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Common fixed points of a pair of multivalued non-self mappings in partial metric spaces

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Abstract

In this paper, we utilize the concept of the partial Hausdorff metric, first introduced by Aydi et al.[4] for partial metric space, to consider a pair of multivalued mappings which are non-self almost contractions on metrically convex partial metric spaces. We establish the existence of fixed point in such mappings.

Keywords

Partial Hausdorff metric, multivalued mapping, almost contraction, partial metric space, non-self mapping.

AMS Subject Classification

47H10, 54H25, 46T99.

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Contents

1. Introduction and Preliminaries.

In 1969, Nadler [11] introduced the study of fixed points using the Hausdorff metric on multivalued mappings. Assad and Kirk [2] proved the Banach fixed point theorem for non-self multivalued mappings. These results were used by Imdad and Kumar [8] to prove common fixed points for a pair of non-self mappings.

Berinde [5] intorduced a class of self-mappings which are known as almost contractions. He developed fixed point theorems for such mappings which generalized several fixed point theorems, including those of Kannan [9] and Chatterjea [7].

Berinde and Berinde [6] also formulated fixed point theorems for almost contractions in multivalued self mappings.

Multivalued non-self almost contractions were introduced by Alghamdi et al. [1], who also proved the existence of fixed points for such type of mappings for metrically convex metric spaces.

Aydi et al. [4] introduced the concept of the partial Hausdorff metric and used it to prove Nadler's theorem on partial metric spaces.

This study formulates a fixed point theorem for pairs of multivalued non-self almost constructions in complete partial metric spaces.

We now introduce preliminaries which will be of use in this paper.

Definition 1.1. [10] A partial metric on a non-empty set X is a mapping $p : X \times X \to [0,\infty)$, such that for all $x,y,z \in X$.
P0: $0 \le p(x,x) \le p(x,y)$,
P1: $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$,
P2: $p(x,y) = p(y,x)$ and
P3: $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.
The pair (X,p) is said to be a partial metric space.

From Definition 1.1 we deduce the following:

$$p(x,y) = 0 \Rightarrow x = y. \tag{1.1}$$

Proof. If $p(x,y) = 0$, then $p(x,x) = 0$ because $0 \le p(x,x) \le p(x,y)$ from P0. Similarly, $p(x,y) = 0$ implies $p(y,y) = 0$ because $0 \le p(y,y) \le p(x,y)$. Hence, $p(x,y) = 0$ implies $p(x,x) = p(x,y) = p(y,y) = 0$. From P1 this means that $x = y$. □

From P3 we deduce that

$$p(x,y) \le p(x,z) + p(z,y). \tag{1.2}$$

As an example, let $X = \mathbb{R}^+$ and let $p(x,y) = \max\{x,y\}$ for all $x,y \in X$. Then (X,p) is a partial metric space .

Each partial metric p on X generates a T_0 topology τ_p on X with a base being the family of open balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$ where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Definition 1.2. [10]

(i) A sequence $\{x_n\}$ in a partial metric space (X,p) converges to $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$,

(ii) A sequence $\{x_n\}$ in a partial metric space (X,p) is called a Cauchy sequence if and only if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.

(iii) A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that

$$p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m).$$

Lemma 1.3. [10] *If p is a partial metric on X, then the mapping $p^s : X\times X \to [0,+\infty)$ given by*

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{1.3}$$

defines a metric on X.

In this paper we will denote p^s as the metric derived from the partial metric p.

We now describe a metrically convex metric space and state some of its properties.

Definition 1.4. [2]

(i) A metric space (X,d) is said to be metrically convex if for all x,y in X with $x \ne y$, there exists a point z in $X, (x \ne z \ne y)$ such that $d(x,y) = d(x,z) + d(z,y)$.

(ii) Let (X,d) be a metrically convex metric space with $x,y \in X, x \ne y$. Then we define

$$\text{seg}[x,y] := \{z : d(x,y) = d(x,z) + d(z,y)\}.$$

As an example, the partial metric space $(\mathbb{R}^+,p)$ where $p(x,y) = \max\{x,y\}$ for all $x,y \in \mathbb{R}^+$ is metrically convex because (X,p^s) where $p^s(x,y) = |x-y|$ is the metric derived from the partial metric p.

Lemma 1.5. [2] *Let C be a closed subset of set X, where (X,d) is a metrically convex metric space. If $x \in C$ and $y \in X\backslash C$ then there exists z in ∂C, (the boundary of C), such that $d(x,y) = d(x,z) + d(z,y)$.*

We now define a metrically convex partial metric space.

Definition 1.6. A partial metric space (X,p) is said to be a metrically convex partial metric space if (X,p^s) is a metrically convex metric space, where p^s is the metric derived from the partial metric p as defined in (1.3).

We deduce the following lemma.

Lemma 1.7. *Let C be a non-empty subset of a metrically convex partial metric space (X,p) which is closed in (X,p^s). If $x \in C$ and $y \in X\backslash C$, then there exists a point $z \in \partial C$ (the boundary of C) such that*

$$p(x,y) + p(z,z) = p(x,z) + p(z,y).$$

Proof. From Definition 1.6, the partial metric space (X,p) is metrically convex if (X,p^s) is a metrically convex metric space. From Lemma 1.5, this means that if $x \in C$ and $y \in X\backslash C$ then there exists z in ∂C, (the boundary of C), such that $p^s(x,y) = p^s(x,z) + p^s(z,y)$. Using (1.3), this means

$$\begin{aligned}
&p^s(x,y) = p^s(x,z) + p^s(z,y)\\
&\Rightarrow 2p(x,y) - p(x,x) - p(y,y) = 2p(x,z) - p(x,x)\\
&\qquad - p(z,z) + 2p(z,y) - p(z,z) - p(y,y)\\
&\Rightarrow 2p(x,y) = 2p(x,z) + 2p(z,y) - 2p(z,z)\\
&\Rightarrow p(x,y) + p(z,z) = p(x,z) + p(z,y)\\
&\Rightarrow p(x,z) + p(z,y) = p(x,y) + p(z,z).
\end{aligned}$$

□

Lemma 1.8. *Given a metrically convex partial metric space (X,p) and let $x,y,z \in X$, with $z \in \text{seg}[x,y]$. Then $p(x,z) \le p(x,y)$.*

Proof. According to Lemma 1.7, in a metrically convex partial metric space (X,p), with $x,y,z \in X$ and $z \in \text{seg}[x,y]$, we have:

$$\begin{aligned}
p(x,z) + p(z,y) &= p(x,y) + p(z,z)\\
\Rightarrow p(x,z) + p(z,y) - p(z,z) &= p(x,y).
\end{aligned}$$

From P0 of Definition 1.1, we have $p(z,y) - p(z,z) \ge 0$. Hence, $p(x,z) \le p(x,y)$. □

2. The Partial Hausdorff Metric

We now describe the partial Hausdorff metric [4].

Let $CB^p(X)$ be a family of all non-empty, closed and bounded subsets of a partial metric space (X,p), induced by the partial metric p. The closedness of the sets is taken from the topology (X,τ_p). Furthermore, the set A is said to be a bounded subset in (X,p) if there exists $x_0 \in X$ and $M \ge 0$ such that for all $a \in A$, we have $a \in B_p(x_0,M)$.

Definition 2.1. [4]
For all $A,B \in CB^p(X)$ and $x \in X$, we define
(i) $p(x,A) = \inf\{p(x,a), a \in A\}$,
(ii) $\delta_p(A,B) = \sup\{p(a,B) : a \in A\}$,
(iii) $\delta_p(B,A) = \sup\{p(b,A) : b \in B\}$,
(iv) $H_p(A,B) = \max\{\delta_p(A,B), \delta_p(B,A)\}$.

The mapping $H_p : CB^p \times CB^p \to [0,+\infty)$ is called the partial Hausdorff metric.

Remark 2.1. Let (X,p) be a partial metric space and $A \in CB^p(X)$, then $A = \bar{A}$ where $\bar{A}$ denotes the closure of A with respect to the partial metric p. In this case, $a \in A$ if and only if $p(a,A) = p(a,a)$.

We now state some properties of mappings δ_p and H_p.

Lemma 2.2. [4] *Let (X,p) be a partial metric space. For any $A,B \in CB^p(X)$ we have*
(i) $\delta_p(A,A) = \sup\{p(a,a) : a \in A\}$,
(ii) $\delta_p(A,A) \leq \delta(A,B)$,
(iii) $\delta_p(A,B) = 0$ *implies that* $A \subseteq B$,
(h1) $H_p(A,A) \leq H_p(A,B)$,
(h2) $H_p(A,B) = H_p(B,A)$,
(h3) $H_p(A,B) = 0$ *implies* $A = B$.

We will also utilize the following lemma in our proofs.

Lemma 2.3. [4] *Let (X,p) be a partial metric space, $A,B \in CB^p(X)$ and $K > 1$. For any $a \in A$, there exists $b = b(a) \in B$ such that $p(a,b) \leq K H_p(A,B)$.*

The following definitions will be useful in the course of our proofs.

Let $T : C \to CB^p(X)$ be a multivalued mapping, where $C \subseteq X$. We say that T is a *self mapping* if $C = X$, otherwise T is called a *non-self mapping*. If there is an element $x \in C$ such that $x \in Tx$, we say that x is a *fixed point* of T in X.

Suppose we have two multivalued mappings $S,T : C \to CB^p(X)$, with $C \subseteq X$. If there is an element $x \in C$ such that $x \in (Sx \cap Tx)$ then we say that x a *common fixed point* of S and T in X.

Aydi et al. proved the following theorem.

Theorem 2.4. [4] *Let (X,p) be a complete partial metric space. If $T : X \to CB^p(X)$ is a multivalued mapping such that for all $x,y \in X$ we have*

$$H_p(Tx,Ty) \leq kp(x,y) \tag{2.1}$$

where $k \in (0,1)$, then T has a fixed point.

3. Common Fixed Point of Multivalued Contraction

We start with proving a generalization of Theorem 2.4 which will then be used to develop Theorem 4.1.

Theorem 3.1. *Let (X,p) be a complete partial metric space. If $S,T : X \to CB^p(X)$ are multivalued mappings such that for all $x,y \in C$ we have*

$$H_p(Tx,Sy) \leq kp(x,y) \tag{3.1}$$

where $k \in (0,1)$, then there is a common fixed point of S and T in X.

Proof. Let $x_0 \in X$ and $x_1 = Tx_0$. From Lemma 2.3 with $K = \frac{1}{\sqrt{k}}$ there exists $x_2 \in Sx_1$ such that
$p(x_1,x_2) \leq \frac{1}{\sqrt{k}} H_p(Tx_0,Sx_1)$.
As $H_p(Tx_0,Sx_1) \leq kp(x_0,x_1)$, it means
$p(x_1,x_2) \leq \sqrt{k}p(x_0,x_1)$. For $x_2 \in Sx_1$ there exists $x_3 \in Tx_2$ such that

$$\begin{aligned} p(x_2,x_3) &\leq \frac{1}{\sqrt{k}} H_p(Sx_1,Tx_2) \\ &= \frac{1}{\sqrt{k}} H_p(Tx_2,Sx_1) \\ &\leq \sqrt{k}p(x_1,x_2). \end{aligned}$$

Continuing this process we obtain a sequence $\{x_n\}$ in X such that $x_{2n+1} \in Tx_{2n}$ and $p(x_{2n},x_{2n+1}) \leq \sqrt{k}p(x_{2n-1},x_{2n})$ for all $n \geq 1$. Similarly $x_{2n+2} \in Sx_{2n+1}$ and $p(x_{2n+1},x_{2n+2}) \leq \sqrt{k}p(x_{2n},x_{2n+1})$ for all n. In general this means

$$p(x_n,x_{n+1}) \leq \sqrt{k}p(x_{n-1},x_n) \text{ for all } n \geq 1. \tag{3.2}$$

From equation (3.2), and using mathematical induction we get

$$p(x_n,x_{n+1}) \leq (\sqrt{k})^n p(x_0,x_1) \text{ for all } n \in \mathbb{N}. \tag{3.3}$$

Using (1.2) and (3.3), for any $m \in \mathbb{N}$ we have

$$\begin{aligned} p(x_n,x_{n+m}) &\leq p(x_n,x_{n+1}) + p(x_{n+1},x_{n+2}) \\ &\qquad + \cdots + p(x_{n+m-1},x_{n+m}) \\ &\leq (\sqrt{k})^n p(x_0,x_1) + (\sqrt{k})^{n+1} p(x_0,x_1) \\ &\qquad + \cdots + (\sqrt{k})^{n+m-1} p(x_0,x_1) \\ &= \left((\sqrt{k})^n + (\sqrt{k})^{n+1} + \cdots + (\sqrt{k})^{n+m-1}\right) p(x_0,x_1) \\ &\leq \left((\sqrt{k})^n + (\sqrt{k})^{n+1} + \cdots\right) p(x_0,x_1) \\ &= \frac{(\sqrt{k})^n}{1-\sqrt{k}} p(x_0,x_1). \end{aligned}$$

As $0 < k < 1$ we have $0 < \sqrt{k} < 1$, hence,

$$\lim_{n,m\to\infty} p(x_n,x_{n+m}) = 0 < +\infty. \tag{3.4}$$

Hence, following Definition 1.2, $\{x_n\}$ is a Cauchy sequence and converges to $x^\star \in X$ because (X,p) is complete. Furthermore,

$$p(x^\star,x^\star) = \lim_{n\to\infty} p(x_n,x^\star) = \lim_{n,m\to\infty} p(x_n,x_m) = 0. \tag{3.5}$$

Since from the assumption, $H_p(Tx_{2n},Sx^\star) \leq kp(x_{2n},x^\star)$, we therefore have

$$\lim_{n\to\infty} H_p(Tx_{2n},Sx^\star) = 0. \tag{3.6}$$

Because $x_{2n+1} \in Tx_{2n}$, we have

$$p(x_{2n+1},Sx^\star) \leq \delta_p(Tx_{2n},Sx^\star) \leq H_p(Tx_{2n},Sx^\star). \tag{3.7}$$

Taking $n \to \infty$ in (3.7) and applying (3.5) and (3.6) we get

$$\lim_{n\to\infty} p(x_{2n+1}, Sx^\star) = \lim_{n\to\infty} H_p(Tx_{2n}, Sx^\star) = 0$$
$$\Rightarrow p(x^\star, Sx^\star) = 0 = p(x^\star, x^\star)$$
$$\Rightarrow x^\star \in Sx^\star. \tag{3.8}$$

This shows that $x^\star$ is a fixed point of S. Using a similar argument we conclude that $x^\star$ is also a fixed point of T. □

Alghamdi et al. introduced the notion of multivalued non-self almost contractions for metric spaces as follows.

Definition 3.2. [1] Let (X,d) be a metric space and K a nonempty subset of X. A map $T : K \to CB(X)$ is called a multivalued almost contraction if there exist a constant $k \in (0,1)$ and some $L > 0$ such that

$$H(Tx,Ty) \le kd(x,y) + Ld(y,Tx) \text{ for all } x,y \in K.$$

Aydi et al. [3] proved the following theorem for almost contractions in metric-like spaces.

Theorem 3.3. ([3]) *Let (X,σ) be a complete metric-like space and C a nonempty closed subset of X such that if $x \in C$ and $y \notin C$, then there exists a point $z \in \partial C$ (the boundary of C) such that*

$$\sigma(x,y) = \sigma(x,z) + \sigma(z,y).$$

Suppose that $T : C \to CB^\sigma(X)$ is a multivalued almost contraction, that is

$$H_\sigma(Tx,Ty) \le k\sigma(x,y) + L\sigma(y,Tx),$$

with $k \in (0,1)$, and some $L \ge 0$ such that $(1+L)(k+2L) < 1$. If also $x \in \partial C$ implies $Tx \subset C$, then there exists $x^\star \in C$ such that $x^\star \in Tx^\star$, that is T has a fixed point in C.

In this paper we modify the Theorem 3.3 so that it can apply to a pair of multivalued mappings in a metrically convex partial metric space.

4. Common Fixed Point of Multivalued Almost Contractions

We intend to provide a proof for the following theorem.

Theorem 4.1. *Let (X,p) be a complete metrically convex partial metric space and C a nonempty closed subset of X with $\partial C \neq \emptyset$. Suppose that $S,T : C \to CB^p(X)$ are multivalued almost contractions, that is,*

$$H_p(Tx,Sy) \le kp(x,y) + Lp(y,Tx) \text{ and}$$
$$H_p(Tx,Sy) \le kp(x,y) + Lp(x,Sy) \text{ for all } x,y \in C \tag{4.1}$$

with $k \in (0,1)$ and some $L \ge 0$ such that $2\left(\sqrt{k} + \frac{L}{\sqrt{k}}\right) < 1$. If $x \in \partial C$ (the boundary of C) implies $Sx \subset C$ and $Tx \subset C$, then there exists a point $x^\star \in C$ such that $x^\star \in Sx^\star \cap Tx^\star$, that is $x^\star$ is a common fixed point of S and T. Furthermore $p(x^\star, x^\star) = 0$.

Proof. We construct a sequence $x_n \in C$ in the following way: Let $x_0 \in C$ and $y_1 \in Tx_0$. If $y_1 \in C$, let $x_1 = y_1$. If $y_1 \notin C$, then by Lemma 1.7, there exists $x_1 \in \partial C$ such that

$$p(x_0,x_1) + p(x_1,y_1) = p(x_0,y_1) + p(x_1,x_1).$$

Because C is closed, $x_1 \in \partial C \Rightarrow x_1 \in C$. Thus by Lemma 2.3, there exists $y_2 \in Sx_1$ such that

$$p(y_1,y_2) \le KH_p(Tx_0,Sx_1), K > 1$$

If $y_2 \in C$, let $x_2 = y_2$. However if $y_2 \notin C$ by Lemma 1.7 there exists $x_2 \in \partial C$ such that

$$p(x_1,x_2) + p(x_2,y_2) = p(x_1,y_2) + p(x_2,x_2).$$

Therefore $x_2 \in C$. From Lemma 2.3, there exists $y_3 \in Tx_2$ such that $p(y_2,y_3) \le KH_p(Sx_1,Tx_2)$.

Continuing in this way, we construct two sequences $\{x_n\}$ and $\{y_n\}$ such that
(i) $y_{2n+1} \in Tx_{2n}, y_{2n+2} \in Sx_{2n+1}$;
(ii) $p(y_{2n+1},y_{2n+2}) \le KH_p(Tx_{2n},Sx_{2n+1})$;
(iii) $p(y_{2n},y_{2n+1}) \le KH_p(Sx_{2n-1},Tx_{2n}), n \ge 1$;
(iv) If $y_n \in C$, then $x_n = y_n$;
(v) If $y_n \notin C$, then $x_n \neq y_n$ and $x_n \in \partial C$ such that

$$p(x_{n-1},x_n) + p(x_n,y_n) = p(x_{n-1},y_n) + p(x_n,x_n).$$

Let $P = \{x_i \in x_n : x_i = y_i, i = 1,2,\dots\}$ and $Q = \{x_i \in x_n : x_i \neq y_i, i = 1,2,\dots\}$. From the construction of proof we note that if $x_n \in Q$ for some n, then $x_{n-1}, x_{n+1} \in P$.

Now for $n \ge 2$, three cases must be considered. Let us relate the $K > 1$ in Theorem 3.1 to $k \in (0,1)$ in Theorem 4.1 by $K = \frac{1}{\sqrt{k}} > 1$.

Case 1: $(x_n,x_{n+1}) \in P \times P$. Then $y_n = x_n, y_{n+1} = x_{n+1}$.
If n is even, that is, if $n = 2m$ for some $m \in \mathbb{N}$, we have

$$\begin{aligned}
p(x_n,x_{n+1}) &= p(x_{2m},x_{2m+1}) \\
&= p(y_{2m},y_{2m+1}) \\
&\le KH_p(Sx_{2m-1},Tx_{2m}) \\
&= \frac{1}{\sqrt{k}}H_p(Tx_{2m},Sx_{2m-1}) \\
&\le \frac{1}{\sqrt{k}}\left[kp(x_{2m},x_{2m-1}) + Lp(x_{2m-1},Tx_{2m})\right] \\
&= \sqrt{k}p(x_{2m},x_{2m-1}) + \frac{L}{\sqrt{k}}p(x_{2m-1},Tx_{2m}) \\
&\le \sqrt{k}p(x_{2m-1},x_{2m}) + \frac{L}{\sqrt{k}}p(x_{2m-1},x_{2m-1}) \\
&\le \left(\sqrt{k} + \frac{L}{\sqrt{k}}\right)p(x_{2m-1},x_{2m}).
\end{aligned}$$

The last step emanates from P0 of Theorem 1.1, which implies $p(x_{2m-1},x_{2m}) \ge p(x_{2m-1},x_{2m-1})$.

Hence, we have

$$p(x_n,x_{n+1}) \le hp(x_{n-1},x_n) \tag{4.2}$$

where $h = \left(\sqrt{k} + \dfrac{L}{\sqrt{k}}\right)$.

We get the same result (4.2) if we assume n is odd.

Case 2: $(x_n, x_{n+1}) \in P \times Q$. Then $x_n = y_n$, $x_{n+1} \neq y_{n+1}$. From the construction of proof, we have $x_{n+1} \in \text{seg}[x_n, y_{n+1}]$. Using Lemma 1.8, this means:

$$\begin{aligned} p(x_n, x_{n+1}) &\leq p(x_n, y_{n+1)} \\ &= p(y_n, y_{n+1}) \\ &\leq hp(x_{n-1}, x_n) \end{aligned}$$

using the argument in Case 1.

Case 3: $(x_n, x_{n+1}) \in Q \times P$. Then we have $x_n \in \partial C$, $x_n \in \text{seg}[x_{n-1}, y_n]$, $x_{n+1} = y_{n+1}$ and $x_{n-1} = y_{n-1}$. If n is even, that is, if $n = 2m$ for some $m \in \mathbb{N}$, we have $y_n = y_{2m} \in Sx_{2m-1}$. Hence, using (1.2) and Case 1, we have

$$\begin{aligned} p(x_n, x_{n+1}) &= p(x_{2m}, x_{2m+1}) \\ &\leq p(x_{2m}, y_{2m}) + p(y_{2m}, x_{2m+1}) \\ &= p(x_{2m}, y_{2m}) + p(y_{2m}, y_{2m+1}) \\ &\leq p(x_{2m}, y_{2m}) + hp(x_{2m-1}, x_{2m}). \end{aligned}$$

We now use the following facts:

(i) As $2h < 1$, we have $hp(x_{2m-1}, x_{2m}) < p(x_{2m-1}, x_{2m})$;

(ii) Because $x_{2m} \in \text{seg}[x_{2m-1}, y_{2m}]$, using Lemma 1.8 we get $p(x_{2m}, y_{2m}) \leq p(x_{2m-1}, y_{2m})$ and $p(x_{2m-1}, x_{2m}) \leq p(x_{2m-1}, y_{2m})$.

Therefore

$$\begin{aligned} p(x_n, x_{n+1}) &= p(x_{2m}, x_{2m+1}) \\ &\leq p(x_{2m}, y_{2m}) + p(x_{2m-1}, x_{2m}) \\ &\leq p(x_{2m-1}, y_{2m}) + p(x_{2m-1}, y_{2m}) \\ &= 2p(x_{2m-1}, y_{2m}) \\ &= 2p(y_{2m-1}, y_{2m}) \\ &\leq 2hp(x_{2m-2}, x_{2m-1}). \end{aligned}$$

We get the same result if we work with n is odd.

Thus, for the above three cases we deduce that, for $n \geq 2$, we have

$$p(x_n, x_{n+1}) \leq 2h \max\{p(x_{n-2}, x_{n-1}), p(x_{n-1}, x_n)\} \quad (4.3)$$

where $h = \sqrt{k} + \dfrac{L}{\sqrt{k}}$ and $2h < 1$. Let

$$\alpha := (2h)^{-1/2} \max\{p(x_{n-2}, x_{n-1}), p(x_{n-1}, x_n)\}$$

Following the method by Assad and Kirk [2], it can be shown by induction that for $n \geq 1$

$$p(x_n, x_{n+1}) \leq \alpha(2h)^{n/2}. \quad (4.4)$$

Let $n > m$. Then from (1.2) we have

$$\begin{aligned} p(x_m, x_n) &\leq \sum_{i=m}^{n-1} p(x_i, x_{i+1}) \\ &\leq \sum_{i=m}^{\infty} p(x_i, x_{i+1}) \\ &\leq \sum_{i=m}^{\infty} \alpha(2h)^{i/2} \\ &= \alpha(2h)^{m/2} \frac{1}{1-(2h)^{1/2}}. \end{aligned}$$

Taking the limit $n, m \to \infty$ we get

$$\lim_{n,m\to\infty} p(x_m, x_n) = 0. \quad (4.5)$$

Hence, $\{x_n\}$ is a Cauchy sequence in C. Because C is a closed subset of the complete partial metric space (X, p), there is $x^\star \in C$ such that $x_n \to x^\star$, that is

$$\lim_{n\to\infty} p(x_n, x^\star) = p(x^\star, x^\star) = \lim_{n,m\to\infty} p(x_n, x_m) = 0. \quad (4.6)$$

We now show that $x^\star$ is a fixed point of T and S.

Consider the subsequence $\{x_{n_j}\}$ of $\{x_n\}$ each of whose terms is in P. This means $x_{n_j} = y_{n_j}$ for $j = 1, 2, \ldots$. Consider the case where n_j is odd, that is $n_j = 2m_j + 1$ for some $m_j \in \mathbb{N}$. Thus by (i) we have $x_{2m_j+1} = y_{2m_j+1} \in Tx_{2m_j}$. Using (1.2), have for all $j = 1, 2, \ldots$

$$\begin{aligned} p(x^\star, Sx^\star) &= \inf_{z\in Sx^\star}\{p(x^\star, z)\} \\ &\leq \inf_{z\in Sx^\star}\{p(x^\star, x_{2m_j+1}) + p(x_{2m_j+1}, z)\} \\ &= p(x^\star, x_{2m_j+1}) + \inf_{z\in Sx^\star}\{p(x_{2m_j+1}, z)\} \\ &= p(x^\star, x_{2m_j+1}) + p(x_{2m_j+1}, Sx^\star) \\ &\leq p(x^\star, x_{2m_j+1}) + H_p(Tx_{2m_j}, Sx^\star) \\ &\leq p(x^\star, x_{2m_j+1}) + kp(x_{2m_j}, x^\star) + Lp(x^\star, Tx_{2m_j}). \end{aligned} \quad (4.7)$$

Note that as $x_{2m_j+1} \in Tx_{2m_j}$ we have

$$\lim_{j\to\infty} p(x^\star, Tx_{2m_j}) \leq \lim_{j\to\infty} p(x^\star, x_{2m_j+1}) = 0.$$

We take limits $j \to \infty$ of (4.7) and note that

$$\begin{aligned} \lim_{j\to\infty} p(x^\star, x_{2m_j+1}) &= \lim_{j\to\infty} p(x_{2m_j}, x^\star) \\ &= \lim_{j\to\infty} p(x^\star, Tx_{2m_j}) \\ &= 0. \end{aligned}$$

Hence, using Remark 2.1 we have

$$p(x^\star, Sx^\star) = 0 = p(x^\star, x^\star) \Rightarrow x^* \in Sx^*, \quad (4.8)$$

making $x^\star$ a fixed point of S. Using a similar argument, we can prove that $x^\star$ is also a fixed point of T. □

Remark 4.1 Theorem 4.1 is valid when we have $S = T$.

When we set $T = f$, where f is a single valued mapping we get the following corollary:

Corollary 4.2. *Let (X, p) be a complete metrically convex partial metric space and Ca nonempty subset of X with $\partial C \neq \emptyset$. Suppose that $f : C \to X$ is a single-valued almost contraction and $S : C \to CB^p(X)$ is a multivalued almost contraction, that is,*

$$H_p(fx,Sy) \le kp(x,y)+Lp(y,fx) \text{ and}$$
$$H_p(fx,Sy) \le kp(x,y)+Lp(x,Sy) \text{ for all } x,y \in C \tag{4.9}$$

with $k \in (0,1)$ and some $L \ge 0$ such that $2\left(\sqrt{k}+\frac{L}{\sqrt{k}}\right) < 1$. If $x \in \partial C$ (the boundary of C) implies $Sx \subset C$ and $fx \in C$, then there exists a point $x^\star \in C$ such that $x^\star = f(x^\star) \subset Sx^\star$, that is $x^\star$ is a common fixed point of f and S. Furthermore $p(x^\star, x^\star) = 0$.

If we set $T = f, S = g$, where both f and g are single valued mappings we get the following corollary:

Corollary 4.3. *Let (X, p) be a complete metrically convex partial metric space and C a nonempty closed subset of X. Suppose that $f, g : C \to X$ are single-valued almost contractions, that is,*

$$H_p(fx,gy) \le kp(x,y)+Lp(y,fx) \text{ and}$$
$$H_p(fx,gy) \le kp(x,y)+Lp(x,gy) \text{ for all } x,y \in C \tag{4.10}$$

with $k \in (0,1)$ and some $L \ge 0$ such that $2\left(\sqrt{k}+\frac{L}{\sqrt{k}}\right) < 1$. If $x \in \partial C$ (the boundary of C) implies $fx, gx \in C$, then there exists a point $x^\star \in C$ which is a common fixed point of f and g. Furthermore $p(x^\star, x^\star) = 0$.

5. Example

Let $X = \mathbb{R}^+, C = [0,1]$. Define $S : T : C \to CB^p(X)$ by $Sx = Tx = \left\{0, \frac{1}{3+x}\right\}$, for all $x \in C$ and $p(x,y) = |x-y|$. We note that the condition $x \in \partial C$ implies $Sx \cap Tx \in C$ holds. We show that S and T are multivalued almost contractions. In fact we have for all $x, y \in [0,1]$,

$$H_p(Tx,Sy) = H_p(Tx,Ty) = \max\{\partial_p(Tx,Ty), \partial_p(Ty,Tx)\}. \tag{5.1}$$

From Definition 2.1, we have

$$\begin{aligned}\partial_p(Tx,Ty) &= \max\{p(a,Ty), a \in Tx\}\\ &= \max\left\{p(0,Ty), p\left(\frac{1}{3+x},Ty\right)\right\}.\end{aligned} \tag{5.2}$$

But

$$\begin{aligned}p(0,Ty) &= \min\left\{p(0,0), p\left(0,\frac{1}{3+y}\right)\right\}\\ &= \min\left\{0,\frac{1}{3+y}\right\}\\ &= 0.\end{aligned}$$

Hence (5.2) becomes

$$\begin{aligned}\partial_p(Tx,Ty) &= p\left(\frac{1}{3+x},Ty\right)\\ &= \min\left\{p\left(\frac{1}{3+x},0\right), p\left(\frac{1}{3+x},\frac{1}{3+y}\right)\right\}\\ &= \min\left\{\frac{1}{3+x},\frac{|x-y|}{(3+y)(3+x)}\right\}.\end{aligned} \tag{5.3}$$

Because $x, y \in [0,1]$, we have

$$\frac{|x-y|}{(3+y)(3+x)} \le \frac{2}{(3+y)(3+x)} \le \frac{2}{3}\times\frac{1}{3+x}. \tag{5.4}$$

Hence (5.3) becomes

$$\partial_p(Tx,Ty) = \frac{|x-y|}{(3+y)(3+x)}. \tag{5.5}$$

Similarly

$$\partial_p(Ty,Tx) = \frac{|x-y|}{(3+y)(3+x)}. \tag{5.6}$$

Therefore (5.1) becomes

$$\begin{aligned}H_p(Tx,Sy) &= \max\left\{\partial_p(Tx,Ty), \partial_p(Ty,Tx)\right\}\\ &= \frac{|x-y|}{(3+y)(3+x)}\\ &\le \frac{1}{9}|x-y|\\ &= \frac{1}{9}p(x,y).\end{aligned} \tag{5.7}$$

Hence we have

$$H_p(Tx,Sy) \le \frac{1}{9}p(x,y)+Lp(y,Tx)$$

and

$$H_p(Tx,Sy) \le \frac{1}{9}p(x,y)+Lp(x,S)$$

where $L \in \left(0,\frac{1}{18}\right)$.

We have shown that for the given data $S, T : C \to CB^p(X)$ are multivariate almost contractions for $k = \frac{1}{9}$ and $L \in \left(0,\frac{1}{18}\right)$. Note that the partial metric space (X, p) is metrically convex and Lemma 1.7 is verified for $z = 1$ if $y \ge 1$. Moreover the additional condition $2\left(\sqrt{k}+\frac{L}{\sqrt{k}}\right) \le 1$ is also satisfied. Then S and T are multivalued almost contractions that satisfy all assumptions of Theorem 4.1 and they have a common fixed point $z = 0$ with $p(z,z) = 0$.

6. Conclusion

We have shown that Theorem 2.4 by Aydi et al. [4] which was developed for single multivalued mappings can be extended to pairs of multivalued mappings. We have also shown that Theorem 3.3 by Aydi et al. [3] which was developed for a non-self multivalued mapping in metric-like spaces can be extended to apply for pairs of multivalued non-self mappings in partial metric spaces.

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