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Two point fuzzy boundary value problem with eigenvalue parameter contained in the boundary condition

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Abstract

In this article two point fuzzy boundary value problem is defined under the approach generalized Hukuhara differentiability (gH-differentiability). We research the solution method of the fuzzy boundary problem with the basic solutions $\widehat{\Phi}(x,\lambda)$ and $\widehat{\chi}(x,\lambda)$ which are defined by the special procuder. We give operator-theoretical formulation, construct fundamental solutions and investigate some properties of the eigenvalues and corresponding eigenfunctions of the considered fuzzy problem. Then results of the proposed method are illustrated with a numerical example.

Keywords

gH-derivative, eigenvalue, fuzzy eigenfunction, fuzzy Hilbert Space.

AMS Subject Classification

03E72, 34A07, 65L05.

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Contents

1. Introduction

In this paper we consider the two point fuzzy boundary value problem

$$L = -\frac{d^2}{dt^2}$$

$$L\widehat{u} = \lambda\widehat{u}, \quad t \in [a,b] \tag{1.1}$$

which satisfy the conditions

$$\widehat{a}_1\widehat{u}(a) = \widehat{a}_2\widehat{u}'(a) \tag{1.2}$$

$$\widehat{b}_1\widehat{u}(b) = \lambda\widehat{b}_2\widehat{u}'(b) \tag{1.3}$$

where $\widehat{a}_1, \widehat{a}_2, \widehat{b}_1, \widehat{b}_2$ are nonnegative triangular fuzzy numbers, $\lambda > 0$ reel number, $\widehat{a}_1 + \widehat{a}_2 \neq 0$, $\widehat{b}_1 + \widehat{b}_2 \neq 0$ and $\widehat{u}(t)$ fuzzy function.

The topic of fuzzy differential equation (FDE) has been rapidly growing in recent years. Many problems such as population, economics, dynamics and physics can be modeled with FDE. The fuzzy boundary value problem depending on the eigenvalue parameter provides benefits in solving these problems. There are many suggestions to define a fuzzy derivative and to study fuzzy differential equation [1, 3, 5, 6, 10, 11, 15]. One of the most well-known definitions of difference and derivative for fuzzy set value functions was given bye Hukuhara in [10]. By using the H-derivative, Kaleva in [14] started to develop a theory for fuzzy differential equations. Many works have been done by several authers in theoretical and applied fields for fuzzy differential equations with the Hukuhara derivative [4, 14]. It soon appeared that the solution of fuzzy differential equation interpreted by Hukuhara derivative has a drawback: it became fuzzier as time goes [3]. So here we use gH-difference and gH-derivative to solve FDE under much less restrictive conditions [11].

The paper is organized as follows; section 2 introduces the basic concept of fuzzy function spaces, fuzzy Hilbert spaces and fuzzy gH- derivative. In section 3, we define operator formulation of the problem in the adequate fuzzy Hilbert spaces. In section 4, we give a numerical example. In section 5, the results are stated gained through the study of the paper.

2. Preliminaries

In this section, we give some concepts and results besides the essential notations which will be used throughout the paper.

Let $\widehat{u}$ be a fuzzy subset on $\mathbb{R}$, i.e. a mapping $\widehat{u}:\mathbb{R}\to[0,1]$ associating with each real number t its grade of membership $\widehat{u}(t)$.

In this paper, the concept of fuzzy real numbers (fuzzy intervals) is considered in the sense of Xiaoand Zhu which is defined below:

Definition 2.1. *[2]. A fuzzy subset $\widehat{u}$ on $\mathbb{R}$ is called a fuzzy real number (fuzzy intervals), whose $\alpha-$ cut set is denoted by $[\widehat{u}]_\alpha$, i.e., $[\widehat{u}]_\alpha=\{t:\widehat{u}(t)\geq\alpha\}$, if it satisfies two axioms:*

(N1) There exists $r\in\mathbb{R}$ such that $\widehat{u}(r)=1$.

(N2) For all $0<\alpha\leq 1$, there exist real numbers $-\infty<u_\alpha^-\leq u_\alpha^+<+\infty$ such that $[\widehat{u}]_\alpha$ is equal to the closed interval $[u_\alpha^-,u_\alpha^+]$.

The set of all fuzzy real numbers (fuzzy intervals) is denoted by $\mathscr{F}(\mathbb{R})$. $\mathscr{F}_K(\mathbb{R})$, the family of fuzzy sets of $\mathbb{R}$ whose $\alpha-$ cuts are nonempty compact subsets of $\mathbb{R}$. If $\widehat{u}\in\mathscr{F}(\mathbb{R})$ and $\widehat{u}(t)=0$ whenever $t<0$, then $\widehat{u}$ is called a non-negative fuzzy real number and $\mathscr{F}^+(\mathbb{R})$ denotes the set of all non-negative fuzzy real numbers. For all $\widehat{u}\in\mathscr{F}^+(\mathbb{R})$ and each $\alpha\in(0,1]$, real number u_α^- is positive.

The fuzzy real number $\widehat{r}\in\mathscr{F}(\mathbb{R})$ defined by

$$\widehat{r}(t)=\begin{cases}1, & t=r\\ 0, & t\neq r,\end{cases}$$

it follows that $\mathbb{R}$ can be embedded in $\mathscr{F}(\mathbb{R})$, that is if $\widehat{r}\in(-\infty,\infty)$, then $\widehat{r}\in\mathscr{F}(\mathbb{R})$ satisfies $\widehat{r}(t)=\widehat{0}(t-r)$ and $\alpha-$ cut of $\widehat{r}$ is given by $[\widehat{r}]_\alpha=[r,r],\alpha\in(0,1]$.

Definition 2.2. *[3] An arbitrary fuzzy number in the parametric form is represented by an ordered pair of functions (u_α^-,u_α^+), $0\leq\alpha\leq 1$, which satisfy the following requirements*

(i) u_α^- is bounded non-decreasing left continuous function on $(0,1]$ and right- continuous for $\alpha=0$,

(ii) u_α^+ is bounded non-increasing left continuous function on $(0,1]$ and right- continuous for $\alpha=0$,

(iii) $u_\alpha^-\leq u_\alpha^+$, $0\leq\alpha\leq 1$.

Definition 2.3. *[13] Let $[a^\alpha b^\alpha]$, $0<\alpha\leq 1$, be a given family of non-empty intervals. Assume that*

(a) $[a^{\alpha_1},b^{\alpha_1}]\supset[a^{\alpha_2},b^{\alpha_2}]$ for all $0<\alpha_1\leq\alpha_2$,

(b) $\left[\lim\limits_{k\to-\infty}a^{\alpha_k},\lim\limits_{k\to\infty}b^{\alpha_k}\right]=[a^\alpha,b^\alpha]$ whenever $\{\alpha_k\}$ is an increasing sequence in $(0,1]$ converging to α,

(c) $-\infty<a^\alpha\leq b^\alpha<+\infty$, for all $\alpha\in(0,1]$.

Then the family $[a^\alpha,b^\alpha]$ represents the $\alpha-$cut sets of a fuzzy real number $\widehat{u}\in\mathscr{F}(\mathbb{R})$. Conversely if $[a^\alpha,b^\alpha]$, $0<\alpha\leq 1$, are the $\alpha-$ cut sets of a fuzzy number $\widehat{u}\in\mathscr{F}(\mathbb{R})$, then the conditions (a), (b) and (c) are satisfied.

Definition 2.4. *[3] For $\widehat{u},\widehat{v}\in\mathscr{F}(\mathbb{R})$, and $\lambda\in\mathbb{R}$, the sum $\widehat{u}\oplus\widehat{v}$ and the product $\lambda\odot\widehat{u}$ are defined by for all $\alpha\in[0,1]$*

$$\begin{aligned}[\widehat{u}\oplus\widehat{v}]^\alpha &= [\widehat{u}]^\alpha+[\widehat{v}]^\alpha=\{x+y:x\in[\widehat{u}]^\alpha,y\in[\widehat{v}]^\alpha\},\\ [\lambda\odot\widehat{u}]^\alpha &= \lambda\odot[\widehat{u}]^\alpha=\{\lambda x:x\in[\widehat{u}]^\alpha\}.\end{aligned}$$

Define $d:\mathscr{F}(\mathbb{R})\times\mathscr{F}(\mathbb{R})\to\mathbb{R}^+\cup\{0\}$ by the equation

$$d(\widehat{u},\widehat{v})=\sup_{0<\alpha\leq 1}\{max[|u_\alpha^--v_\alpha^-|,|u_\alpha^+-v_\alpha^+|]\}$$

where $[\widehat{u}]^\alpha=[u_\alpha^-,u_\alpha^+]$, $[\widehat{v}]^\alpha=[v_\alpha^-,v_\alpha^+]$.

Let $T\subset\mathbb{R}$ be an interval. We denote by the $\mathscr{F}(C(T;\mathbb{R}^n))$ space of all continuous fuzzy functions on T. For $\widehat{u}$, $\widehat{v}\in\mathscr{F}(C(T;\mathbb{R}^n))$, define a metric

$$D^*(\widehat{u},\widehat{v})=\sup_{t\in T}d(\widehat{u}(t),\widehat{v}(t)).$$

From [3], $\mathscr{F}(C(T;\mathbb{R}^n),D^*)$ is a complete metric space.

A fuzzy function $\widehat{F}:T\to\mathscr{F}(\mathbb{R})$ is measurable if for all $\alpha\in[0,1]$, the set valued mapping $F_\alpha:T\to\mathscr{F}_K(\mathbb{R})$ defined bye $F_\alpha(t)=\left[\widehat{F}(t)\right]^\alpha$ is measurable.

Let $L^1(T;\mathbb{R})$ denote the space of Lebesque integrable functions. We denote bye S_F^1 the set of all Lebesque integrable selections of $F_\alpha:T\to\mathscr{F}_K(\mathbb{R})$, that is

$$S_F^1=\left\{f\in L^1(T;\mathbb{R}):f(t)\in F_\alpha(t)\ \text{a.e}\right\} \tag{2.1}$$

A fuzzy function $\widehat{F}:T\to\mathscr{F}(\mathbb{R})$ is integrably bounded if there exists an integrable function h such that $\|x\|\leq h(t)$ for all $x\in F_0(t)$. A measurable and integrably bounded fuzzy functions $F_\alpha:T\to\mathscr{F}_K(\mathbb{R})$ is said to be integrable over T if there exists $\widehat{F}\in\mathscr{F}(\mathbb{R})$ such that

$$\int_T F_\alpha(t)\,dt=\left\{\int_T f(t)\,dt:f\in S_F^1\right\},$$

for all $\alpha\in[0,1]$ [7].

For measurable functions $f:T\to\mathbb{R}$ define the norm

$$\|f\|_{L_T^p} = \left\{ \begin{array}{ll} (\int_T |f(t)|^p dt)^{\frac{1}{p}}, & 1 \le p < \infty \\ \inf_{\mu(T_0)=0} \sup_{t \in T - T_0} |f(t)|, & p = \infty \end{array} \right\},$$

where $\mu(T)$ is the Lebesque measure on T and $T_0 \subset T$. Let $L^p(T;\mathbb{R})$ be the Banach space of all measurable functions $f : T \to \mathbb{R}$ with $\|f\|_{L_T^p} < \infty$. By $\mathscr{F}(L^p(T;\mathbb{R}))$, $1 \le p < \infty$, it is denoted the space of all functions $\widehat{u} : T \to \mathscr{F}(\mathbb{R})$ such that the function $t \to d(\widehat{u}(t), \widehat{0})$ belong to $L^p(T;\mathbb{R}_+)$. Then

$$d_p(\widehat{u},\widehat{v}) = \left(\int_0^1 (d_H([\widehat{u}]^\alpha, [\widehat{v}]^\alpha))^p \, d\alpha \right)^{\frac{1}{p}} \tag{2.2}$$

is ametric on $\mathscr{F}(L^p(T;\mathbb{R}))$, and for $1 \le p < \infty$

$$d_\infty(\widehat{u},\widehat{v}) = d(\widehat{u},\widehat{v}) = \sup_{0 \le \alpha < 1} d_H([\widehat{u}]^\alpha, [\widehat{v}]^\alpha) \tag{2.3}$$

is a metric on $\mathscr{F}(L^\infty(T;\mathbb{R}))$ [7].

So for $p = 2$, let $L^2(T;\mathbb{R})$ be the Banach space of all measurable functions $f : T \to \mathbb{R}$ with $\|f\|_{L_T^2} < \infty$. By $\mathscr{F}\left(L^2(T;\mathbb{R})\right)$, we denote the space of all square-integrable fuzzy functions $\widehat{u} : T \to \mathscr{F}(\mathbb{R})$ such that the function $t \to d(\widehat{u}(t), \widehat{0})$ belong to $L^2(T;\mathbb{R}_+)$. Then from (2.2)

$$d_2(\widehat{u},\widehat{v}) = \left(\int_0^1 (d_H([\widehat{u}]^\alpha, [\widehat{v}]^\alpha))^2 \, d\alpha \right)^{\frac{1}{2}}$$

is a metric on $\mathscr{F}\left(L^2(T;\mathbb{R})\right)$, for $p = 2$.

Definition 2.5. *[9] Let X be a vector space over $\mathbb{R}$. A Felbin-fuzzy inner product on X is a mapping $\langle .,. \rangle : X \times X \to \mathscr{F}^+(\mathbb{R})$ such that for all vectors $x,y,z \in X$ and all $r \in \mathbb{R}$, have*

$(FIP1)$ $\langle x+y, z \rangle = \langle x,z \rangle \oplus \langle y,z \rangle$
$(FIP2)$ $\langle rx, y \rangle = |r| \langle x,y \rangle$
$(FIP3)$ $\langle x,y \rangle = \langle y,x \rangle$
$(FIP4)$ $x \ne 0 \Rightarrow \langle x,x \rangle (t) = 0,$ *for all* $t < 0$
$(FIP5)$ $\langle x,x \rangle = 0$ *if and only if* $x = 0$.

Remark 2.6. *Condition (FIP4) in the above definition is equivalent to the condition for all $(0 \ne) x \in X$ $\langle x,x \rangle_\alpha^- > 0$, for each $\alpha \in (0,1]$ where $[\langle x,x \rangle]^\alpha = \left[\langle x,x \rangle_\alpha^- \langle x,x \rangle_\alpha^+\right]$ see in [9].The vector space X equipped with a Felbin-fuzzy inner product is called a Felbin-fuzzy inner space. A Felbin-fuzzy inner product on X defines a fuzzy number*

$$\|x\| = \sqrt[2]{\langle x,x \rangle}, \text{ for all } x \in X. \tag{2.4}$$

A fuzzy Hilbert space is a complete Felbin-fuzzy inner product space with the fuzzy norm defined by (2.4).

Example 2.7. *Consider the lineer space $L^2([0,1];\mathbb{R})$ of all square-integrable functions. Define*

$$\langle f,g \rangle (t) = \sup \left\{ \alpha \in (0,1] \middle| t \in \left[\int_0^1 f_\alpha^- g_\alpha^- dx, \int_0^1 f_\alpha^+ g_\alpha^+ dx \right] \right\} \tag{2.5}$$

for $f,g \in L^2[0,1]$ such that $\widehat{F},\widehat{G} \in \mathscr{F}\left(L^2([0,1];\mathbb{R})\right)$. It can be showed that $\langle f,g \rangle$ is Felbin-fuzzy inner product space on $L^2[0,1]$ and α-cut set of $\langle f,g \rangle$ is given bye

$$\begin{aligned} [\langle f,g \rangle]_\alpha &= \left[\langle f,g \rangle_\alpha^-, \langle f,g \rangle_\alpha^+\right] \\ &= \left[\int_0^1 f_\alpha^-(x) g_\alpha^-(x) dx, \int_0^1 f_\alpha^+(x) g_\alpha^+(x) dx \right] \end{aligned} \tag{2.6}$$

for all $\alpha \in [0,1]$. From theorem 6 of [8], it is clear that $\langle f,g \rangle$ is a fuzzy real number. So $L^2([0,1];\mathbb{R})$ is a fuzzy Hilbert space.

Definition 2.8. *[10]. Let $\widehat{u},\widehat{v} \in \mathscr{F}(\mathbb{R})$. If there exist $\widehat{w} \in \mathscr{F}(\mathbb{R})$ such that $\widehat{u} = \widehat{v} \oplus \widehat{w}$, then $\widehat{w}$ is called the Hukuhara difference of $\widehat{u}$ and $\widehat{v}$ and it is denoted by $\widehat{u} \ominus_h \widehat{v}$. If $\widehat{u} \ominus_h \widehat{v}$ exists, its $\alpha-$cuts are*

$$[\widehat{u} \ominus_H \widehat{v}]^\alpha = \left[u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+\right]$$

for all $\alpha \in [0,1]$.

Definition 2.9. *The generalized Hukuhara difference of two fuzzy numbers $\widehat{u},\widehat{v} \in \mathscr{F}(\mathbb{R})$ is defined as follows*

$$[\widehat{u} \ominus_{gH} \widehat{v}] = \widehat{w} \Leftrightarrow \left\{ \begin{array}{ll} (i) & \widehat{u} = \widehat{v} \oplus \widehat{w} \\ or\ (ii) & \widehat{v} = \widehat{u} \oplus (-1)\widehat{w}. \end{array} \right.$$

In terms of α-cuts we have

$$\begin{aligned} [\widehat{u} \ominus_{gH} \widehat{v}]^\alpha \quad = \quad & \left[\min\left\{u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+\right\}, \right. \\ & \left. \max\left\{u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+\right\}\right] \end{aligned}$$

and if the H -difference exists, then $\widehat{u} \ominus_h \widehat{v} = \widehat{u} \ominus_{gH} \widehat{v}$; the conditions for the existence of $\widehat{w} = \widehat{u} \ominus_{gH} \widehat{v} \in \mathscr{F}(\mathbb{R})$ are

$$case\ (i) \left\{ \begin{array}{l} w_\alpha^- = u_\alpha^- - v_\alpha^- \text{ and } w_\alpha^+ = u_\alpha^+ - v_\alpha^+, \forall \alpha \in [0,1] \\ \text{with } w_\alpha^- \text{ increasing, } w_\alpha^+ \text{ decreasing,} w_\alpha^- \le w_\alpha^+ \end{array} \right.$$

$$case\ (ii) \left\{ \begin{array}{l} w_\alpha^- = u_\alpha^+ - v_\alpha^+ \text{ and } w_\alpha^+ = u_\alpha^- - v_\alpha^-, \forall \alpha \in [0,1] \\ \text{with } w_\alpha^- \text{ increasing, } w_\alpha^+ \text{ decreasing,} w_\alpha^- \le w_\alpha^+. \end{array} \right.$$

It is easy to show that (i) and (ii) are both valid if and only if w is a crisp number [11].

Remark 2.10. *Throughout the rest of this paper, we assume that $\widehat{u} \ominus_{gH} \widehat{v} \in \mathscr{F}(\mathbb{R})$ and α-cut representation of fuzzy valued function $\widehat{f} : (a,b) \to \mathscr{F}(\mathbb{R})$ is expressed by $\left[\widehat{f}(t)\right]^\alpha = [(f_\alpha^-)(t), (f_\alpha^+)(t)]$, $t \in [a,b]$ for each $\alpha \in [0,1]$.*

Definition 2.11. *[11] Let $t_0 \in (a,b)$ and h be such that $t_0 + h \in (a,b)$, then the gH-derivative of a function $\widehat{f} : (a,b) \to \mathscr{F}(\mathbb{R})$ at t_0 is defined as*

$$\widehat{f}'_{gH}(t_0) = \lim_{h \to 0} \frac{f(t_0+h) \ominus_{gH} f(t_0)}{h}. \tag{2.7}$$

If $\widehat{f}'_{gH}(t_0) \in \mathscr{F}(\mathbb{R})$ satisfying (2.7) exists, we say that $\widehat{f}$ is generalized Hukuhara differentiable (gH-differentiable for short) at t_0.

Definition 2.12. *[11] Let $\widehat{f}:[a,b]\to\mathscr{F}(\mathbb{R})$ and $t_0\in(a,b)$ with $f_\alpha^-(t)$ and $f_\alpha^+(t)$ both differentiable at t_0. We say that*

- *$\widehat{f}$ is $[(i)-gH]$-differentiable at t_0 if*

$$\left[\widehat{f}'_{gH}(t_0)\right]^\alpha=\left[\left(f_\alpha^-\right)'(t),\left(f_\alpha^+\right)'(t)\right] \tag{2.8}$$

- *$\widehat{f}$ is $[(ii)-gH]$-differentiable at t_0 if*

$$\left[\widehat{f}'_{gH}(t_0)\right]^\alpha=\left[\left(f_\alpha^+\right)'(t),\left(f_\alpha^-\right)'(t)\right] \tag{2.9}$$

for all $\alpha\in[0,1]$.

Definition 2.13. *The second generalized Hukuhara derivative of a fuzzy function $\widehat{f}:[a,b]\to\mathscr{F}(\mathbb{R})$ at t_0 is defined as*

$$\widehat{f}''_{gH}(t_0)=\lim_{h\to0}\frac{\widehat{f}'(t_0+h)\ominus_{gH}\widehat{f}'(t_0)}{h}.$$

if $\widehat{f}''_{gH}(t_0)\in\mathscr{F}(\mathbb{R})$, we say that $\widehat{f}'_{gH}(t)$ is generalized Hukuhara derivative at t_0.

Also we say that $\widehat{f}'_{gH}(t)$ is $[(i)-gH]-$differentiable at t_0 if

$$\widehat{f}''_{i.gH}(t_0,\alpha)=\begin{cases}\left[\left(f_\alpha^-\right)''(t_0),\left(f_\alpha^+\right)''(t_0)\right], & \text{if } \widehat{f} \text{ be } [(i)-gH]\\ \left[\left(f_\alpha^+\right)''(t_0),\left(f_\alpha^-\right)''(t_0)\right], & \text{if } \widehat{f} \text{ be } [(ii)-gH]\end{cases}$$

for all $\alpha\in[0,1]$, and that $\widehat{f}'_{gH}(t)$ is $[(i)-gH]-$differentiable at t_0 if

$$\widehat{f}''_{ii.gH}(t_0,\alpha)=\begin{cases}\left[\left(f_\alpha^+\right)''(t_0),\left(f_\alpha^-\right)''(t_0)\right], & \text{if } \widehat{f} \text{ be } [(i)-gH]\\ \left[\left(f_\alpha^-\right)''(t_0),\left(f_\alpha^+\right)''(t_0)\right], & \text{if } \widehat{f} \text{ be } [(ii)-gH]\end{cases}$$

for all $\alpha\in[0,1]$ [12].

3. Operator-Theoretic Formulation of the Problem

Consider the $(1.1)-(1.3)$ eigenvalue problem.

This is not the usual type of eigenvalue problem because the eigenvalue appears in the boundary conditions; so, we cannot put $L=-\frac{d^2}{dt^2}$ and consider problem as a special case of $L\widehat{u}=\lambda\widehat{u}$ because D, the domain of L, depends upon λ. So we enlarge our definition of L. To do this we formulate a theoretic approach to the problem.

From theorem 6 of [8], we define a fuzzy Hilbert space $\widehat{H}=L^2([0,1];\mathbb{R})$ with a Felbin-fuzzy inner product

$$\begin{aligned}\langle f,g\rangle \;=\; &\left[\int_0^1 f_\alpha^-(t)g_\alpha^-(t)dt+(f_1)_\alpha^-(g_1)_\alpha^-,\right.\\ &\left.\int_0^1 f_\alpha^+(t)g_\alpha^+(t)dt+(f_1)_\alpha^+(g_1)_\alpha^+\right]\end{aligned}$$

where $F\in\left[\widehat{F}\right]^\alpha=\begin{pmatrix}[f_\alpha^-(t),f_\alpha^+(t)]\\ \left[(f_1)_\alpha^-,(f_1)_\alpha^+\right]\end{pmatrix}\subset H$ and $G\in\left[\widehat{G}\right]^\alpha=\begin{pmatrix}[g_\alpha^-(t),g_\alpha^+(t)]\\ \left[(g_1)_\alpha^-,(g_1)_\alpha^+\right]\end{pmatrix}\subset H$ and $\widehat{F},\widehat{G}\in\mathscr{F}^+\left(L^2([0,1];\mathbb{R})\right)$ such that $f_\alpha^-(t),f_\alpha^+(t)$, $g_\alpha^-(t),g_\alpha^+(t)\in L^2([0,1];\mathbb{R})$ and $(f_1)_\alpha^-,(f_1)_\alpha^+$, $(g_1)_\alpha^-,(g_1)_\alpha^+\in\mathbb{R}$.

Consider the space of two- component fuzzy vectors $\widehat{F}\in\widehat{H}$,whose first component is a twice gH-differentiable fuzzy function $\widehat{f}(t)\in\mathscr{F}^+\left(L^2([0,1];\mathbb{R})\right)$ and whose second component is a fuzzy real number $\widehat{f}_1\in F^+(\mathbb{R})$. Here $\widehat{F}$, $\widehat{f}(x)$ and $\widehat{f}_1$ are pozitive fuzzy numbers and their $\alpha-$cut sets are respectively $[F_\alpha^-,F_\alpha^+]$, $[f_\alpha^-(t),f_\alpha^+(t)]$ and $\left[(f_1)_\alpha^-,(f_1)_\alpha^+\right]$.

In this fuzzy Hilbert space we construct the lineer operator $A:\widehat{H}\to\widehat{H}$ with domain

$$D(A)=\left\{\begin{pmatrix}\widehat{f}(t)\\ \widehat{f}_1\end{pmatrix}\;\middle|\;\begin{matrix}\widehat{f}(t),\widehat{f}'(t)\text{ fuzzy absolutely}\\ \text{continuous (see [6]) in }[0,1];\\ \widehat{a}_1\widehat{f}(a)=\widehat{a}_2\widehat{f}'(a),\ \widehat{f}_1=\widehat{b}_2\widehat{f}'(b)\end{matrix}\right\}$$

which acts by the rule $A\widehat{F}=A\begin{pmatrix}\widehat{f}(t)\\ \widehat{f}_1\end{pmatrix}=A\begin{pmatrix}\widehat{f}(t)\\ \widehat{b}_2\widehat{f}'(b)\end{pmatrix}=\begin{pmatrix}-\widehat{f}''(t)\\ \widehat{b}_1\widehat{f}(b)\end{pmatrix}$ with $\widehat{F}=\begin{pmatrix}\widehat{f}\\ \widehat{f}_1\end{pmatrix}\in D(A)$.

Thus the problem $(1.1)-(1.3)$ can be written in the form

$$A\widehat{F}=\lambda\widehat{F}$$

where $\widehat{F}=\begin{pmatrix}\widehat{f}\\ \widehat{f}_1\end{pmatrix}\in D(A)$. Then the eigenvalues and the eigenfunctions of the problem $(1.1)-(1.3)$ are defined as the eigenvalues and the first components of the corresponding eigenelements of the operator A, respectively.

4. Solution Method of the Fuzzy Problem

In this section we concern with fuzzy eigenvalues and eigenfunctions of two-Point fuzzy boundary value problems with eigenvalue parameter contained in the boundary condition. To do this, at first we need to use fuzzy derivatives. So here we use gH-difference and gH-derivative to solve fuzzy problem [11].

Definition 4.1. *Let $\widehat{u}:[a,b]\subset\mathbb{R}\to\mathscr{F}(\mathbb{R})$ be a fuzzy function and $[\widehat{u}(t,\lambda)]^\alpha=[u_\alpha^-(t,\lambda),u_\alpha^+(t,\lambda)]$ be the $\alpha-$cut representation of the fuzzy function $\widehat{u}(t)$ for all $t\in[a,b]$ and $\alpha\in[0,1]$. If the fuzzy differential equation (1.1) has the nontrivial solutions such that $u_\alpha^-(t,\lambda)\neq0$, $u_\alpha^+(t,\lambda)\neq0$ then λ is eigenvalue of (1.1).*

Consider the eigenvalues of the fuzzy boundary value problem $(1.1)-(1.3)$. Using the $\alpha-$cut sets and $[i-gH]$-differentiability of $\widehat{u}$, $[ii-gH]$-differentiability of $\widehat{u}'$; we get from $(1.1)-(1.3)$ for $\lambda=k^2\,(k>0)$:

$$\left[-\left(u_\alpha^-\right)''(t),-\left(u_\alpha^+\right)''(t)\right]=\lambda\left[u_\alpha^-(t),u_\alpha^+(t)\right] \tag{4.1}$$

$$\begin{aligned} &\left[(a_1)_\alpha^-,(a_1)_\alpha^+\right]\left[u_\alpha^-(a),u_\alpha^+(a)\right] \\ =\ &\left[(a_2)_\alpha^-,(a_2)_\alpha^+\right]\left[\left(u_\alpha^-\right)'(a),\left(u_\alpha^+\right)'(a)\right] \end{aligned} \tag{4.2}$$

$$\begin{aligned} &\left[(b_1)_\alpha^-,(b_1)_\alpha^+\right]\left[u_\alpha^-(b),u_\alpha^+(b)\right] \\ =\ &\lambda\left[(b_2)_\alpha^-,(b_2)_\alpha^+\right]\left[\left(u_\alpha^-\right)'(b),\left(u_\alpha^+\right)'(b)\right]. \end{aligned} \tag{4.3}$$

Suppose that the two linearly independent solutions of $u''+\lambda u=0$ classic differential equation are $u_1(t,\lambda)$ and $u_2(t,\lambda)$. Using these solutions the general solution of the fuzzy differential equation (4.1) is

$$[\widehat{u}(t,\lambda)]^\alpha=\left[u_\alpha^-(t,\lambda),u_\alpha^+(t,\lambda)\right]$$

where

$$\begin{aligned} u_\alpha^-(t,\lambda) &= C_1(\alpha,\lambda)u_1(t,\lambda)+C_2(\alpha,\lambda)u_2(t,\lambda) \\ u_\alpha^+(t,\lambda) &= C_3(\alpha,\lambda)u_1(t,\lambda)+C_4(\alpha,\lambda)u_2(t,\lambda). \end{aligned}$$

Let $\left[\widehat{\Phi}(x,\lambda)\right]^\alpha$ be a solution which is satisfying

$$\begin{aligned} \left[u_\alpha^-(a),u_\alpha^+(a)\right] &= \left[(a_2)_\alpha^-,(a_2)_\alpha^+\right] \\ \left[\left(u_\alpha^-\right)'(a),\left(u_\alpha^+\right)'(a)\right] &= \left[(a_1)_\alpha^-,(a_1)_\alpha^+\right] \end{aligned} \tag{4.4}$$

initial conditions of fuzzy differential equations (4.1). This solution can be expressed as

$$\begin{aligned} \Phi_\alpha^-(t,\lambda) &= C_{11}(\alpha,\lambda)u_1(t,\lambda)+C_{12}(\alpha,\lambda)u_2(t,\lambda) \\ \Phi_\alpha^+(t,\lambda) &= C_{13}(\alpha,\lambda)u_1(t,\lambda)+C_{14}(\alpha,\lambda)u_2(t,\lambda). \end{aligned} \tag{4.5}$$

and we write (4.4) in (4.5) such that

$$\begin{aligned} \Phi_\alpha^-(a,\lambda) &= C_{11}(\alpha,\lambda)\cos(ka)+C_{12}(\alpha,\lambda)\sin(ka)=(a_2)_\alpha^- \\ (\Phi_\alpha^-)'(a,\lambda) &= C_{13}(\alpha,\lambda)\cos(ka)+C_{14}(\alpha,\lambda)\sin(ka)=(a_1)_\alpha^-. \end{aligned}$$

From the determinant of the coefficients matrix of the above linear system, we get $C_{11}(\alpha,\lambda)$ and $C_{12}(\alpha,\lambda)$ such that

$$\begin{aligned} C_{11} &= (a_2)_\alpha^-\cos(ka)-(a_1)_\alpha^-\frac{\sin(ka)}{k}, \\ C_{12} &= (a_2)_\alpha^-\sin(ka)+(a_1)_\alpha^-\frac{\cos(ka)}{k}. \end{aligned}$$

Substituting this C_{11} and C_{12} coefficients the above equations in (4.5), the general solution is obtained as

$$\begin{aligned} \Phi_\alpha^-(t,\lambda) = &\left((a_2)_\alpha^-\cos(ka)-(a_1)_\alpha^-\frac{\sin(ka)}{k}\right)\cos(kt) \\ &+\left((a_2)_\alpha^-\sin(ka)+(a_1)_\alpha^-\frac{\cos(ka)}{k}\right)\sin(kt). \end{aligned} \tag{4.6}$$

Similarly we find $\Phi_\alpha^+(t,\lambda)$ as

$$\begin{aligned} \Phi_\alpha^+(t,\lambda) = &\left((a_2)_\alpha^+\cos(ka)-(a_1)_\alpha^+\frac{\sin(ka)}{k}\right)\cos(kt) \\ &+\left((a_2)_\alpha^+\sin(ka)+(a_1)_\alpha^+\frac{\cos(ka)}{k}\right)\sin(kt). \end{aligned} \tag{4.7}$$

Let $[\widehat{\chi}(x,\lambda)]^\alpha$ be a solution which is satisfying

$$\begin{aligned} \left[u_\alpha^-(b),u_\alpha^+(b)\right] &= \left[(b_2)_\alpha^-,(b_2)_\alpha^+\right] \\ \left[\left(u_\alpha^-\right)'(b),\left(u_\alpha^+\right)'(b)\right] &= \left[(b_1)_\alpha^-,(b_1)_\alpha^+\right] \end{aligned} \tag{4.8}$$

initial conditions of fuzzy differential equations (4.1). This solution can be expressed as

$$\begin{aligned} \chi_\alpha^-(t,\lambda) &= C_{21}(\alpha,\lambda)u_1(t,\lambda)+C_{22}(\alpha,\lambda)u_2(t,\lambda) \\ \chi_\alpha^+(t,\lambda) &= C_{23}(\alpha,\lambda)u_1(t,\lambda)+C_4(\alpha,\lambda)u_2(t,\lambda). \end{aligned} \tag{4.9}$$

and we write (4.8) in (4.9) such that

$$\begin{aligned} \chi_\alpha^-(a,\lambda) &= C_{21}(\alpha,\lambda)\cos(ka)+C_{22}(\alpha,\lambda)\sin(ka)=k^2(b_2)_\alpha^- \\ (\chi_\alpha^-)'(a,\lambda) &= C_{23}(\alpha,\lambda)\cos(ka)+C_{24}(\alpha,\lambda)\sin(ka)=(b_1)_\alpha^- \end{aligned}.$$

From the determinant of the coefficients matrix of the above linear system, we get $C_{21}(\alpha,\lambda)$ and $C_{22}(\alpha,\lambda)$ such that

$$\begin{aligned} C_{21} &= =k^2(b_2)_\alpha^-\cos(kb)-(b_1)_\alpha^-\frac{\sin(kb)}{k}, \\ C_{22} &= k^2(b_2)_\alpha^-\sin(kb)+(b_1)_\alpha^-\frac{\cos(kb)}{k}. \end{aligned}$$

Substituting this C_{21} and C_{22} coefficients the above equations in (4.9), the general solution is obtained as

$$\begin{aligned} \chi_\alpha^-(t,\lambda) = &\left(k^2(b_2)_\alpha^-\cos(kb)-(b_1)_\alpha^-\frac{\sin(kb)}{k}\right)\cos(kt) \\ &+\left(k^2(b_2)_\alpha^-\sin(kb)+(b_1)_\alpha^-\frac{\cos(kb)}{k}\right)\sin(kt). \end{aligned} \tag{4.10}$$

Similarly we find $\chi_\alpha^+(t,\lambda)$ as

$$\chi_\alpha^+(t,\lambda) = \left(k^2(b_2)_\alpha^+ cos(kb) - (b_1)_\alpha^+ \frac{sin(kb)}{k}\right)cos(kt) + \left(k^2(b_2)_\alpha^+ sin(kb) + (b_1)_\alpha^+ \frac{cos(kb)}{k}\right)sin(kt). \tag{4.11}$$

Then from (4.6) – (4.7) and (4.10) – (4.11) we find $W(\Phi_\alpha^-,\chi_\alpha^-)(t,\lambda)$ and $W(\Phi_\alpha^+,\chi_\alpha^+)(t,\lambda)$ Wronskian functions as

$$\begin{aligned} & W(\Phi_\alpha^-,\chi_\alpha^-)(t,\lambda) \\ = \; & \Phi_\alpha^-(t,\lambda)(\chi_\alpha^-)'(t,\lambda) - (\Phi_\alpha^-)'(t,\lambda), \chi_\alpha^-(t,\lambda) \\ = \; & \left((a_2)_\alpha^-(b_1)_\alpha^- - k^2(a_1)_\alpha^-(b_2)_\alpha^-\right)cos(k(a-b)) \\ & -\left(k^3(a_2)_\alpha^-(b_2)_\alpha^- + \frac{(a_1)_\alpha^-(b_1)_\alpha^-}{k}\right)sin(k(a-b)) \end{aligned}$$

and

$$\begin{aligned} & W(\Phi_\alpha^+,\chi_\alpha^+)(t,\lambda) \\ = \; & \Phi_\alpha^+(t,\lambda)(\chi_\alpha^+)'(t,\lambda) - (\Phi_\alpha^+)'(t,\lambda), \chi_\alpha^+(t,\lambda) \\ = \; & \left((a_2)_\alpha^+(b_1)_\alpha^+ - k^2(a_1)_\alpha^+(b_2)_\alpha^+\right)cos(k(a-b)) \\ & -\left(k^3(a_2)_\alpha^+(b_2)_\alpha^+ + \frac{(a_1)_\alpha^+(b_1)_\alpha^+}{k}\right)sin(k(a-b)) \end{aligned}$$

Theorem 4.2. *The Wronskian functions $W(\Phi_\alpha^-,\chi_\alpha^-)(t,\lambda)$ and $W(\Phi_\alpha^+,\chi_\alpha^+)(t,\lambda)$ are independent of variable t for $t \in [a,b]$, where functions $\Phi_\alpha^-, \chi_\alpha^-, \Phi_\alpha^+, \chi_\alpha^+$ are the solution of the fuzzy boundary value problem (1.1) – (1.3) and it is show that $W_\alpha^-(\lambda) = W(\Phi_\alpha^-,\chi_\alpha^-)(t,\lambda)$ and $W_\alpha^+(\lambda) = W(\Phi_\alpha^+,\chi_\alpha^+)(t,\lambda)$ for each $\alpha \in [0,1]$ [4].*

Theorem 4.3. *The eigenvalues of the fuzzy boundary value problem (1.1) – (1.3) are coincided zeros of the functions $W_\alpha^-(\lambda)$ and $W_\alpha^+(\lambda)$ [4].*

Example 4.4. *Consider the two point fuzzy boundary problem*

$$-\widehat{u}'' = \lambda\widehat{u} \tag{4.12}$$
$$\widehat{u}(0) = 0 \tag{4.13}$$
$$\left[\widehat{1}\right]\widehat{u}(1) = \lambda\widehat{u}'(1) \tag{4.14}$$

where $[1]^\alpha = [\alpha, 2-\alpha]$, $\lambda = k^2\,(k>0)$ and $\widehat{u}(t)$ is $[i-gH]$-differentiable and $\widehat{u}'(t)$ is $[ii-gH]$-differentiable fuzzy functions.

Let $\left[\widehat{\Phi}(t,\lambda)\right]^\alpha$ be the solution which is satisfying

$$\left[u_\alpha^-(0), u_\alpha^+(0)\right] = 0$$

initial condition of fuzzy differential equations (4.12). Then we find $\left[\widehat{\Phi}(t,k)\right]^\alpha$ as

$$\left[\widehat{\Phi}(t,\lambda)\right]^\alpha = \sin(kx). \tag{4.15}$$

Similarly let $[\widehat{\chi}(t,\lambda)]^\alpha$ be the solution which is satisfying

$$\begin{aligned} \left[u_\alpha^-(1), u_\alpha^+(1)\right] &= \lambda \\ \left[(u_\alpha^-)'(1), (u_\alpha^+)'(1)\right] &= [\alpha, 2-\alpha] \end{aligned}$$

initial conditions of fuzzy differential equations (4.12). Then we find $[\widehat{\chi}(t,\lambda)]^\alpha$ as

$$\begin{aligned} \chi_\alpha^-(t,\lambda) &= k^2\cos(kt-k) + \frac{\alpha}{k}\sin(kt-k) \\ \chi_\alpha^+(t,\lambda) &= k^2\cos(kt-k) + \frac{(2-\alpha)}{k}\sin(kt-k). \end{aligned} \tag{4.16}$$

From theorem 4.3, , eigenvalues of the fuzzy problem (4.12)-(4.14) are zeros of the functions $W_\alpha^-(\lambda)$ and $W_\alpha^+(\lambda)$. So we get

$$W_\alpha^-(\lambda) = \alpha\sin k - k^3\cos k \tag{4.17}$$
$$W_\alpha^+(\lambda) = (2-\alpha)\sin k - k^3\cos k. \tag{4.18}$$

For each $\alpha \in [0,1]$ if the k values satisfying (4.17) and (4.18) equations compute with Matlab Program, then an infinite number of eigenvalues such that $\lambda_n = (k_n)^2$ are obtained. So we show first four k values of (4.17) with $k_{1,n}$ in table 1 such that

Table 1. For $W_\alpha^-(\lambda)$ $k_{1,n}$ values corresponding to α

$\alpha \in [0,1]$	$k_{1,0}$	$k_{1,1}$	$k_{1,2}$	$k_{1,3}$
$\alpha = 0$	4.7124	7.8540	10.9956	14.1372
$\alpha = 0.2$	4.7105	7.8536	10.9954	14.1371
$\alpha = 0.5$	4.7176	7.8529	10.9952	14.1370
$\alpha = 0.8$	4.7147	7.8523	10.9950	14.1369
$\alpha = 1$	4.7028	7.8519	10.9948	14.1368

and first four k values of (4.18) with $k_{1,n}$ in table 1 such that

Table 2. For $W_\alpha^+(\lambda)$ $k_{2,n}$ values corresponding to α

$\alpha \in [0,1]$	$k_{2,0}$	$k_{2,1}$	$k_{2,2}$	$k_{2,3}$
$\alpha = 0$	4.6930	7.8498	10.9941	14.1364
$\alpha = 0.2$	4.6950	7.8503	10.9942	14.1365
$\alpha = 0.5$	4.6979	7.8509	10.9944	14.1366
$\alpha = 0.8$	4.7008	7.8515	10.9947	14.1367
$\alpha = 1$	4.7028	7.8519	10.9948	14.1368

For each k_n values and $\alpha \in [0,1]$ graphic of $W_\alpha^-(\lambda)$ and $W_\alpha^+(\lambda)$ functions are given figures 1.

So if we write $k_{1,n}$ and $k_{2,n}$ values in (4.15) and (4.16) equations, then $\left[\widehat{\Phi}_n(t,\lambda)\right]^\alpha$ and $[\widehat{\chi}_n(t,\lambda)]^\alpha$ are

$$\left[\widehat{\Phi}_n(t,\lambda)\right]^\alpha = \left[(\Phi_n)_\alpha^-(t), (\Phi_n)_\alpha^+(t)\right] = [\sin(k_{1,n}t), \sin(k_{2,n}t)]$$

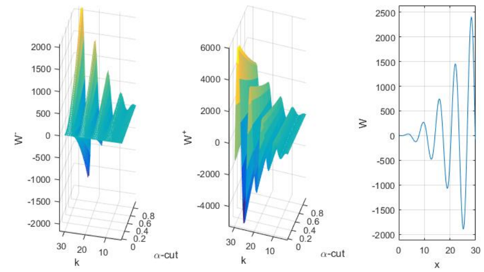

Figure 1. $W_\alpha^-(\lambda)$, $W_\alpha^+(\lambda)$ and $W(\lambda)$

(4.19)

and

$$\begin{aligned}&[\widehat{\chi}_n(t,\lambda)]^\alpha\\ &=\left[(\chi_n)_\alpha^-(t),(\chi_n)_\alpha^+(t)\right]\\ &=\left[\left((k_{1,n})^2\cos(k_{1,n}t-k_{1,n})+\frac{\alpha}{k_{1,n}}\sin(k_{1,n}t-k_{1,n})\right),\right.\\ &\left.\left((k_{2,n})^2\cos(k_{2,n}t-k_{2,n})+\frac{(2-\alpha)}{k_{2,n}}\sin(k_{2,n}t-k_{2,n})\right)\right].\end{aligned} \tag{4.20}$$

Then for all $\alpha\in[0,1]$, (4.19) are eigenfunctions corresponding to $\lambda_{1,n}=(k_{1,n})^2$ eigenvalues satisfying (4.17) equation and (4.20) are eigenfunctions corresponding to $\lambda_{2,n}=(k_{2,n})^2$ eigenvalues satisfying (4.18) equation.

Consider that eigenvalues of $\left[\widehat{\Phi}_n(t,\lambda)\right]^\alpha$ and $[\widehat{\chi}_n(t,\lambda)]^\alpha$ eigenfunctions depend on $\alpha-$cut. So if we change α, then this eigenvalues change and eigenfunctions corresponding to λ change. In particular, we select $k_{1,1}=7.8540$ in table 1 and $k_{2,1}=7.8498$ in table 2 for $\alpha=0$. If we substitute this values respectively in (4.19) and (4.20), we have the following figures for $\widehat{\chi}$ and $\widehat{\Phi}$ functions.

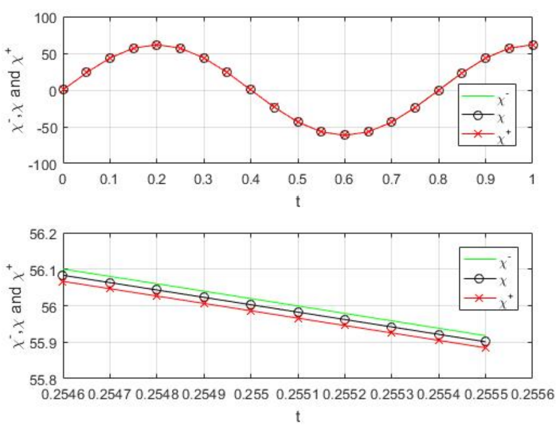

Figure 2. $\widehat{\chi}$ Eigenfunctions and fuzzy interval of $\widehat{\chi}$

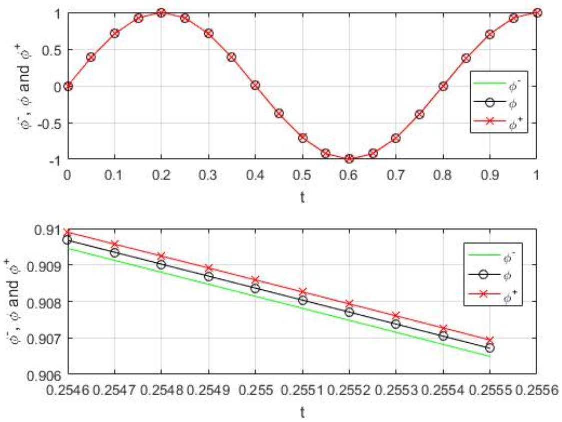

Figure 3. $\widehat{\Phi}$ Eigenfunctions and fuzzy interval of $\widehat{\Phi}$

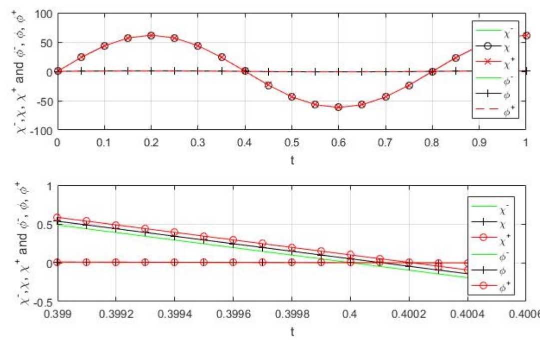

Figure 4. $\alpha-$ cut of $\widehat{\chi}$ and $\widehat{\Phi}$ Eigenfunctions

Also $\widehat{\chi}$ and $\widehat{\Phi}$ functions are given same grafic with figure 4.

In figure 2 χ_α^-, χ, χ_α^+ functions and in figure 3 Φ_α^-, Φ, Φ_α^+ functions are seem to have the same graphic. But because of eigenvalues have very close to each other, this eigenfunctions are different. Also $\left[\widehat{\Phi}_n(t,\lambda)\right]^\alpha$ and $[\widehat{\chi}_n(t,\lambda)]^\alpha$ eigenfunctions are fuzzy at certain intervals see figures.

5. Conclusion

Eigenvalues and eigenfunctions of the fuzzy boundary problem are introduced and examined by using generalized Hukuhara differentiability concept. In this problem, the eigenvalue parameter is contained in the boundary condition at b. So we define lineer operator in fuzzy Hilbert space. To solve this problem, we use some initial value problems. Then we give a numerical example.

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