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Interval-valued intuitionistic fuzzy k-ideal in semi-rings

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Abstract

The concept of interval-valued intuitionistic fuzzy set was introduced by Atanassov and Gargov [2] in the year 1989.In this paper the Cartesian product of interval-valued intuitionistic fuzzy k-ideals is introduced. Also some basic properties are derived. The relationship between interval-valued intuitionistic fuzzy k-ideal A,B and $A\times B$ are proposed. Some theorems related to the above concepts are stated and proved.

Keywords

Semi-ring, Interval-valued Intuitionistic Fuzzy k-ideal.

AMS Subject Classification

08A72, 06D72.

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Contents

1. Introduction

The notion of fuzzy was introduced by [13] in 1965. Atanassov [1] introduced the concept of intuitionistic fuzzy sets in 1986. Atanassov et al.[2] introduced the concept of interval-valued intuitionistic fuzzy sets which is a generalization of both intuitionistic fuzzy sets and interval-valued fuzzy sets. Several mathematicians applied the concept of interval-valued intuitionistic fuzzy sets to algebraic structures. Biswas [5] studied on Rosenfeld's fuzzy subgroups with interval-valued membership function. Das and Dutta [6] developed the concept of extension of fuzzy ideals in semirings. Dutta and Biswas [7–10] introduced and studied some properties of fuzzy prime, fuzzy semi-prime, fuzzy completely prime ideals in semiring. Balasubramanian and Raja [3, 4] introduced intuitionistic fuzzy k-ideal and interval-valued intuitionistic fuzzy ideal on semi-rings. In this paper, the cartisian product of interval-valued intuitionistic fuzzy k-ideal in semi-rings are studied. Investigate the relationship between A,B and $A\times B$.

2. Preliminaries

In this section, we recall some definitions and basic results of interval-valued intuitionistic fuzzy ideal.

Definition 2.1. *A non-empty set S together with two binary operation $+$ and . is said to be a semi-ring. if*

- $(S,+)$ *is a commutative semigroup,*
- $(S,.)$ *is a semigroup,*
- $a(b+c)=ab+ac$ *and* $(a+b)c=ac+bc\ \forall a,b,c\in S$.

Let $(S,+,.)$ be a semi-ring. If there exists an element $0_s\in$ Ssuch that $a+0_s=a=0_s+a$ and $a.0_s=0_s=0_s.a$ for all $a\in S$; then 0_s is called the zero element of S. If there exists an element $1_s\in S$ such that $a.1_s=a=1_s.a$ for all $a\in S$, then 1_s is called the identity element of S.

Note 2.2. *A semiring may or may not have a zero and an identity element. We say that a semiring S has a zero.if there exists an element $0\in S$ such that $0x=x0=0$ and $0+x=x+0=x$ for all $x\in S$.*

Definition 2.3. *An interval number on $[0,1]$, denoted by $\tilde{A}_M$, is defined as the closed subinterval of $[0,1]$, where $\tilde{A}=[A^-,A^+]$ satisfying $0\le A^-\le A^+\le 1$.*
For any two interval numbers $\tilde{A}=[A^-,A^+]$ and $\tilde{B}=[B^-,B^+]$,we define:

- $\tilde{A} \leq \tilde{B}$ *if and only if* $A^- \leq B^-$ *and* $A^+ \leq B^+$
- $\tilde{A} = \tilde{B}$ *if and only if* $A^- = B^-$ *and* $A^+ = B^+$
- $\tilde{A} \leq \tilde{B}$ *if and only if* $A^- \leq B^-$ *and* $A^+ \leq B^+$
- $\tilde{A} < \tilde{B}$ *if and only if* $\tilde{A} \neq \tilde{B}$ *and* $\tilde{A} \leq \tilde{B}$

Definition 2.4. *The interval* min*-norm is a function* $\min^i : D[0,1] \times D[0,1] \to D[0,1]$ *defined by* $\min^i(\tilde{A}, \tilde{B}) = [\min(A^-, B^-), \min(A^+, B^+)]$ *for all* $\tilde{A}, \tilde{B} \in D[0,1]$, *where* $\tilde{A} = [A^-, A^+]$ *and* $\tilde{B} = [B^-, B^+]$.

Definition 2.5. *The interval* max*-norm is a function* $\max^i : D[0,1] \times D[0,1] \to D[0,1]$ *defined by* $\max^i(\tilde{A}, \tilde{B}) = [\max(A^-, B^-), \max(A^+, B^+)]$ *for all* $\tilde{A}, \tilde{B} \in D[0,1]$, *where* $\tilde{A} = [A^-, A^+]$ *and* $\tilde{B} = [B^-, B^+]$.

Definition 2.6. *An interval-valued intuitionistic fuzzy set A in a semiring S is called an intuitionistic fuzzy left ideal of S if it satisfies*
$\tilde{M}_A(x+y) \geq \min^i\{\tilde{M}_A(x), \tilde{M}_A(y)\}$
$\tilde{N}_A(x+y) \leq \max^i\{\tilde{N}_A(x), \tilde{N}_A(y)\}$ $\forall x, y \in S$ *and*
$\tilde{M}_A(x+y) \geq \tilde{M}_A(y)$,
$\tilde{N}_A(x+y) \leq \tilde{N}_A(y)$, $\forall x, y \in S$.

Definition 2.7. *If A is an interval-valued intuitionistic fuzzy set in a set S, the strongest interval-valued intuitionistic fuzzy relation on S that is an interval-valued intuitionistic fuzzy relation on A is* $A_s = (\tilde{M}_{A_s}, \tilde{N}_{A_s})$, *given by*
$\tilde{M}_{A_s}(x,y) = \min^i\{\tilde{M}_A(x), \tilde{M}_A(y)\}$ *and*
$\tilde{N}_{A_s}(x,y) = \max^i\{\tilde{N}_A(x), \tilde{N}_A(y)\}$,
$\forall x, y \in S$.

Definition 2.8. *A non-empty interval-valued intuitionistic fuzzy subset* $A = (\tilde{M}_A, \tilde{N}_A)$ *of a semi-group S is called an interval-valued intuitionistic fuzzy left(right) ideal of S if*

1. $\tilde{M}_A(xy) \geq \tilde{M}_A(y)(resp. \tilde{M}_A(xy) \geq \tilde{M}_A(x)), \forall x, y \in S$,
2. $\tilde{N}_A(xy) \leq \tilde{N}_A(y)(resp. \tilde{N}_A(xy) \leq \tilde{N}_A(x)), \forall x, y \in S$

Definition 2.9. *A non-empty interval-valued intuitionistic fuzzy subset* $A = (\tilde{M}_A, \tilde{N}_A)$ *of a semi-group S is called an interval-valued intuitionistic fuzzy two-sided ideal or an interval-valued intuitionistic fuzzy ideal of S if it is both an interval-valued intuitionistic fuzzy left and an interval-valued intuitionistic fuzzy right ideal of S.*

Definition 2.10. *An interval-valued intuitionistic fuzzy left ideal A of a semiring S is called an interval-valued intuitionistic fuzzy left k-ideal of S if for any* $x, y, z \in S, x+y = z$ *implies* $\tilde{M}_A(x) \geq \min\{\tilde{M}_A(y), \tilde{M}_A(z)\}$ *and* $\tilde{N}_A(x) \leq \max\{\tilde{N}_A(y), \tilde{N}_A(z)\}$

3. Main Results

Proposition 3.1. *For a given interval-valued intuitionistic fuzzy set A in a semiring S with the zero element,* A_s *can be the strongest interval-valued intuitionistic fuzzy relation on S. If* A_s *is an interval-valued intuitionistic fuzzy left k-ideal of* $S \times S$, *then* $\tilde{M}_A(a) \leq \tilde{M}_A(0); \tilde{N}_A(a) \geq \tilde{N}_A(0)$ *for all* $a \in S$.

Proof. If A_s is an interval-valued intuitionistic fuzzy left $k-$ideal of $S \times S$, then $\tilde{M}_{A_s}(a,a) \leq \tilde{M}_{A_s}(0,0); \tilde{N}_{A_s}(a,a) \geq \tilde{N}_{A_s}(0,0)$ for all $a \in S$.
$\Longrightarrow$ $\min^i\{\tilde{M}_A(a), \tilde{M}_A(a)\} \leq \min^i\{\tilde{M}_A(0), \tilde{M}_A(0)\}$;
and $\max^i\{\tilde{N}_A(a), \tilde{N}_A(a)\} \geq \max^i\{\tilde{N}_A(0), \tilde{N}_A(0)\}$,
which implies that $\tilde{M}(a) \leq \tilde{M}(0); \tilde{N}(a) \geq \tilde{N}(0)$. □

Theorem 3.2. *Let* $A = (\tilde{M}_A, \tilde{N}_A)$ *and* $B = (\tilde{M}_B, \tilde{N}_B)$ *be interval-valued intuitionistic fuzzy left k-ideals of a semiring S. Then* $A \times B$ *is an interval-valued intuitionistic fuzzy left k-ideal of* $S \times S$.

Proof. Let $(x_1, x_2), (y_1, y_2) \in S \times S$. Then

$$\begin{aligned}
&(\tilde{M}_A \times \tilde{M}_B)((x_1,x_2)+(y_1,y_2)) \\
&= (\tilde{M}_A \times \tilde{M}_B)(x_1+y_1, x_2+y_2) \\
&= \overset{i}{\min}\{\tilde{M}_A(x_1+y_1), \tilde{M}_B(x_2+y_2)\} \\
&\geq \overset{i}{\min}\{\overset{i}{\min}\{\tilde{M}_A(x_1), \tilde{M}_A(y_1)\}, \\
&\overset{i}{\min}\{\tilde{M}_B(x_2), \tilde{M}_B(y_2)\}\} \\
&= \overset{i}{\min}\{\overset{i}{\min}\{\tilde{M}_A(x_1), \tilde{M}_B(x_2)\}, \\
&\overset{i}{\min}\{\tilde{M}_A(y_1), \tilde{M}_B(y_2)\}\} \\
&= \overset{i}{\min}\{(\tilde{M}_A \times \tilde{M}_B)(x_1,x_2), \\
&(\tilde{M}_A \times \tilde{M}_B)(y_1,y_2)\}
\end{aligned} \tag{3.1}$$

Similarly,

$$\begin{aligned}
&(\tilde{N}_A \times \tilde{N}_B)((x_1,x_2)+(y_1,y_2)) \\
&= (\tilde{N}_A \times \tilde{N}_B)(x_1+y_1, x_2+y_2) \\
&= \overset{i}{\max}\{\tilde{N}_A(x_1+y_1), \tilde{N}_B(x_2+y_2)\} \\
&\leq \overset{i}{\max}\{\overset{i}{\max}\{\tilde{N}_A(x_1), \tilde{N}_A(y_1)\}, \\
&\overset{i}{\max}\{\tilde{N}_B(x_2), \tilde{N}_B(y_2)\}\} \\
&= \overset{i}{\max}\{\overset{i}{\max}\{\tilde{N}_A(x_1), \tilde{N}_B(x_2)\}, \\
&\overset{i}{\max}\{\tilde{N}_A(y_1), \tilde{N}_B(y_2)\}\} \\
&= \overset{i}{\max}\{(\tilde{N}_A \times \tilde{N}_B)(x_1,x_2), \\
&(\tilde{N}_A \times \tilde{N}_B)(y_1,y_2)\}
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&(\tilde{M}_A \times \tilde{M}_B)((x_1,x_2)(y_1,y_2)) \\
&= (\tilde{M}_A \times \tilde{M}_B)(x_1y_1, x_2y_2) \\
&= \overset{i}{\min}\{\tilde{M}_A(x_1y_1), \tilde{M}_B(x_2y_2)\} \\
&\geq \overset{i}{\min}\tilde{M}_A(y_1), \tilde{M}_B(y_2) \\
&= (\tilde{M}_A \times \tilde{M}_B)(y_1,y_2)
\end{aligned} \tag{3.3}$$

Similarly,

$$\begin{aligned}&(\tilde{N}_A\times\tilde{N}_B)((x_1,x_2)(y_1,y_2))\\&=(\tilde{N}_A\times\tilde{N}_B)(x_1y_1,x_2y_2)\\&=\max^i\{\tilde{N}_A(x_1y_1),\tilde{N}_B(x_2y_2)\}\\&\leq\max^i\{\tilde{N}_A(y_1),\tilde{N}_B(y_2)\}\\&=(\tilde{N}_A\times\tilde{N}_B)(y_1,y_2)\end{aligned}\tag{3.4}$$

Hence $A\times B$ is an interval-valued intuitionistic fuzzy left ideal of $S\times S$.
Now let $(a_1,a_2),(b_1,b_2),(x_1,x_2)\in S\times S$ be such that $(x_1,x_2)+(a_1,a_2)=(b_1,b_2)$
i.e., $(x_1+a_1,x_2+a_2)=(b_1,b_2)$.
It follows that $x_1+a_1=b_1$ and $x_2+a_2=b_2$
Therefore,

$$\begin{aligned}&(\tilde{M}_A\times\tilde{M}_B)(x_1,x_2)\\&=\min^i\{\tilde{M}_A(x_1),\tilde{M}_B(x_2)\}\\&\geq\min^i\{\min^i\{\tilde{M}_A(a_1),\tilde{M}_A(b_1)\},\\&\min^i\{\tilde{M}_B(a_2),\tilde{M}_B(b_2)\}\}\\&=\min^i\{\min^i\{\tilde{M}_A(a_1),\tilde{M}_B(a_2)\},\\&\min^i\{\tilde{M}_A(b_1),\tilde{M}_B(b_2)\}\}\\&=\min^i\{(\tilde{M}_A\times\tilde{M}_B)(a_1,a_2),\\&(\tilde{M}_A\times\tilde{M}_B)(b_1,b_2)\}\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}&(\tilde{N}_A\times\tilde{N}_B)(x_1,x_2)\\&=\max^i\{\tilde{N}_A(x_1),\tilde{N}_B(x_2)\}\\&\leq\max^i\{\max^i\{\tilde{N}_A(a_1),\tilde{N}_A(b_1)\},\\&\max^i\{\tilde{N}_B(a_2),\tilde{N}_B(b_2)\}\}\\&=\max^i\{\max^i\{\tilde{N}_A(a_1),\tilde{N}_B(a_2)\},\\&\max^i\{\tilde{N}_A(b_1),\tilde{N}_B(b_2)\}\}\\&=\max^i\{(\tilde{N}_A\times\tilde{N}_B)(a_1,a_2),\\&(\tilde{N}_A\times\tilde{N}_B)(b_1,b_2)\}\end{aligned}\tag{3.6}$$

Hence $A\times B$ is an interval-valued intuitionistic fuzzy left $k-$ideal of $S\times S$. □

Theorem 3.3. *Let $A=(\tilde{M}_A,\tilde{N}_A)$ and $B=(\tilde{M}_B,\tilde{N}_B)$ be two intuitionistic fuzzy sets in a semiring S with the zero element such that $A\times B$ is an interval-valued intuitionistic fuzzy left $k-$ideal of $S\times S$. Then*

1. Either $\tilde{M}_A(x)\leq\tilde{M}_A(0)$ and $\tilde{N}_A(x)\geq\tilde{N}_A(0)$ or $\tilde{M}_B(x)\leq\tilde{M}_B(0)$ and $\tilde{N}_B(x)\geq\tilde{N}_B(0)$ for all $x\in S$.
2. If $\tilde{M}_A(x)\leq\tilde{M}_A(0)$ and $\tilde{M}_B(x)\geq\tilde{M}_B(0)$ for all $x\in S$, then either $\tilde{M}_A(x)\leq\tilde{M}_B(0);\tilde{N}_A(x)\geq\tilde{N}_B(0)$ or $\tilde{M}_B(x)\leq\tilde{M}_B(0);\tilde{N}_B(x)\geq\tilde{N}_B(0)$.
3. If $\tilde{M}_B(x)\leq\tilde{M}_B(0);\tilde{N}_B(x)\geq\tilde{N}_B(0)$ for all $x\in S$, then either $\tilde{M}_A(x)\leq\tilde{M}_A(0);\tilde{N}_A(x)\leq\tilde{N}_A(0)$ or $\tilde{M}_B(x)\leq\tilde{M}_A(0);\tilde{N}_B(x)\geq\tilde{N}_A(0)$
4. If $\tilde{M}_B(x)\leq\tilde{M}_A(0);\tilde{N}_B(x)\geq\tilde{N}_A(0)$ for any $x\in S$, then B is an interval-valued intuitionistic fuzzy left k-ideal of S.

Proof. (i). Suppose that $\tilde{M}_A(x)>\tilde{M}_A(0);\tilde{N}_A(x)<\tilde{N}_A(0)$ and $\tilde{M}_B(x)>\tilde{M}_B(0);\tilde{N}_B(x)<\tilde{N}_B(0)$. Then
$(\tilde{M}_A\times\tilde{M}_B)(x,y)>\min^i\{\tilde{M}_A(x),\tilde{M}_B(y)\}=(\tilde{M}_A\times\tilde{M}_B)(0,0)$
$(\tilde{N}_A\times\tilde{N}_B)(x,y)<\max^i\{\tilde{N}_A(x),\tilde{N}_B(y)\}=(\tilde{N}_A\times\tilde{N}_B)(0,0)$
Which is a contradiction. Hence we obtain (i).
(ii). Let us assume that there exist $x,y\in S$ such that $\tilde{M}_A(x)>\tilde{M}_B(0);\tilde{N}_A(x)>\tilde{N}_B(0)$. Then $(\tilde{M}_A\times\tilde{M}_B)(0,0)=\min^i\{\tilde{M}_A(0),\tilde{M}_B(0)\}=\tilde{M}_B(0)$ and $(\tilde{N}_A\times\tilde{N}_B)(0,0)=\max^i\{\tilde{N}_A(0),\tilde{N}_B(0)\}=\tilde{N}_B(0)$
hence, $(\tilde{M}_A\times\tilde{M}_B)(x,y)=\min^i\{\tilde{M}_A(x),\tilde{M}_B(x)\}>\tilde{M}_B(0)=(\tilde{M}_A\times\tilde{M}_B)(0,0)$,
$(\tilde{N}_A\times\tilde{N}_B)(x,y)=\max^i\{\tilde{N}_A(x),\tilde{N}_B(x)\}<\tilde{N}_B(0)=(\tilde{N}_A\times\tilde{N}_B)(0,0)$
This is a contradiction. Hence (ii) holds.
Similarly we can prove (iii).
(iv). If $\tilde{M}_B(x)\leq\tilde{M}_A(0);\tilde{N}_B(x)\geq\tilde{N}_B(0)$ for any $x\in S$, then

$$\begin{aligned}\tilde{M}_B(x+y)&=\min^i\{\tilde{M}_A(0),\tilde{M}_B(x+y)\}\\&=(\tilde{M}_A\times\tilde{M}_B)(0,x+y)\\&=(\tilde{M}_A\times\tilde{M}_B)((0,x)+(0,y))\\&\geq\min^i\{(\tilde{M}_A\times\tilde{M}_B)(0,x),(\tilde{M}_A\times\tilde{M}_B)(0,y)\}\\&=\min^i\{\min^i\{\tilde{M}_A(0),\tilde{M}_B(x)\},\min^i\{\tilde{M}_A(0),\tilde{M}_B(y\}\}\\&=\min^i\{\tilde{M}_B(x),\tilde{M}_B(y)\}\end{aligned}\tag{3.7}$$

$$\begin{aligned}\tilde{N}_B(x+y)&=\max^i\{\tilde{N}_A(0),\tilde{N}_B(x+y)\}\\&=(\tilde{N}_A\times\tilde{N}_B)(0,x+y)\\&=(\tilde{N}_A\times\tilde{N}_B)((0,x)+(0,y))\\&\leq\max^i\{(\tilde{N}_A\times\tilde{N}_B)(0,x),(\tilde{N}_A\times\tilde{N}_B)(0,y)\}\\&=\max^i\{\max^i\{\tilde{N}_A(0),\tilde{N}_B(x)\},\max^i\{\tilde{N}_A(0),\tilde{N}_B(y\}\}\\&=\max^i\{\tilde{N}_B(x),\tilde{N}_B(y)\}\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}\tilde{M}_B(xy) &= \min^i\{\tilde{M}_A(0), \tilde{M}_B(xy)\} \\ &= (\tilde{M}_A \times \tilde{M}_B)(0, xy) \\ &= (\tilde{M}_A \times \tilde{M}_B)((0,x)(0,y)) \\ &\geq (\tilde{M}_A \times \tilde{M}_B)(0,y) \\ &= \min^i\{\tilde{M}_A(0), \tilde{M}_B(y)\} \\ &= \tilde{M}_B(y)\end{aligned} \tag{3.9}$$

$$\begin{aligned}\tilde{N}_B(xy) &= \max^i\{\tilde{N}_A(0), \tilde{N}_B(xy)\} \\ &= (\tilde{N}_A \times \tilde{N}_B)(0, xy) \\ &= (\tilde{N}_A \times \tilde{N}_B)((0,x)(0,y)) \\ &\leq (\tilde{N}_A \times \tilde{N}_B)(0,y) \\ &= \max^i\{\tilde{N}_A(0), \tilde{N}_B(y)\} \\ &= \tilde{N}_B(y)\end{aligned} \tag{3.10}$$

for all $x, y \in S$. Hence B is an interval-valued intuitionistic fuzzy left ideal of S. Now let $a, b, x \in S$ be such that $x + a = b$. Then $(0,x) + (0,a) = (0,b)$ and so

$$\begin{aligned}\tilde{M}_B(x) &= \min^i\{\tilde{M}_A(0), \tilde{M}_B(x)\} \\ &= (\tilde{M}_A \times \tilde{M}_B)(0,x) \\ &\geq \min^i\{(\tilde{M}_A \times \tilde{M}_B)(0,a), (\tilde{M}_A \times \tilde{M}_B)(0,b)\} \\ &= \min^i\{\min^i\{\tilde{M}_A(0), \tilde{M}_B(a)\}, \min^i\{\tilde{M}_A(0), \tilde{M}_B(b)\}\} \\ &= \min^i\{\tilde{M}_B(a), \tilde{M}_B(a)\}\end{aligned} \tag{3.11}$$

$$\begin{aligned}\tilde{N}_B(x) &= \max^i\{\tilde{N}_A(0), \tilde{N}_B(x)\} \\ &= (\tilde{N}_A \times \tilde{N}_B)(0,x) \\ &\leq \max^i\{(\tilde{N}_A \times \tilde{N}_B)(0,a), (\tilde{N}_A \times \tilde{N}_B)(0,b)\} \\ &= \max^i\{\max^i\{\tilde{N}_A(0), \tilde{N}_B(a)\}, \max^i\{\tilde{N}_A(0), \tilde{N}_B(b)\}\} \\ &= \max^i\{\tilde{N}_B(a), \tilde{N}_B(a)\}\end{aligned} \tag{3.12}$$

Hence $\tilde{M}_B$ is an interval-valued intuitionistic fuzzy left k-ideal of S.

(v). Assume that $\tilde{M}_A(x) \leq \tilde{M}_A(0); \tilde{N}_A(x) \geq \tilde{N}_A(0)$ for all $x \in S$ and $\tilde{M}_B(y) > \tilde{M}_A(0); \tilde{N}_B(y) < \tilde{N}_A(0)$ for some $y \in S$. Then $\tilde{M}_B(0) \geq \tilde{M}_B(x) > \tilde{M}_A(0); \tilde{N}_B(0) \leq \tilde{N}_B(x) < \tilde{N}_A(0)$. Since $\tilde{M}_A(0) \geq \tilde{M}_A(x); \tilde{M}_A(0) \geq \tilde{M}_A(x)$. Hence
$(\tilde{M}_A \times \tilde{M}_B)(0,x) = \min^i\{\tilde{M}_A(x), \tilde{M}_B(0)\} = \tilde{M}_A(x)$
$(\tilde{N}_A \times \tilde{N}_B)(0,x) = \max^i\{\tilde{N}_A(x), \tilde{N}_B(0)\} = \tilde{N}_A(x)$ for all $x \in S$. Thus

$$\begin{aligned}\tilde{M}_A(x+y) &= (\tilde{M}_A \times \tilde{M}_B)(x+y, 0) \\ &= (\tilde{M}_A \times \tilde{M}_B)((x,0) + (y,0)) \\ &\geq \min^i\{(\tilde{M}_A \times \tilde{M}_B)(x,0), (\tilde{M}_A \times \tilde{M}_B)(y,0)\} \\ &= \min^i\{\tilde{M}_A(x), \tilde{M}_A(y)\}\end{aligned} \tag{3.13}$$

$$\begin{aligned}\tilde{N}_A(x+y) &= (\tilde{N}_A \times \tilde{N}_B)(x+y, 0) \\ &= (\tilde{N}_A \times \tilde{N}_B)((x,0) + (y,0)) \\ &\leq \max^i\{(\tilde{N}_A \times \tilde{N}_B)(x,0), (\tilde{N}_A \times \tilde{N}_B)(y,0)\} \\ &= \max^i\{\tilde{N}_A(x), \tilde{N}_A(y)\}\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\tilde{M}_A(xy) &= (\tilde{M}_A \times \tilde{M}_B)(xy, 0) \\ &= (\tilde{M}_A \times \tilde{M}_B)((x,0)(y,0)) \\ &\geq \min^i\{(\tilde{M}_A \times \tilde{M}_B)(y,0) = \tilde{M}_B(y)\end{aligned} \tag{3.15}$$

$$\begin{aligned}\tilde{N}_A(xy) &= (\tilde{N}_A \times \tilde{N}_B)(xy, 0) \\ &= (\tilde{N}_A \times \tilde{N}_B)((x,0)(y,0)) \\ &\leq \max^i\{(\tilde{N}_A \times \tilde{N}_B)(y,0) = \tilde{N}_B(y)\end{aligned} \tag{3.16}$$

for all $x, y \in S$. Now let $a, b, x \in S$ be such that $x + a = b$ and so $(x,0) + (a,0) = (b,0)$. Then

$$\begin{aligned}\tilde{M}_A(x) &= (\tilde{M}_A \times \tilde{M}_B)(x,0) \\ &\geq \min^i\{(\tilde{M}_A \times \tilde{M}_B)(a,0), (\tilde{M}_A \times \tilde{M}_B)(b,0)\} \\ &= \min^i\{\tilde{M}_A(a), \tilde{M}_B(b)\}\end{aligned} \tag{3.17}$$

$$\begin{aligned}\tilde{N}_A(x) &= (\tilde{N}_A \times \tilde{N}_B)(x,0) \\ &\leq \max^i\{(\tilde{N}_A \times \tilde{N}_B)(a,0), (\tilde{N}_A \times \tilde{N}_B)(b,0)\} \\ &= \max^i\{\tilde{N}_A(a), \tilde{N}_B(b)\}\end{aligned} \tag{3.18}$$

Consequently, A is an interval-valued intuitionistic fuzzy left $k-$ideal of S. Hence the proof. □

Theorem 3.4. *Let A be an interval-valued intuitionistic fuzzy set in a semiring S and let A_s be the strongest interval-valued intuitionistic fuzzy relation on S. Then A is an interval-valued intuitionistic fuzzy left k-ideal of S if and only if A_s is an interval-valued intuitionistic fuzzy left k-ideal of $S \times S$.*

Proof. Let $A = (\tilde{M}, \tilde{N})$ be an interval-valued intuitionistic fuzzy left k-ideal of S.

Let $(x_1, x_2, (y_1, y_2) \in S \times S$. Then

$$\begin{aligned}
&\tilde{M}_{A_s}((x_1,x_2)+(y_1,y_2)) \\
&= \tilde{M}_{A_s}(x_1+y_1, x_2+y_2) \\
&= \overset{i}{\min}\{\tilde{M}(x_1+y_1), \tilde{M}(x_2+y_2)\} \\
&\geq \overset{i}{\min}\{\overset{i}{\min}\{\tilde{M}(x_1), \tilde{M}(y_1)\}, \overset{i}{\min}\{\tilde{M}(x_2), \tilde{M}(y_2)\}\} \\
&= \overset{i}{\min}\{\overset{i}{\min}\{\tilde{M}(x_1), \tilde{M}(x_2)\}, \overset{i}{\min}\{\tilde{M}(y_1), \tilde{M}(y_2)\}\} \\
&= \overset{i}{\min}\{\tilde{M}_{A_s}(x_1,x_2), \tilde{M}_{A_s}(y_1,y_2)\}
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
&\tilde{N}_{A_s}((x_1,x_2)+(y_1,y_2)) \\
&= \tilde{N}_{A_s}(x_1+y_1, x_2+y_2) \\
&= \overset{i}{\max}\{\tilde{N}(x_1+y_1), \tilde{N}(x_2+y_2)\} \\
&\leq \overset{i}{\max}\{\overset{i}{\max}\{\tilde{N}(x_1), \tilde{N}(y_1)\}, \overset{i}{\max}\{\tilde{N}(x_2), \tilde{N}(y_2)\}\} \\
&= \overset{i}{\max}\{\overset{i}{\max}\{\tilde{N}(x_1), \tilde{N}(x_2)\}, \overset{i}{\max}\{\tilde{N}(y_1), \tilde{N}(y_2)\}\} \\
&= \overset{i}{\max}\{\tilde{N}_{A_s}(x_1,x_2), \tilde{N}_{A_s}(y_1,y_2)\}
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
\tilde{M}_{A_s}((x_1,x_2)(y_1,y_2)) &= \tilde{M}_{A_s}(x_1y_1, x_2y_2) \\
&= \overset{i}{\min}\{\tilde{M}(x_1y_1), \tilde{M}(x_2y_2)\} \\
&\geq \overset{i}{\min}\{\tilde{M}(y_1), \tilde{M}(y_2)\} \\
&= \tilde{M}_{A_s}(y_1,y_2)
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
\tilde{N}_{A_s}((x_1,x_2)(y_1,y_2)) &= \tilde{N}_{A_s}(x_1y_1, x_2y_2) \\
&= \overset{i}{\max}\{\tilde{N}(x_1y_1), \tilde{N}(x_2y_2)\} \\
&\leq \overset{i}{\max}\{\tilde{N}(y_1), \tilde{N}(y_2)\} \\
&= \tilde{N}_{A_s}(y_1,y_2)
\end{aligned} \tag{3.22}$$

Let $(a_1,a_2),(b_1,b_2),S \times S$ be such that $(x_1,x_2)+(a_1,a_2) = (b_1,b_2)$
Then $(x_1+a_1, x_2+a_2) = (b_1,b_2)$,it follows that $x_1+a_1 = b_1$ and $x_2+a_2 = b_2$.
Thus

$$\begin{aligned}
\tilde{M}_{A_s}(x_1,x_2) &= \overset{i}{\min}\{\tilde{M}(x_1), \tilde{M}(x_2)\} \\
&\geq \overset{i}{\min}\{\overset{i}{\min}\{\tilde{M}(a_1), \tilde{M}(b_1)\}, \overset{i}{\min}\{\tilde{M}(a_2), \tilde{M}(b_2)\}\} \\
&= \overset{i}{\min}\{\overset{i}{\min}\{\tilde{M}(a_1), \tilde{M}(a_2)\}, \overset{i}{\min}\{\tilde{M}(b_1), \tilde{M}(b_2)\}\} \\
&= \overset{i}{\min}\{\tilde{M}_{A_s}(a_1,a_2), \tilde{M}_{A_s}(b_1,b_2)\}
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
\tilde{N}_{A_s}(x_1,x_2) &= \overset{i}{\max}\{\tilde{N}(x_1), \tilde{N}(x_2)\} \\
&\leq \overset{i}{\max}\{\overset{i}{\max}\{\tilde{N}(a_1), \tilde{N}(b_1)\}, \overset{i}{\max}\{\tilde{N}(a_2), \tilde{N}(b_2)\}\} \\
&= \overset{i}{\max}\{\overset{i}{\max}\{\tilde{N}(a_1), \tilde{N}(a_2)\}, \overset{i}{\max}\{\tilde{N}(b_1), \tilde{N}(b_2)\}\} \\
&= \overset{i}{\max}\{\tilde{N}_{A_s}(a_1,a_2), \tilde{N}_{A_s}(b_1,b_2)\}
\end{aligned} \tag{3.24}$$

HenceA_s is an interval-valued intuitionistic fuzzy left $k-$ ideal of $S \times S$.

Conversely, suppose that A_s is an interval-valued intuitionistic fuzzy left $k-$ ideal of $S \times S$. Let $x_1, x_2, y_1, y_2 \in S$. Then

$$\begin{aligned}
&\overset{i}{\min}\{\tilde{M}(x_1+y_1), \tilde{M}(x_2+y_2)\} \\
&= \tilde{M}_{A_s}(x_1+y_1, x_2+y_2) \\
&\geq \overset{i}{\min}\{\tilde{M}_{A_s}(x_1,x_2), \tilde{M}_{A_s}(y_1,y_2)\} \\
&= \overset{i}{\min}\{\overset{i}{\min}\{\tilde{M}(x_1), \tilde{M}(x_2)\}, \\
&\overset{i}{\min}\{\tilde{M}(y_1), \tilde{M}(y_2)\}\}
\end{aligned} \tag{3.25}$$

$\implies \tilde{M}(x_1 + y_1) \geq \min^i\{\min^i\{\tilde{M}(x_1), \tilde{M}(x_2)\}, \min^i\{\tilde{M}(y_1), \tilde{M}(y_2)\}\}$
Similarly,

$$\begin{aligned}
&\overset{i}{\max}\{\tilde{N}(x_1+y_1), \tilde{N}(x_2+y_2)\} \\
&= \tilde{N}_{A_s}(x_1+y_1, x_2+y_2) \\
&\leq \overset{i}{\max}\{\tilde{N}_{A_s}(x_1,x_2), \tilde{N}_{A_s}(y_1,y_2)\} \\
&= \overset{i}{\max}\{\overset{i}{\max}\{\tilde{N}(x_1), \tilde{N}(x_2)\}, \\
&\overset{i}{\max}\{\tilde{N}(y_1), \tilde{N}(y_2)\}\}
\end{aligned} \tag{3.26}$$

$\implies \tilde{N}(x_1 + y_1) \leq \max^i\{\max^i\{\tilde{N}(x_1), \tilde{N}(x_2)\}, \max^i\{\tilde{N}(y_1), \tilde{N}(y_2)\}\}$
In this inequality, we choose the values of x_1, x_2, y_1 and y_2 as follows:
$x_1 = x, x_2 = 0, y_1 = y$ and $y_2 = 0$.
Then we have
$\tilde{M}(x+y) \geq \min^i\{\min^i\{\tilde{M}(x), \tilde{M}(0)\}, \min^i\{\tilde{M}(y), \tilde{M}(0)\}\}$
$= \min^i\{\tilde{M}(x), \tilde{M}(y)\}$
$\tilde{N}(x+y) \leq \max^i\{\max^i\{\tilde{N}(x), \tilde{N}(0)\}, \max^i\{\tilde{N}(y), \tilde{N}(0)\}\}$
$= \max^i\{\tilde{N}(x), \tilde{N}(y)\}$
by using Proposition 3.1. Next, we have

$$\begin{aligned}
\overset{i}{\min}\{\tilde{M}(x_1y_1), \tilde{M}(x_2y_2)\} &= \tilde{M}_{A_s}(x_1y_1, x_2y_2) \\
&= \tilde{M}_{A_s}((x_1,x_2)(y_1,y_2)) \\
&\geq \tilde{M}_{A_s}(y_1,y_2) \\
&= \overset{i}{\min}\{\tilde{M}(y_1), \tilde{M}(y_2)\}
\end{aligned} \tag{3.27}$$

$$\begin{aligned}\overset{i}{\max}\{\tilde{N}(x_1y_1),\tilde{N}(x_2y_2)\} &= \tilde{N}_{A_s}(x_1y_1,x_2y_2)\\ &= \tilde{N}_{A_s}((x_1,x_2)(y_1,y_2))\\ &\leq \tilde{N}_{A_s}(y_1,y_2)\\ &= \overset{i}{\max}\{\tilde{N}(y_1),\tilde{N}(y_2)\}\end{aligned} \tag{3.28}$$

and so $\tilde{M}(x_1y_1) \geq \min^i\{\tilde{M}(y_1),\tilde{M}(y_2)\}$. Taking $x_1 = x, y_1 = y$ and $y_2 = 0$ and using Proposition 3.1, we get $\tilde{M}(xy) \geq \min^i\{\tilde{M}(y),\tilde{M}(0)\} = \tilde{M}(y)$ $\tilde{N}(xy) \leq \max^i\{\tilde{N}(y),\tilde{N}(0)\} = \tilde{N}(y)$ Hence A is an interval-valued intuitionistic fuzzy left ideal of S. Let $a,b,x \in S$ be such that $x+a = b$. Then $(x,0)+(a,0) = (b,0)$. Since A_s is an interval-valued intuitionistic fuzzy left k-ideal of $S\times S$, it follows from Proposition 2.1 that

$$\begin{aligned}\tilde{M}(x) &= \overset{i}{\min}\{\tilde{M}(x),\tilde{M}(0)\}\\ &= \tilde{M}_{A_s}(x,0)\\ &\geq \overset{i}{\min}\{\tilde{M}_{A_s}(a,0),\tilde{M}_{A_s}(b,0)\}\\ &= \overset{i}{\min}\{\overset{i}{\min}\{\tilde{M}(a),\tilde{M}(0)\},\overset{i}{\min}\{\tilde{M}(b),\tilde{M}(0)\}\}\\ &= \overset{i}{\min}\{\tilde{M}(a),\tilde{M}(b)\}\end{aligned} \tag{3.29}$$

$$\begin{aligned}\tilde{N}(x) &= \overset{i}{\max}\{\tilde{N}(x),\tilde{N}(0)\}\\ &= \tilde{N}_{A_s}(x,0)\\ &\leq \overset{i}{\max}\{\tilde{N}_{A_s}(a,0),\tilde{N}_{A_s}(b,0)\}\\ &= \overset{i}{\max}\{\overset{i}{\max}\{\tilde{N}(a),\tilde{N}(0)\},\overset{i}{\max}\{\tilde{N}(b),\tilde{N}(0)\}\}\\ &= \overset{i}{\max}\{\tilde{N}(a),\tilde{N}(b)\}\end{aligned} \tag{3.30}$$

Consequently, A is an interval-valued intuitionistic fuzzy left k-ideal of S. This completes the proof. □

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