}\tag{3.4}$$

where

$$F_\lambda(\varphi_1,\dots,\varphi_n)=\sum_{i=1}^{n}\left(\frac{\alpha_i+1}{p_i}\left[\int_\Omega|\varphi_i|^{p_i}dx-\lambda\int_\Omega m_i|\Lambda\varphi_i|^{p_i}dx\right]\right)-\int_\Omega m\prod_{i=1}^{n}|\Lambda\varphi_i|^{\alpha_i+1}dx,$$
$$R(\varphi_1,\dots,\varphi_n)=\sum_{i=1}^{n}\frac{\alpha_i+1}{p_i}\|\Lambda\varphi_i\|_{p_i}^{p_i}.$$

Lemma 3.3. *If* $((u_1,\dots,u_n),\mu(\lambda))$ *is a solution of* (Q_λ) *then* $-\Delta u_i\in C(\bar{\Omega})$ *and* $u_i\in C^{1,\nu}(\bar{\Omega})$*, for* $1\le i\le n$ *and for all* $\nu\in(0,1)$.

Proof. An easy adaptation of Lemma 3.2 in [17]. ■

Lemma 3.4. $((\varphi_{1,1},\dots,\varphi_{1,n}),\mu_1(\lambda))\in L(\Omega)$ *is a solution of problem* $(Q^{'}_{\lambda})$*, if and only if*

$$G_\lambda(\varphi_{1,1},\dots,\varphi_{1,n})=0=\min_{(\varphi_1,\dots,\varphi_n)\in L^*(\Omega)}G_\lambda(\varphi_1,\dots,\varphi_n)\tag{3.5}$$

where

$$G_\lambda(\varphi_1,\dots,\varphi_n)=F_\lambda(\varphi_1,\dots,\varphi_n)-\mu_1(\lambda)R(\varphi_1,\dots,\varphi_n)\ \textit{and}\ L^*(\Omega)=\left[\prod_{i=1}^{n}L^{p_i}(\Omega)\right]\setminus\{(0,\dots,0)\}.$$

Proof. Assume that $((\varphi_{1,1},\dots,\varphi_{1,n}),\mu_1(\lambda))\in L(\Omega)$ is a solution of problem $(Q^{'}_{\lambda})$.

Then $F_\lambda(\varphi_{1,1},\dots,\varphi_{1,n})=\mu_1(\lambda)R(\varphi_{1,1},\dots,\varphi_{1,n})$.

Hence $G_\lambda(\varphi_{1,1},\dots,\varphi_{1,n})=F_\lambda(\varphi_{1,1},\dots,\varphi_{1,n})-\mu_1(\lambda)R(\varphi_{1,1},\dots,\varphi_{1,n})=0$.

Put $\bar{\varphi}_i=\dfrac{\varphi_i}{[R(\varphi_1,\dots,\varphi_n)]^{\frac{1}{p_i}}}$ for every $(\varphi_1,\dots,\varphi_n)\in L^*(\Omega)$ and $1\le i\le n$. Then $R(\bar{\varphi}_1,\dots,\bar{\varphi}_n)=1$. We deduce that

$$\mu_1(\lambda)\le F_\lambda(\bar{\varphi}_1,\dots,\bar{\varphi}_n)=\frac{F_\lambda(\varphi_1,\dots,\varphi_n)}{R(\varphi_1,\dots,\varphi_n)}.\tag{3.6}$$

and

$$G_\lambda(\varphi_1,\dots,\varphi_n)=F_\lambda(\varphi_1,\dots,\varphi_n)-\mu_1(\lambda)R(\varphi_1,\dots,\varphi_n)\ge 0\tag{3.7}$$

for all $(\varphi_1,\dots,\varphi_n)\in L^*(\Omega)$. We claim that (3.5) holds.

Now suppose that (3.5) holds. We deduce that $\nabla G_\lambda(\varphi_{1,1}, \ldots, \varphi_{1,n}) = (0, \ldots, 0)$. Then

$$\langle \frac{\partial G_\lambda}{\partial \varphi_i}(\varphi_{1,1}, \ldots, \varphi_{1,n}), \Psi_i \rangle = 0, \text{ for } 1 \leq i \leq n, \tag{3.8}$$

for all $(\Psi_1, \ldots, \Psi_n) \in \prod_{i=1}^{n} L^{p_i}(\Omega)$. Hence, $((\varphi_{1,1}, \ldots, \varphi_{1,n}), \mu_1(\lambda)) \in L(\Omega)$ is a solution of (Q'_λ). ■

Lemma 3.5. *If (H_m) holds and $((\varphi_{1,1}, \ldots, \varphi_{1,n}), \mu_1(\lambda)) \in L_0(\Omega)$ is a solution of problem (Q'_λ) then $((|\varphi_{1,1}|, \ldots, |\varphi_{1,n}|), \mu_1(\lambda)) \in L_0(\Omega)$ is a solution of problem (Q'_λ).*

Proof. Assume that (H_m) holds and $((\varphi_{1,1}, \ldots, \varphi_{1,n}), \mu_1(\lambda)) \in L_0(\Omega)$ is a solution of problem (Q'_λ). Then $G_\lambda(\varphi_{1,1}, \ldots, \varphi_{1,n}) = 0$, $\mu_1(\lambda) = 0$ and $(|\varphi_{1,1}|, \ldots, |\varphi_{1,n}|) \in \left[\prod_{i=1}^{n} L^{p_i}(\Omega)\right] \setminus \{(0, \ldots, 0)\}$. Hence $G_\lambda(|\varphi_{1,1}|, \ldots, |\varphi_{1,n}|) \geq 0$.

Additionally, one has $|\Lambda(|\varphi_i|)|^r \geq |\Lambda\varphi_i|^r$ for $1 \leq i \leq n$ and for all $r \in (1; \infty)$. We deduce that $F_\lambda(|\varphi_{1,1}|, \ldots, |\varphi_{1,n}|) \leq F_\lambda(|\varphi_{1,1}|, \ldots, |\varphi_{1,n}|)$ and $G_\lambda(|\varphi_{1,1}|, \ldots, |\varphi_{1,n}|) \leq G_\lambda(|\varphi_{1,1}|, \ldots, |\varphi_{1,n}|) = 0$.

Thus $G_\lambda(|\varphi_{1,1}|, \ldots, |\varphi_{1,n}|) = 0$ and $((|\varphi_{1,1}|, \ldots, |\varphi_{1,n}|), \mu_1(\lambda))$ is solution of (Q'_λ). ■

Lemma 3.6. *[17].*

Let A, B, C and r be real numbers satisfying $A \geq 0$, $B \geq 0$, $C \geq \max\{B - A, 0\}$ and $r \in [1, +\infty)$. Then

$$|A + C|^r + |B - C|^r \geq A^r + B^r.$$

Lemma 3.7. *Let a_i and b_i be real numbers and $I_n = \{1, 2, \ldots, n\}$, then*

$$\prod_{i=1}^{n}(a_i + b_i) = \sum_{J \subset I_n} \left(\prod_{i \in J} a_i\right) \left(\prod_{i \in I_n \setminus J} b_i\right).$$

Proof. Straightforward by recurrence on n. ■

Lemma 3.8. *Suppose that (H_m) holds.*

If $(\varphi_{1,1}, \ldots, \varphi_{1,n})$ and $(\varphi_{2,1}, \ldots, \varphi_{2,n})$ are positive eigenfunctions of problem (Q'_λ) associated with $\mu_1(\lambda) = 0$, then $(w_{k,1}, \ldots, w_{l,s}, \ldots, w_{j,n})$ with:

$$\begin{cases} w_{1,i}(x) := \max\{\varphi_{1,i}(x), \varphi_{2,i}(x)\} = \varphi_{1,i}(x) + (\varphi_{2,i} - \varphi_{1,i})^+(x), \\ w_{2,i}(x) := \min\{\varphi_{1,i}(x), \varphi_{2,i}(x)\} = \varphi_{2,i}(x) - (\varphi_{2,i} - \varphi_{1,i})^+(x), \end{cases}$$

for all $x \in \Omega$, k, l, $j \in \{1, 2\}$, $s \in \{2, \ldots, n-1\}$ and $i \in \{1, \ldots, n\}$, are eigenfunctions of (Q'_λ) associated with $\mu_1(\lambda) = 0$.

Proof. Assume that (H_m) holds and $(\varphi_{1,1}, \ldots, \varphi_{1,n})$ and $(\varphi_{2,1}, \ldots, \varphi_{2,n})$ are positive eigenfunctions of problem (Q'_λ) associated with $\mu_1(\lambda) = 0$. By Lemma 3.6 we get

$$\begin{cases} |\Lambda w_{1,i}|^{p_i} + |\Lambda w_{2,i}|^{p_i} \geq |\Lambda\varphi_{1,i}|^{p_i} + |\Lambda\varphi_{2,i}|^{p_i} \\ |\Lambda w_{1,i}|^{\alpha_i+1} + |\Lambda w_{2,i}|^{\alpha_i+1} \geq |\Lambda\varphi_{1,i}|^{\alpha_i+1} + |\Lambda\varphi_{2,i}|^{\alpha_i+1}. \end{cases}.$$

Then, one has:

$$-\lambda \int_\Omega m_i|\Lambda w_{1,i}|^{p_i} dx - \lambda \int_\Omega m_i|\Lambda w_{2,i}|^{p_i} dx \leq -\lambda \int_\Omega m_i|\Lambda\varphi_{1,i}|^{p_i} dx - \lambda \int_\Omega m_i|\Lambda\varphi_{2,i}|^{p_i} dx. \tag{3.9}$$

Likewise, we have

$$\begin{aligned} Z_1(w_1,\ldots,w_i,\ldots,w_n) &\le Z_1(\varphi_1,\ldots,\varphi_i,\ldots,\varphi_n) \\ &\le -\int_\Omega m\prod_{i=1}^n|\Lambda\varphi_{1,i}|^{\alpha_i+1}dx-\int_\Omega m\prod_{i=1}^n|\Lambda\varphi_{2,i}|^{\alpha_i+1}dx \end{aligned} \tag{3.10}$$

with

$$Z_1(w_1,\ldots,w_i,\ldots,w_n)=-\sum_{J\subset I_n}\int_\Omega m\left(\prod_{i\in J}|\Lambda w_{1,i}|^{\alpha_i+1}\right)\left(\prod_{i\in I_n\setminus J}|\Lambda w_{2,i}|^{\alpha_i+1}\right)dx$$

and

$$Z_1(\varphi_1,\ldots,\varphi_i,\ldots,\varphi_n)=-\sum_{J\subset I_n}\int_\Omega m\left(\prod_{i\in J}|\Lambda\varphi_{1,i}|^{\alpha_i+1}\right)\left(\prod_{i\in I_n\setminus J}|\Lambda\varphi_{2,i}|^{\alpha_i+1}\right)dx.$$

Additionally, we have:

$$\int_\Omega|w_{1,i}|^{p_i}dx+\int_\Omega|w_{2,i}|^{p_i}dx=\int_\Omega|\varphi_{1,i}|^{p_i}dx+\int_\Omega|\varphi_{2,i}|^{p_i}dx. \tag{3.11}$$

By (3.9), (3.10) and (3.11) we deduce that:

$$\sum_{i\in J\subset I_n\setminus\{1,n\},\,\mathrm{k,\,l,\,j}\in\{1,2\}}F_\lambda(w_{k,1},\ldots,w_{l,i},\ldots,w_{j,n})\le F_\lambda(\varphi_{1,1},\ldots,\varphi_{1,n})+F_\lambda(\varphi_{2,1},\ldots,\varphi_{2,n})$$

and

$$\sum_{i\in J\subset I_n\setminus\{1,n\},\,\mathrm{k,\,l,\,j}\in\{1,2\}}G_\lambda(w_{k,1},\ldots,w_{l,i},\ldots,w_{j,n})\le G_\lambda(\varphi_{1,1},\ldots,\varphi_{1,n})+G_\lambda(\varphi_{2,1},\ldots,\varphi_{2,n})=0.$$

It follows that

$$G_\lambda(w_{k,1},\ldots,w_{l,i},\ldots,w_{j,n})=0,\ \text{with } i\in J\subset I_n\setminus\{1,n\} \text{ and } k,\,l,\,j\in\{1,2\}.$$

Hence $(w_{k,1},\ldots,w_{l,s},w_{j,n})$ with $s\in\{2,\ldots,n-1\}$ and $k,\,l,\,j\in\{1,2\}$, are eigenfunctions of $(Q_\lambda^{'})$ associated with $\mu_1(\lambda)=0$. ■

We are now in position to summarize the main existence result of this section in the following, which generalizes and extends result of Theorem 3.1 in [17].

Theorem 3.9. *Assume that (H_m) is satisfied. We have the following results:*

1. *If $\mu_1(\lambda)=0$ then λ is a semitrivial principal eigenvalue or strictly principal eigenvalue of problem (Q) and simple.*
2. *The lowest eigenvalue of problem (Q) is the value*

$$\lambda_1:=\min_{(u_1,\ldots,u_n)\in\mathcal{S}}E_m(u_1,\ldots,u_n). \tag{3.12}$$

where

$$\mathcal{S}=\{(u_1,\ldots,u_n)\in W:\quad M(u_1,\ldots,u_n)=1\}.$$

Moreover, λ_1 is unique, positive, strictly principal eigenvalue or strictly principal eigenvalue and simple.

Proof. Assume that (H_m) is satisfied.

1. If $\mu_1(\lambda) = 0$ then λ is an eigenvalue of the problem (Q) associated with $(u_1, \dots, u_n) \in W \setminus \{(0, \dots, 0)\}$.

• If $u_i \not\equiv 0$, for $1 \le i \le n$, we deduce that

$$((\varphi_1, \dots, \varphi_n), \mu_1(\lambda)) \in L_0(\Omega) \quad \text{and} \quad ((|\varphi_1|, \dots, |\varphi_n|), \mu_1(\lambda)) \in L_0(\Omega)$$

are solution of problem (Q'_λ) with $\varphi_i = -\Delta u_i \not\equiv 0$, for $1 \le i \le n$. Since $|\varphi_i| \ge 0$, then $\Lambda(|\varphi_i|) > 0$, for $1 \le i \le n$. Therefore $N_{p_i}(\Lambda|\varphi_i|) > 0$ for $1 \le i \le n$,

$$\prod_{j=1, i\neq j}^{n} \left(\Lambda(|\varphi_j|)|^{\alpha_i+1} |\Lambda(|\varphi_i|)|^{\alpha_i}\right) > 0$$

and

$$\begin{cases} |\varphi_i| = N_{p'_i}\left(\Lambda\left[\lambda m_i N_{p_i}(\Lambda|\varphi_i|) + m \displaystyle\prod_{j=1, i\neq j}^{n} \left(\Lambda(|\varphi_j|)|^{\alpha_i+1} |\Lambda(|\varphi_i|)|^{\alpha_i}\right)\right]\right) > 0 \\ \text{for } 1 \le i \le n. \end{cases}$$

We then conclude that $((\varphi_1, \dots, \varphi_n), \mu_1(\lambda))$ is solution of problem (Q'_λ) with $\varphi_i > 0$ or $\varphi_i < 0$, for $1 \le i \le n$. Since by Lemma 3.3, $\varphi_i \in C(\bar{\Omega})$, we have $u_i = \Lambda\varphi_i > 0$ or $u_i = \Lambda\varphi_i < 0$, for $1 \le i \le n$, from Lemma 1.1. Then λ is a strictly principal eigenvalue of (Q).

• If $\exists i, j \in \{1, \dots, n\}$ such that $[u_i \equiv 0$ and $u_j \not\equiv 0]$, then we also prove that $[u_i \equiv 0$ and $u_j > 0$ in Ω or $u_j < 0$ in $\Omega]$. Then λ is a semitrivial principal eigenvalue of (Q).

It is now left with the simplicity and we argue by cases:

Case (1) λ is a strictly principal eigenvalue of (Q).

Let $(u_{1,1}, \dots, u_{1,n})$ and $(u_{2,1}, \dots, u_{2,n})$ be two eigenfunctions of (Q) associated with λ.

Then, $((\varphi_{1,1}, \dots, \varphi_{1,n}), 0)$, $((\varphi_{2,1}, \dots, \varphi_{2,n}), 0)$, $((|\varphi_{1,1}|, \dots, |\varphi_{1,n}|), 0)$, $((|\varphi_{2,1}|, \dots, |\varphi_{2,n}|), 0) \in L_0(\Omega)$, are solutions of (Q'_λ) where $\varphi_{j,i} = -\Delta u_{j,i}$ with $\varphi_{j,i} > 0$ or $\varphi_{j,i} < 0$, for $j \in \{1, 2\}$ and $i \in \{1, \dots, n\}$.

For $x_0 \in \Omega$, we set $k_i = \dfrac{\varphi_{2,i}(x_0)}{\varphi_{1,i}(x_0)}$, $w_{1,i}(x) = \max\{\varphi_{2,i}(x), k_i\varphi_{1,i}(x)\}$ for all $x \in \Omega$. From Lemma 3.8, $((w_{1,1}, \dots, w_{1,n}), 0)$ is a solution of problem (Q'_λ) . We deduce that $N_{p_i}(\varphi_{1,i})$, $N_{p_i}(\varphi_{2,i})$, $N_{p_i}(w_{1,i}) \in C^{1,\nu}(\bar{\Omega})$ and $\dfrac{N_{p_i}(\varphi_{2,i})}{N_{p_i}(\varphi_{1,i})} \in C^1(\Omega)$.

For any unit vector $e = (0, \dots, e_j, \dots, 0)$ with $j \in \{1, \dots, N\}$ and $t \in \mathbb{R}$, we have

$$\begin{cases} N_{p_i}(\varphi_{2,i})(x_0 + te) - N_{p_i}(\varphi_{2,i})(x_0) \le N_{p_i}(w_{1,i})(x_0 + te) - N_{p_i}(w_{1,i})(x_0) \\ N_{p_i}(k\varphi_{1,i})(x_0 + te) - N_{p_i}(k\varphi_{1,i})(x_0) \le N_{p_i}(w_{1,i})(x_0 + te) - N_{p_i}(w_{1,i})(x_0) \end{cases}$$

Dividing these inequalities by $t > 0$ and $t < 0$ and letting t tend to $0^{\pm}$, we get

$$\begin{cases} \dfrac{\partial}{\partial x_j}[N_{p_i}(\varphi_{2,i})](x_0) \le \dfrac{\partial}{\partial x_j}[N_{p_i}(w_{1,i})](x_0) \\ \dfrac{\partial}{\partial x_j}[N_{p_i}(k\varphi_{1,i})](x_0) \le \dfrac{\partial}{\partial x_j}[N_{p_i}(w_{1,i})](x_0) \end{cases}$$

and

$$\begin{cases} \dfrac{\partial}{\partial x_j}[N_{p_i}(\varphi_{2,i})](x_0) \geq \dfrac{\partial}{\partial x_j}[N_{p_i}(w_{1,i})](x_0) \\ \dfrac{\partial}{\partial x_j}[N_{p_i}(k\varphi_{1,i})](x_0) \geq \dfrac{\partial}{\partial x_j}[N_{p_i}(w_{1,i})](x_0) \end{cases}$$

for all $j \in \{1, \ldots, N\}$. Thus,

$$\begin{cases} \dfrac{\partial}{\partial x_j}[N_{p_i}(\varphi_{2,i})](x_0) = \dfrac{\partial}{\partial x_j}[N_{p_i}(w_{1,i})](x_0) \\ \dfrac{\partial}{\partial x_j}[N_{p_i}(k\varphi_{1,i})](x_0) = \dfrac{\partial}{\partial x_j}[N_{p_i}(w_{1,i})](x_0) \end{cases}$$

for all $j \in \{1, \ldots, N\}$. Hence,

$$\nabla N_{p_i}(\varphi_{2,i})(x_0) = \nabla N_{p_i}(w_{1,i})(x_0) = \nabla N_{p_i}(k\varphi_{1,i})(x_0) = k^{p_i-1}\nabla N_{p_i}(\varphi_{1,i})(x_0).$$

Furthermore

$$\begin{aligned} \nabla\left(\frac{N_{p_i}(\varphi_{2,i})}{N_{p_i}(\varphi_{1,i})}\right)(x_0) &= \frac{\nabla(N_{p_i}(\varphi_{2,i}))(x_0)N_{p_i}(\varphi_{1,2})(x_0) - N_{p_i}(\varphi_{2,i})(x_0)\nabla(N_{p_i}(\varphi_{2,i}))(x_0)}{[N_{p_i}(\varphi_{1,i})(x_0)]^2} \\ &= \frac{[N_{p_i}(\varphi_{1,2})(x_0) - k_i^{1-p_i}N_{p_i}(\varphi_{2,i})(x_0)]\nabla(N_{p_i}(\varphi_{2,1}))(x_0)}{[N_{p_i}(\varphi_{1,i})(x_0)]^2} = 0. \end{aligned}$$

Then,

$$\frac{N_{p_i}(\varphi_{2,i})}{N_{p_i}(\varphi_{1,i})}(x) = const = \frac{N_{p_i}(\varphi_{2,i})}{N_{p_i}(\varphi_{1,i})}(x_0) = \left(\frac{\varphi_{2,i}(x_0)}{\varphi_{1,i}(x_0)}\right)^{p_i-1} = k_i^{p_i-1},$$

for all $x \in \Omega$. Consequently $\varphi_{2,i} = k_i\varphi_{1,i}$.

Accordingly, $(\varphi_{2,1}, \ldots, \varphi_{2,n}) = (k_1\varphi_{1,1}, \ldots, k_n\varphi_{1,n})$.

We deduce that $(u_{2,1}, \ldots, u_{2,n}) = (k_1u_{1,1}, \ldots, k_nu_{1,n})$ and the result follows.

Case (2) λ is a semitrivial principal eigenvalue of (Q).

Let $(\cdots, u_{1i}, \cdots)$ and $(\cdots, u_{2i}, \cdots)$ be two eigenfunctions of (Q) associated with λ (with $u_{1i} \not\equiv 0$, $u_{2i} \not\equiv 0$ and $i \in \{1, \cdots, n\}$). It is easy to see that there exist $k_i \neq 0$ real numbers such that $u_{1i} = k_iu_{2i}$.

2. By Lemma 3.1, $\mu_1(0) > 0$ and $\mu_1(\lambda) = 0$ if and only if $\lambda > 0$ is an eigenvalue of (Q).

   From Proposition 2.5, there exists a unique real $\lambda_1 \in (0, \infty)$ satisfying $\mu_1(\lambda_1) = 0$ and $\mu_1'(\lambda_1) = -M(u_{1,0}, \ldots, u_{n,0}) < 0$. On the other hand, $0 = \mu_1(\lambda_1) = E_m(u_{1,0}, \ldots, u_{n,0}) - \lambda_1M(u_{1,0}, \ldots, u_{n,0})$ with $(u_{1,0}, \ldots, u_{n,0}) \in \mathcal{M}$. Then,

$$E_m(u_{1,0}, \ldots, u_{n,0}) = \lambda_1M(u_{1,0}, \ldots, u_{n,0}) > 0$$

   and we can set

$$\overline{u}_{i,0} = \frac{u_{i,0}}{[M(u_{1,0}, \ldots, u_{n,0})]^{\frac{1}{p_i}}}.$$

   Thus, $(\overline{u}_{1,0}, \ldots, \overline{u}_{n,0}) \in \mathcal{S}$ and $E_m(\overline{u}_{1,0}, \ldots, \overline{u}_{n,0}) = \lambda_1$. Additionally, for every $(u_1, \ldots, u_n) \in \mathcal{S}$, one has

$$E_m\left(\frac{u_1}{[I(u_1, \ldots, u_n)]^{\frac{1}{p_1}}}, \ldots, \frac{u_n}{[I(u_1, \ldots, u_n)]^{\frac{1}{p_n}}}\right) \geq \lambda_1M\left(\frac{u_1}{[I(u_1, \ldots, u_n)]^{\frac{1}{p_1}}}, \ldots, \frac{u_n}{[I(u_1, \ldots, u_n)]^{\frac{1}{p_n}}}\right)$$

   i.e. $E_m(u_1, \ldots, u_n) \geq \lambda_1$. Consequently (3.12) holds. Moreover, from what has been previously proved, λ_1 is a strictly principal eigenvalue or strictly principal eigenvalue and simple.

■

References

[1] R. A. ADAMS, Sobolev Spaces, *Academic Press, New York*, (1975).

[2] G. BARLETTA AND R. LIVREA Infinitely many solutions for a class of differential inclusions involving the p-biharmonic, *Differ. Integral Equ.*, **26** (2013), 1157–1167.

[3] J. BENEDIKT, Uniqueness theorem for p-biharmonic equations, *Electron. J. Differential Equations*, **53** (2002), 1-17.

[4] J. BENEDIKT, On the discretness of the spectra of the Dirichlet and Neumann p-biharmonic problem, *Abstr. Appl. Anal.*, **293** (2004), 777-792.

[5] P. A. BINDING AND Y. X. HUANG, The eigencurve for the p-laplacian, *Differential Integral Equations*, **8(2)**(1995), 405-414.

[6] G. BONANNO AND B. DI BELLA, A boundary value problem for fourth-order elastic beam equations, *J. Math. Anal. Appl.*, **343** (2008), 1166–1176.

[7] M. CUESTA AND L. LEADI, Weighted eigenvalue problems for quasilinear elliptic operator with mixed Robin-Dirichlet boundary conditions, *J. Math. Anal. Appl.*, **422** (2015), 1-26.

[8] M. CUESTA AND HUMBERTO RAMOS QUOIRIN, A weighted eigenvalue problem for the p-laplacian plus a potential, *NoDEA Nonlinear Equations Appl.*, **16(4)** (2009), 469-491.

[9] P. DRÁBEK AND M. ÔTANI, Global bifurcation result for the p-biharmonic operator, *Electron. J. Differential Equations*, **48** (2001), 1-19.

[10] A. EL KHALIL, S. KELLATI AND A. TOUZANI, On the spectrum of the p-biharmonic operator In: 2002-Fez Conference on Partial Differential Equations, *Electron. J. Differential Equations*, **09** (2002), 161-170.

[11] D. GILBARG AND N. S. TRUDINGER, Elliptic Partial Differential Equations of Second Order, *2nd ed., Springer, New York*, (1983).

[12] J. R. GRAEF AND S. HEIDARKHANI, Multiple solutions for a class of $(p_1, \ldots, p_n)$-biharmonic systems, *Communications on Pure and Applied Analysis*, **12(3)**(2013), doi:10.3934/cpaa.2013.12.1393.

[13] EL MILOUD HSSINI, MOHAMMED MASSAR AND NAJIB TSOULI, Infinitely many solutions for the Navier boundary (p,q)-biharmonic systems, *Electron. J. Differential Equations*, **2012(163)** (2012), 1-9.

[14] A. C. LAZER AND P. J. MCKENNA, Large amplitude periodic oscillations in suspension bridges: Some new connections with nonlinear analysis, *SIAM Rev.*, **32** (1990), 537–578.

[15] LIAMIDI LEADI AND HUMBERTO RAMOS QUOIRIN, Principal eigenvalue for quasilinear cooperative elliptic systems, *Differential and integral equations*, **24(11-12)**(2011), 1107–1124.

[16] L. LEADI AND A. MARCOS, A weighted eigencurve for Steklov problems with a potential, *NoDEA Nonlinear Differential Equations Appl.*, **16** (2013), 687-713.

[17] L. LEADI AND R. L. TOYOU, Principal Eigenvalue for Cooperative (p, q)-biharmonic Systems, *J. Partial Diff. Eq.*, **32** (2019), 33-51.

[18] LIN LI AND YU FU, Existence of Three Solutions for $(p_1, \ldots, p_n)$-biharmonic Systems, *International Journal of Nonlinear Sciences*, **10(4)**(2010), 495-506.

[19] M. TALBI AND N. TSOULI, On the spectrum of the weighted p-biharmonic operator with weight, *Mediterr. J. Math.*, **4** (2007), 73-86.

[20] WILLEM, Minimax Theorems, *Birkhäuser, Boston*, (1996).