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V_k-Super vertex magic labeling of graphs

Sivagnanam Mutharasu[1] and Duraisamy Kumar[2]*

Abstract

Let G be a simple graph with p vertices and q edges. A V-super vertex magic labeling is a bijection $f: V(G) \cup E(G) \rightarrow \{1,2,\ldots,p+q\}$ such that $f(V(G)) = \{1,2,\ldots,p\}$ and for each vertex $v \in V(G)$, $f(v) + \sum_{u \in N(v)} f(uv) = M$ for some positive integer M. A V_k-super vertex magic labeling (V_k-SVML) is a bijection $f: V(G) \cup E(G) \rightarrow \{1,2,\ldots,p+q\}$ with the property that $f(V(G)) = \{1,2,\ldots,p\}$ and for each $v \in V(G)$, $f(v) + w_k(v) = M$ for some positive integer M. A graph that admits a V_k-SVML is called V_k-super vertex magic. This paper contains several properties of V_k-SVML in graphs. A necessary and sufficient condition for the existence of V_k-SVML in graphs has been obtained. Also, the magic constant for E_k-regular graphs has been obtained. Further, we study some classes of graphs such as cycles, complement of cycles, prism graphs and a family of circulant graphs which admit V_2-SVML.

Keywords

Vertex magic total labeling, super vertex magic total labeling, V_k-super vertex magic labeling, E_k-regular graphs, circulant graphs.

AMS Subject Classification

05C78.

[1,2] *Department of Mathematics, C. B. M. College, Coimbatore - 641 042, Tamil Nadu, India.*

***Corresponding author**: [2] dkumarcnc@gmail.com; [1] skannanmunna@yahoo.com

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Contents

1. Introduction

Throughout this paper, we consider only finite, simple and undirected graphs. The set of vertices and edges of a graph $G(p,q)$ will be denoted by $V(G)$ and $E(G)$ respectively, $p = |V(G)|$ and $q = |E(G)|$. For graph theoretic terminology, we follow [2].

A graph labeling is a mapping or a function that carries a set of graph elements (usually vertices and/or edges) into a set of numbers (usually integers). Lot of labelings have been defined and studied by many authors and an excellent survey of graph labeling can be found in [1].

In 2002, MacDougall et al. [3] introduced the notion of vertex magic total labeling (VMTL) in graphs. A VMTL of the graph G is a bijection $f: V(G) \cup E(G) \rightarrow \{1,2,\ldots,p+q\}$ such that for each vertex $v \in V(G)$, $f(v) + \sum_{u \in N(v)} f(uv) = M$ for some positive integer M [3]. This constant is called as the magic constant of VMTL of G. They studied some basic properties of vertex magic graphs and showed some families of graphs having a VMTL.

In 2004, MacDougall et al. [4] defined the super vertex-magic total labeling (SVMTL) in graphs. They call a VMTL is super if $f(V(G)) = \{1,2,\ldots,p\}$. In this labeling, the smallest labels are assigned to the vertices.

This paper generalizes the definition of SVMTL and define a new labeling called V_k-super vertex magic labeling. Let $G(V,E)$ be a graph and k be an integer such that $1 \leq k \leq \mathrm{diam}(G)$. For $e \in E(G)$, we define $E_k(e)$ as the set of all vertices which are at a distance at most k from e. Also $E_k(v)$ denotes the set of all edges which are at a distance at most k from v. Note that if uv is an edge, then the vertices u and v are at distance 1 from the edge uv. The graph G is said to be E_k-regular with regularity r if and only if $|E_k(e)| = r$ for some integer $r \geq 1$ and for all $e \in E(G)$. Note that all nontrivial graphs are E_1-regular. For a vertex $v \in V(G)$, we denote $w_k(v) = \sum_{e \in E_k(v)} f(e)$. Consider the following graph $G(V,E)$, where $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

and $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$.

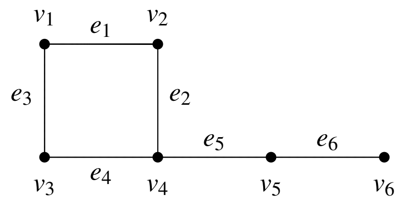


Fig 1: G

Table 1 gives $E_k(v)$ and $E_k(e)$ for $k = 2$.

$E_2(v)$	$E_2(e)$
$E_2(v_1) = \{e_1, e_2, e_3, e_4\}$	$E_2(e_1) = \{v_1, v_2, v_3, v_4\}$
$E_2(v_2) = \{e_1, e_2, e_3, e_4, e_5\}$	$E_2(e_2) = \{v_1, v_2, v_3, v_4, v_5\}$
$E_2(v_3) = \{e_1, e_2, e_3, e_4, e_5\}$	$E_2(e_3) = \{v_1, v_2, v_3, v_4\}$
$E_2(v_4) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$	$E_2(e_4) = \{v_1, v_2, v_3, v_4, v_5\}$
$E_2(v_5) = \{e_2, e_4, e_5, e_6\}$	$E_2(e_5) = \{v_2, v_3, v_4, v_5, v_6\}$
$E_2(v_6) = \{e_5, e_6\}$	$E_2(e_6) = \{v_4, v_5, v_6\}$

Table 1

A V_k-super vertex magic labeling (V_k-SVML) is a bijection $f : V(G) \cup E(G) \to \{1, 2, \dots, p+q\}$ with the property that $f(V(G)) = \{1, 2, \dots, p\}$ and for every $v \in V(G)$, $f(v) + w_k(v) = M$ for some positive integer M. This constant is called as the magic constant of V_k-SVML of G. A graph that admits a V_k-SVML is called V_k-super vertex magic (V_k-SVM).

This paper contains several properties of V_k-SVML in graphs. A necessary and sufficient condition for the existence of V_k-SVML in graphs has been obtained. Also, the magic constant for E_k-regular graphs has been obtained. Further, we study some classes of graphs such as cycles, complement of cycles, prism graphs and a family of circulant graphs which admit V_2-SVML.

2. Main Results

In this section, we obtain some basic properties of V_k-SVML.

Let G be a connected graph of order $p (\geq 2)$. Suppose $E_k(u) = E_k(v)$ for two different vertices u and v of G. Then $f(u) + w_k(u) \neq f(v) + w_k(v)$ for any V_k-SVML f of G (since f is one to one). In this case, G does not admit V_k-SVML and hence the next result follows.

Lemma 2.1. *Let G be a connected graph of order $p (\geq 2)$. If $E_k(u) = E_k(v)$ for some $u, v \in V(G)$ $(u \neq v)$, then the graph G does not admit V_k-SVML.*

Corollary 2.2. *The star graph S_n does not admit V_k-SVML for $k \geq 2$.*

If a graph G admits V_k-SVML, then $1 \leq k \leq \text{diam}(G)$ (otherwise, $E_k(u) = E_k(v)$ for any two different vertices $u, v \in V(G)$).

Definition 2.3. *In a graph G, a vertex of degree $|V(G)| - 1$ is called a full vertex of G.*

Corollary 2.4. *Let G be a connected graph of order $p (\geq 2)$ and u be a full vertex of G. Then G does not admit V_k-SVML for $k \geq 3$.*

Lemma 2.5. *If a graph $G(p,q)$ is V_k-SVM and G is E_k-regular with regularity r, then the magic constant is given by $M = \frac{p+1}{2} + rq + \frac{rq(q+1)}{2p}$.*

Proof. Let f be a V_k-SVML of G with the magic constant M. Then $f(V(G)) = \{1, 2, \dots, p\}$, $f(E(G)) = \{p+1, p+2, \dots, p+q\}$ and $M = f(v) + w_k(v)$ for all $v \in V(G)$. By summing over all $v \in V(G)$, $pM = \sum_{v \in V(G)} f(v) + \sum_{v \in V(G)} w_k(v)$. The first sum is $\frac{p(p+1)}{2}$ and the second sum is $\sum_{v \in V(G)} w_k(v) = \sum_{v \in V(G)} \sum_{e \in E_k(v)} f(e) = r \sum_{e \in E(G)} f(e) = r(pq) + \frac{rq(q+1)}{2}$, where the second equality uses from E_k-regular that each edge is in exactly r of the sets $E_k(v)$. Thus $pM = \frac{p(p+1)}{2} + r(pq) + \frac{rq(q+1)}{2}$ and hence $M = \frac{p+1}{2} + rq + \frac{rq(q+1)}{2p}$. ☐

In Lemma 2.5, we give the magic constant only for E_k-regular graphs which admit V_k-SVML for $k \geq 1$. MacDougall et. al obtained the following result which gives the magic constant of V-SVML for any graph.

Lemma 2.6. *[4] If G has a super-vertex magic total labeling, then $M = 2q + \frac{(p+1)}{2} + \frac{q(q+1)}{p}$.*

When $k = 1$, we have $r = |E_1(e)| = 2$ for all $e \in E(G)$. Thus if we put $k = 1$ in Lemma 2.5, then it gives the proof of Lemma 2.6.

Lemma 2.7. *For $k \geq 2$, there dose not exist a tree, which is E_k-regular and V_k-SVM.*

Proof. Let T be a tree and $\text{diam}(T) = d (\geq 3)$. Let $P = u_0 u_1 \dots u_{d-1} u_d$ be a path of length d. Then $u_0 u_1$ and $u_{d-1} u_d$ must be pendent edges. When $k = d$, we have $E_k(u_0) = E_k(u_d)$ and hence T is not V_k-SVM. Also, $k \leq d-1$, we have $E_k(u_1 u_2) > E_k(u_0 u_1)$ and hence T is not E_k-regular. Thus $\text{diam}(T) \leq 2$ and hence T is a star graph. By Corollary2.2, the star graph S_n does not admit V_k-SVML for $k \geq 2$. ☐

Theorem 2.8. *If $G(p,q)$ is a connected E_k-regular graph with regularity r, then*
$M \geq \frac{7p-5}{2}$ if $k = 1$ and $M \geq \frac{(p+1)(r+1)}{2} + rp$ if $k \geq 2$.

Proof. For $k = 1$, we have $r = 2$. Since G is connected, $q \geq p-1$. Thus by Lemma 2.5, $M \geq \frac{p+1}{2} + 2(p-1) + (p-1)$ $= \frac{7p-5}{2}$ (This is already proved in [4]).
Let $k \geq 2$. If $q = p-1$, then G must be a tree and hence by

Lemma 2.7, there dose not exist a tree T, which is E_k-regular and V_k-SVM. Hence assume that $q \geq p$. By Lemma 2.5, $M \geq \frac{p+1}{2} + rp + \frac{r(p+1)}{2} = \frac{(p+1)(r+1)}{2} + rp$. □

Remark 2.9. For $k \geq 2$, the lower bound for the magic constant M obtained in Theorem 2.8 is sharp. For example, consider the following V_2-SVML of C_5 (see Figure 2).

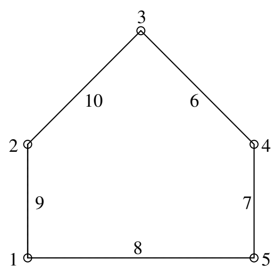


Fig 2: V_2**-SVML of** C_5

Note that the cycle C_5 is E_2-regular with regularity 4. Here the magic constant $M = 35$. In Theorem 2.8, we proved that $M \geq 35$.

Theorem 2.10. *Let G be a (p,q) graph and g be a bijection from $E(G)$ onto $\{p+1, p+2, \ldots, p+q\}$. Then g can be extended to a V_k-SVML of G if and only if $\{w_k(u)/u \in V(G)\}$ consists of p sequential integers.*

Proof. Assume that $\{w_k(u)/u \in V(G)\}$ consists of p sequential integers. Let $t = \min\ \{w_k(u)/u \in V(G)\}$. Define $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ as $f(xy) = g(xy)$ for $xy \in E(G)$ and $f(x) = t + p - w_k(x)$. Then $f(E(G)) = \{p+1, p+2, \ldots, p+q\}$ and $f(V(G)) = \{1, 2, \ldots, p\}$ (since $\{w_k(x) - t : x \in V(G)\}$ is a set of consecutive integers). Hence f is V_k-SVML with $M = t + p$.
Conversely, suppose g can be extended to a V_k-SVML f of G with a magic constant M. Since $f(u) + w_k(u) = M$ for every $u \in V(G)$, we have $w_k(u) = M - f(u)$. Thus $\{w_k(u)/u \in V(G)\} = \{M-p, M-p+1, \ldots, M-1\}$, which is a set of p consecutive integer. □

3. V_2-SVML of cycles and prisms

In this section, we identified some classes of graphs such as cycles, complement of cycles, prism graphs and a family of circulant graphs, which admit V_2-SVML.

Lemma 3.1. *[5] For any integers a and b, we have* $gcd(a,b) = gcd(b,a) = gcd(\pm a, \pm b) = gcd(a, b-a) = gcd(a, b+a)$.

By Lemma 2.1, the cycles C_3 and C_4 are not V_2-SVM.

Theorem 3.2. *For an integer $n (\geq 5)$, the cycle C_n is V_2-SVM if and only if n is odd.*

Proof. Suppose there exists a V_2-SVML f of C_n. Since $|E_2(e)| = r = 4$ for all $e \in E(C_n)$, by taking $k = 2$, $p = q = n$ and $r = 4$ in Lemma 2.5, we get $M = \frac{13n+5}{2}$. Since M is an integer, n must be odd.
Conversely, assume that n is odd and $n \geq 5$. Let $V(C_n) = \{a_i : 1 \leq i \leq n\}$ and $E(C_n) = \{a_i a_{i \oplus_n 1} : 1 \leq i \leq n\}$, where the operation $\oplus_n$ stands for addition modulo n.
Case A: Suppose $n = 4\ell + 1$ for some integer $\ell \geq 1$.
Define a function $f : V(C_n) \cup E(C_n) \to \{1, 2, \ldots, 2n\}$ as follows: $f(a_i) = i - 3$ when $4 \leq i \leq n$ and $f(a_i) = (n-3) + i$ when $1 \leq i \leq 3$; $f(a_i a_{i \oplus_n 1}) = [(i-1)\ell \oplus_n 1] + n$, where $(i-1)\ell \oplus_n 1$ is the positive residue when $(i-1)\ell + 1$ divides n.
Now we prove that $f(E(C_n)) = \{n+1, n+2, \ldots, 2n\}$. By taking $b = n$ and $a = \ell$ in Lemma 3.1, we get $gcd(\ell, n) = gcd(\ell, 4\ell+1) = gcd(\ell, 3\ell+1) = gcd(\ell, 2\ell+1) = gcd(\ell, \ell+1) = gcd(\ell, 1) = 1$. Thus ℓ is a generator for the finite cyclic group $(Z_n, \oplus_n)$ and hence $f(E(C_n)) = \{n+1, n+2, \ldots, 2n\}$.
Claim 1: $w_2(a_i) = 26\ell + 12 - i$ for $4 \leq i \leq n$.
Case i: Suppose $i = 4x$ for some $1 \leq x \leq \ell$. Now $w_2(a_i) = f(a_{i-2}a_{i-1}) + f(a_{i-1}a_i) + f(a_i a_{i+1}) + f(a_{i+1}a_{i+2})$.
Since $f(a_{i-2}a_{i-1}) = [(i-3)\frac{n-1}{4} \oplus_n 1] + n = [nx - x - \frac{3n}{4} + \frac{3}{4} \oplus_n 1] + n = [-x - \frac{3n}{4} + \frac{3}{4} \oplus_n 1] + n = [-x - 3\ell \oplus_n 1] + n$, by the definition of f, we have $w_2(a_i) = [-x - 3\ell \oplus_n 1] + [-x - 2\ell \oplus_n 1] + [-x - \ell \oplus_n 1] + [-x \oplus_n 1] + 4n$.
Since $1 \leq x \leq \ell$, in the above four terms(brackets), all the residues are not positive, we have
$w_2(a_i) = 4n + [n - x - 3\ell + 1] + [n - x - 2\ell + 1] + [n - x - \ell + 1] + [n - x + 1]$. By taking $n = 4\ell + 1$, we get $w_2(a_i) = 26\ell + 12 - i$.
Case ii: Suppose $i = 4x + 1$ for some $1 \leq x \leq \ell$. In this case, $w_2(a_i) = [-x - \frac{n}{2} + \frac{1}{2} \oplus_n 1] + n + [-x - \frac{n}{4} + \frac{1}{4} \oplus_n 1] + n + [-x \oplus_n 1] + n + [-x + \frac{n}{4} - \frac{1}{4} \oplus_n 1] + n. \ = 4n + [-x - 2\ell \oplus_n 1] + [-x - \ell \oplus_n 1] + [-x \oplus_n 1] + [-x + \ell \oplus_n 1]$. Since $1 \leq x \leq \ell$, the first three terms are not positive, we have $w_2(a_i) = 4n + [n - x - 2\ell \oplus_n 1] + [n - x - \ell \oplus_n 1] + [n - x \oplus_n 1] + [-x + \ell \oplus_n 1] = 26\ell + 12 - i$. Similarly, we can show that $w_2(a_i) = 26\ell + 12 - i$ when $i = 4x + 2$ and $i = 4x + 3$.
Claim 2: $w_2(a_i) = (2\ell + 1)11 - i$ for $1 \leq i \leq 3$.
Consider the vertex a_1. $w_2(a_1) = f(a_{n-1}a_n) + f(a_n a_1) + f(a_1 a_2) + f(a_2 a_3)$. Since $f(a_{n-1}a_n) = [(n-2)\frac{(n-1)}{4} \oplus_n 1)] + n = [(4\ell - 1)\frac{(n-1)}{4} \oplus_n 1] + n = [-2\ell \oplus_n 1] + n$, we have $w_2(a_1) = [-2\ell \oplus_n 1] + [-\ell \oplus_n 1] + [\ell \oplus_n 1] + 4n + 1$. Here, the first two terms are not positive. Thus $w_2(a_1) = [n - 2\ell + 1] + [n - \ell + 1] + [\ell \oplus_n 1] + 4n + 1 = (2\ell + 1)11 - 1$. Similarly, we can prove $w_2(a_2) = (2\ell + 1)11 - 2$ and $w_2(a_3) = (2\ell + 1)11 - 3$.
Note that $\ell = \frac{n-1}{4}$. Thus by Claim 1, $f(a_i) + w_2(a_i) = i - 3 + 26\ell + 12 - i = \frac{13n+5}{2} = M$ for $4 \leq i \leq n$. Also by Claim 2, $f(a_i) + w_2(a_i) = (n-3) + i + 11(2\ell + 1) - i = \frac{13n+5}{2} = M$ for $i = 1, 2, 3$.
Case B: Suppose $n = 4\ell + 3$ for some integer $\ell \geq 1$.
Define $f : V(C_n) \cup E(C_n) \to \{1, 2, \ldots, 2n\}$ as follows: $f(a_i) = n - i$ when $1 \leq i \leq n-1$ and $f(a_n) = n$; $f(a_i a_{i \oplus_n 1}) = [(i-1)(\ell+1) \oplus_n 1] + n$, where $[(i-1)(\ell+1) \oplus_n 1] + n$ is the positive residue $(i-1)(\ell+1) + 1$ divides n. By Lemma 3.1, $gcd(\ell+1, n) = gcd(\ell+1, 4\ell+3) = gcd(\ell+1, 3\ell+2) = gcd(\ell+1, 2\ell+1) = gcd(\ell+1, \ell) = gcd(\ell, \ell+1) = gcd(\ell, 1) =$

1. Hence $\ell+1$ is a generator for the finite cyclic group $(Z_n, \oplus_n)$ and hence $f(E(C_n)) = \{n+1, n+2, \ldots, 2n\}$. As proved in Case A, we can prove that the above labeling is a V_2-SVML with magic constant $M = \frac{13n+5}{2}$. □

Theorem 3.3. *Let $G = \overline{C_n}$ be the complement of the cycle C_n, where $n(\geq 5)$ is an integer. Then G is V_2-SVM with the magic constant $\frac{n^4-2n^3-n^2-14n}{8}$.*

Proof. Define $f : V(\overline{C_n}) \cup E(\overline{C_n}) \to \{1, 2, \ldots, \frac{n^2-n}{2}\}$ as follows: First we label the n edges $\{a_1a_3, a_2a_4, \ldots, a_na_2\}$ by $f(a_{i\oplus n-1}, a_{i\oplus 1}) = n+i$ for $1 \leq i \leq n$. And the remaining $\frac{n^2-3n}{2} - n$ edges are randomly labeled with the labels $\{2n+1, 2n+2, \ldots, \frac{n^2-n}{2}\}$. The vertices are labeled as $f(a_i) = i$. Note that for each vertex a_i, the only edge with label $n+i$, is not in $E_2(a_i)$. Thus for each a_i with $1 \leq i \leq n$, we have $f(a_i) + w_2(a_i) = i + [\frac{n^4-2n^3-n^2-6n}{8} - (n+i)] = \frac{n^4-2n^3-n^2-14n}{8}$. □

Definition 3.4. *Let D_n be a prism graph of order n with $|V(D_n)| = 2n$ and $|E(D_n)| = 3n$. We take $V(D_n) = \{a_i, b_i / 1 \leq i \leq n\}$ and $E(D_n) = \{a_ib_i / 1 \leq i \leq n\} \cup \{a_ia_{i\oplus_n 1}, b_ib_{i\oplus_n 1} / 1 \leq i \leq n\}$.*

Theorem 3.5. *For an integer $n(\geq 3)$, the prism D_n is V_2-SVM if and only if n is even.*

Proof. Suppose there exists a V_2-SVML f of D_n with the magic constant M. Since $|E_2(e)| = r = 6$ for all $e \in E(D_n)$, by taking $k = 2$, $p = 2n$, $q = 3n$ and $r = 6$ in Lemma 2.5, we get $M = \frac{65n+10}{2}$. Since M is an integer, n must be even.
Conversely, assume that n is even. Let $V(D_n) = \{a_i, b_i / 1 \leq i \leq n\}$ and $E(D_n) = \{a_ib_i / 1 \leq i \leq n\} \cup \{a_ia_{i\oplus_n 1}, b_ib_{i\oplus_n 1} / 1 \leq i \leq n\}$. Define $f : V(D_n) \cup E(D_n) \to \{1, 2, \ldots, 5n\}$ as follows:
$f(a_i) = n + \frac{n}{2} - \frac{i-1}{2}$ if i is odd; The range is given by $\{n+1, n+2, \ldots, n+\frac{n}{2}\}$,
$f(a_i) = 2n - (\frac{i}{2} - 2)$ if $i \geq 4$ and i is even; $\{n+\frac{n}{2}+2, n+\frac{n}{2}+3, \ldots, 2n\}$,
$f(a_2) = n + \frac{n}{2} + 1$; $\{n+\frac{n}{2}+1\}$,
$f(b_i) = \frac{i+1}{2}$ if i is odd; $\{1, 2, \ldots, \frac{n}{2}\}$,
$f(b_i) = \frac{n}{2} + \frac{i}{2} - 1$ if $i \geq 4$ and i is even; $\{\frac{n}{2}+1, \ldots, n-1\}$,
$f(b_2) = n$; $\{n\}$,
$f(a_ib_i) = 2n + \frac{i+1}{2}$ if i is odd; $\{2n+1, 2n+2, \ldots, 2n+\frac{n}{2}\}$,
$f(a_ib_i) = 2n + \frac{n}{2} + \frac{i}{2}$ if i is even; $\{2n+\frac{n}{2}+1, \ldots, 3n\}$,
$f(a_ia_{i\oplus_n 1}) = 3n + \frac{n}{2} - \frac{i-1}{2}$ if i is odd; $\{3n+1, \ldots, 3n+\frac{n}{2}\}$,
$f(b_ib_{i\oplus_n 1}) = 4n - (\frac{i}{2} - 1)$ if i is even; $\{3n+\frac{n}{2}+1, 3n+\frac{n}{2}+2, \ldots, 4n\}$,
$f(a_ia_{i\oplus_n 1}) = 4n + \frac{i}{2}$ if i is even; $\{4n+1, 4n+2, \ldots, 4n+\frac{n}{2}\}$,
$f(b_ib_{i\oplus_n 1}) = 5n - \frac{i-1}{2}$ if i is odd; $\{4n+\frac{n}{2}+1, \ldots, 5n\}$.
It is easily seen that f is a V_2-SVML with the magic constant $M = \frac{65n+10}{2}$. □

Let Γ be a finite group with e as the identity. A generating set of Γ is a subset A such that every element of Γ can be expressed as a product of finitely many elements of A. Assume that $e \notin A$ and $a \in A$ implies $a^{-1} \in A$ (A is called as symmetric generating set). A Cayley graph is a graph $G = (V, E)$, where $V(G) = \Gamma$ and $E(G) = \{(x, a)/x \in V(G), a \in A\}$, denoted by $Cay(\Gamma, A)$. Since A is a generating set for Γ, G is a connected regular graph of degree $|A|$. When $\Gamma = Z_n$, the corresponding Cayley graph is called as a circulant graph, denoted by $Cir(n, A)$.

In Lemma 2.5, we find the magic constant of E_k-regular graphs which admit V_k-SVML. When $A = \{1, 2, n-1, n-2\}$, the circulant graph $Cir(n, A)$ is not E_2-regular. In the next result, we find the magic constant of this family of circulant graphs.

Theorem 3.6. *For an integer $n \geq 7$, the graph $G = Cir(n, \{1, 2, n-1, n-2\})$ is V_2-SVM with the magic constant $M = 27n+7$.*

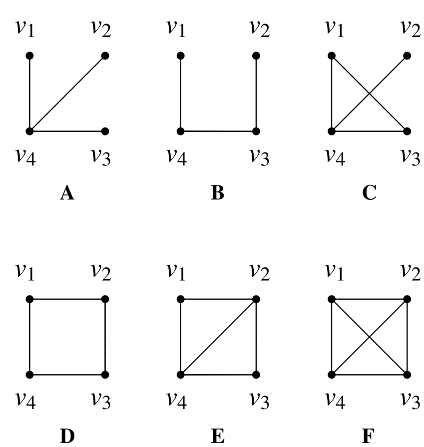
Proof. Let $V(G) = \{a_1, a_2, \ldots, a_n\}$ and $E(G) = \{a_ia_{i\oplus_n 1}, a_ia_{i\oplus_n 2} : 1 \leq i \leq n\}$. Define $f : V(G) \cup E(G) \to \{1, 2, \ldots, 3n\}$ as follows:
$f(a_i) = i - 4$ for $5 \leq i \leq n$;
$f(a_i) = n + i - 4$ for $1 \leq i \leq 4$;
$f(a_ia_{i\oplus_n 1}) = n + i$ for $1 \leq i \leq n$ and
$f(a_ia_{i\oplus_n 2}) = 3n + 1 - i$ for $1 \leq i \leq n$.
Let $v \in V(G)$. Suppose $v = a_i$ for some integer i with $5 \leq i \leq n$. Then $f(a_i) + w_2(a_i) = f(a_i) + f(a_{i-3}a_{i-2}) + f(a_{i-2}a_{i-1}) + f(a_{i-1}a_i) + f(a_ia_{i\oplus_n 1}) + f(a_{i\oplus_n 1}a_{i\oplus_n 2}) + f(a_{i\oplus_n 2}a_{i\oplus_n 3}) + f(a_{i-4}a_{i-2}) + f(a_{i-3}a_{i-1}) + f(a_{i-2}a_i) + f(a_{i-1}a_{i\oplus_n 1}) + f(a_ia_{i\oplus_n 2}) + f(a_{i\oplus_n 1}a_{i\oplus_n 3}) + f(a_{i\oplus_n 2}a_{i\oplus_n 4})$
$= [i-4] + [n+i-3] + [n+i-2] + [n+i-1] + [n+i] + [n+i+1] + [n+i+2] + [3n+1-(i-4)] + [3n+1-(i-3)] + [3n+1-(i-2)] + [3n+1-(i-1)] + [3n+1-i] + [3n+1-(i+1)] + [3n+1-(i+2)] = 27n+7 = M$. Similarly, we can prove that $f(a_i) + w_2(a_i) = 27n+7$ for $i = 1, 2, 3, 4$. □

4. Some Results on V-SVML

In this section, we obtained some results on V-SVML.

Lemma 4.1. *Any connected graph on four vertices is not V-SVM.*

Proof. Suppose there exists a V-SVML with magic constant M. All the non-isomorphic connected graphs on four vertices are given below.



By Lemma 2.6, $M = 2q + \frac{p+1}{2} + \frac{q(q+1)}{p}$. For the graphs A, B, C and D, the magic constant is not an integer and hence they are not V-SVM.
Suppose the graph E admits a V-SVML, say f. Then $M = 20$, $f(V(E)) = \{1,2,3,4\}$ and $f(E(E)) = \{5,6,7,8,9\}$.
Note that the vertices v_1 and v_3 are of degree two and $f(v_1)$, $f(v_3) \in \{1,2,3,4\}$. Since $M = 20$, both $w(v_1)$ and $w(v_3)$ must be greater than or equal to 16, which is not possible since $f(E(E))$= {5, 6, 7, 8, 9}. Thus E is not E-SVM.
Next, we consider the graph F. Suppose the graph F admits V-SVML, say f. Then $M = 25$, $f(V(F)) = \{1,2,3,4\}$ and $f(E(F)) = \{5,6,7,8,9,10\}$. With out loss of generality, we take $f(v_1) = 1$, $f(v_2) = 2$, $f(v_3) = 3$ and $f(v_4) = 4$. Consider

$f(v)$	incident edges of v	possible edge labelings	$w(v)$
$f(v_1) = 1$	v_1v_2, v_1v_3, v_1v_4	$\{(10,9,5),(10,8,6),(9,8,7)\}$	$w(v_1) = 24$
$f(v_2) = 2$	v_2v_1, v_2v_3, v_2v_4	$\{(10,8,5),(10,7,6),(9,8,6)\}$	$w(v_2) = 23$
$f(v_3) = 3$	v_3v_1, v_3v_2, v_3v_4	$\{(10,7,5),(9,8,5),(9,7,6)\}$	$w(v_3) = 22$
$f(v_4) = 4$	v_4v_1, v_4v_2, v_4v_3	$\{(10,6,5),(9,7,5),(8,7,6)\}$	$w(v_1) = 21$

Table 2

the vertex v_4. Suppose the edges incident with v_4 receive the labels $\{10,6,5\}$. In this case the edges incident with v_1 cannot be labeled with $\{(10,9,5),(10,8,6)\}$. Since v_1 is adjacent with v_4, one of the edge incident with v_1 must be labeled with 10 or 6 or 5. Thus the the edges incident with v_1 cannot be labeled with $\{(9,8,7)\}$. Hence f is not SVML, a contradiction. We can get the same contradictions when the edges incident with v_4 receive the labels $\{9,7,5\}$ and $\{8,7,6\}$. □

Theorem 4.2. *Let G be a (p,q) graph. If $q = p+1$, then G is not V-SVM.*

Proof. Suppose $q = p+1$. Then by Lemma 2.6, $M = 2q + \frac{p+1}{2} + \frac{q(q+1)}{p} = 2(p+1) + \frac{p+1}{2} + \frac{(p+1)(p+2)}{p} = 3p + 5 + \frac{1}{2} + \frac{p}{2} + \frac{2}{p}$, which is an integer only when $p = 4$. Thus by Lemma 4.1 G is not V-SVM. □

Corollary 4.3. *For $n \geq 4$, the cycle with one chord is not V-SVM.*

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