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Strong convergence of the one step implicit iteration for α-demicontractive mappings

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Abstract

A new class of demicontractivity called α-demicontractivity is introduced by L. Maruster and S. Maruster [6] in which they proved strong convergence theorem using Mann iteration process in Hilbert space. In this paper, we have introduced one step implicit iteration process for α-demicontractive mapping and proved strong convergence theorem in Hilbert space.

Keywords

Fixed point, α-Demicontractive mapping, Strong convergence, Hilbert space, One step implicit iteration.

AMS Subject Classification

47H09, 47H10.

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Contents

1. Introduction

Let H be a Hilbert space and K is a closed convex subset of H. Suppose $T : K \to K$ is a (possibly nonlinear) mapping and the set of fixed points of T is denoted by $F(T)$. The mapping T is said to be

- Nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \ \ \forall x, y \in K;$$

- Pseudocontractive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \ \ \forall x, y \in K;$$

- Strongly pseudocontractive if there exists $k \in (0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \ \ \forall x, y \in K;$$

- Demicontractive if $F(T) \neq \phi$ and

$$\|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2, \ \ \forall x \in K \text{ and } p \in F(T);$$

- Hemicontractive if $F(T) \neq \phi$ and

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \|x - Tx\|^2, \ \ \forall x \in K \text{ and } p \in F(T).$$

It is well known that every nonexpansive mapping is a pseudocontractive mapping but the converse need not be true. The class of pseudocontractive mappings with fixed points is a subclass of the class of hemicontractive mappings and every demicontractive mappings with fixed points is hemicontractive mappings. Thus it is clear that the class of demicontractive mapping is more general than the class of mappings defined above. In the previous literatures of demicontractive type mappings, it is obsereved that the iterative methods for this type of mappings were less developed other than the mappings defined above. Therefore it is interesting to the researchers to find the iterative methods for approximation of fixed points of demicontractive mappings.

2. Preliminaries

In this section we illustrate the famous Mann and Ishikawa iteration schemes and also with their error schemes. To approximate the fixed points of demicontractive mappings, Mann type iteration is very significantly used. Recall that the Mann [7] iteration formula is given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \ \ x_0 \in K,$$

where $\{\alpha_n\}$ is real sequence in [0,1] satisfying some appropriate conditions.

The following iteration process is given by Ishikawa [4] and formerly known as Ishikawa iteration process.

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n \end{cases}$$

for each $n \geq 1$ where the two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1] satisfy some appropriate conditions.

The following iteration is due to Liu [5]. The sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n + v_n \end{cases}$$

for each $n \geq 1$ where $\{\alpha_n\}, \ \{\beta_n\} \in [0,1]$ satisfying appropriate conditions and $\sum \|u_n\| < \infty, \ \sum \|v_n\| < \infty$, known as Ishikawa iteration process with errors and the sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n + u_n, \end{cases}$$

for each $n \geq 1$ where $\{\alpha_n\} \in [0,1]$ satisfying appropriate conditions and $\sum \|u_n\| < \infty$, known as Mann iteration process with errors.

In 1998, Xu [11] introduced the following iteration process:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = a_n x_n + b_n T y_n + c_n u_n, \\ y_n = a'_n x_n + b'_n T x_n + c'_n v_n \end{cases}$$

for each $n \geq 1$ where $\{u_n\}, \ \{v_n\}$ are the bounded sequences in K and $\{a_n\}, \ \{b_n\}, \ \{c_n\}, \ \{a'_n\}, \ \{b'_n\}$ and $\{c'_n\}$ are the sequences in [0,1] such that $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$ for each $n \geq 1$ is known as Ishikawa iteration with errors in the sense of Xu.

The following Theorem is proved by Chidume and Moore [2].

Theorem 2.1. *[2] Let K be a compact convex subset of a real Hilbert space H and $T : K \to K$ be a continous hemicontractive mapping. Let $\{a_n\}, \ \{b_n\}, \ \{c_n\}, \ \{a'_n\}, \ \{b'_n\}$ and $\{c'_n\}$ be the real sequences in [0,1] satisfying the following conditions:*

(i) $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$

(ii) $\lim b_n = \lim b'_n = 0$;

(iii) $\sum c_n < \infty, \ \sum c'_n < \infty$;

(iv) $\sum \alpha_n \beta_n = \infty$ *and* $\sum \alpha_n \beta_n \delta_n < \infty$, *where* $\delta_n := \|Tx_n - Ty_n\|^2$;

(v) $0 \leq \alpha_n \leq \beta_n < 1$ *for each* $n \geq 1$, *where* $\alpha_n := b_n + c_n$ *and* $\beta_n := b'_n + c'_n$

For arbitrary $x_1 \in K$, the sequence $\{x_n\}$ defined by

$$\begin{cases} x_{n+1} = a_n x_n + b_n T y_n + c_n u_n, \\ y_n = a'_n x_n + b'_n T x_n + c'_n v_n \end{cases}$$

for each $n \geq 1$ where $\{u_n\}, \ \{v_n\}$ are the arbitrary sequences in K. Then $\{x_n\}$ converges strongly to a fixed point of T.

In 2007, Rafiq [8] proved the following result by defining the Mann type implicit iteration process for hemicontractive mapping in real Hilbert space.

Theorem 2.2. *[8] Let K be a compact convex subset of a real Hilbert space H and $T : K \to K$ be a hemicontractive mapping. Let $\{\alpha_n\}$ be a real sequence in [0,1] satifying $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. For arbitrary $x_0 \in K$, the sequence $\{x_n\}$ is defined by*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T x_n,$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

Song [9] proved the following theorem in real Banach space.

Theorem 2.3. *[9] Suppose K is a compact convex subset of a real Banach space E and $T : K \to K$ is a continuous pseudocontractive mapping such that $F(T) \neq \phi$. Assume that $\{\alpha_n\} \subset (0,1)$ is a real sequence satisfying the condition* $\lim_{n\to\infty} \alpha_n = 0$. *Let $x_0 \in K$ and let $\{x_n\}$ be defined by*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T x_n, \ n \geq 0.$$

Then $\{x_n\}$ strongly converges to a fixed point of T.

In 2013, Hussain et al. [3] introduced the following Mann-type implicit iteration associated with a family of continuous hemicontractive mappings to prove a strong convergence result in Hilbert spaces.

$$\begin{cases} x_0 \in K, \\ x_n = \alpha_n x_{n-1} + \sum_{i=1}^{m} \beta_n^i T_i x_n \end{cases} \tag{2.1}$$

for each $n \geq 1$ where $\alpha_n, \ \beta_n^i \in [0,1], \ i = 1,2,\ldots,m$, are such that $\alpha_n + \sum_{i=1}^{m} \beta_n^i = 1$ and some appropriate conditions hold.

Theorem 2.4. *[3] Let K be a compact convex subset of a real Hilbert space H and $T_i : K \to K, \ i = 1,2,\ldots m$, be a family of continuous hemicontractive mappings. Let $\alpha_n, \ \beta_n^i \in [0,1]$ be such that $\alpha_n + \sum_{i=1}^{m} \beta_n^i = 1$ and satisfying $\{\alpha_n\}, \ \beta_n^i \subset [\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1), \ i = 1,2,\ldots,m$. Then, for arbitrary $x_0 \in K$, the sequence $\{x_n\}$ defined by (2.1) converges strongly to a common fixed point in $\bigcap_{i=1}^{m} F(T_i) \neq \phi$.*

In 2011, L. Maruster and S. Maruster [6] introduced a new concept of demicontractivity called α-demicontractivity in Hilbert spaces and proved strong convergence theorem for this mapping.

Definition 2.5. *[6] Let K be a closed convex subset of Hilbert space H, then a mapping from $T : K \to K$ is said to be α-demicontractive if for some $\alpha \geq 1$,*

$$\|Tx - \alpha p\|^2 \leq \|x - \alpha p\|^2 + k\|x - Tx\|^2, \;\; k \in (0,1) \quad (2.2)$$

for all $x \in K$ and $p \in F(T)$.

Remark 2.6. *[6] If T is α-demicontractive then αp is a fixed point of T for all $p \in F(T)$ such that αp remains in the domain $D(T)$ of T. Note that if T is demicontractive, then $F(T)$ is closed and convex, it follows that if T is both demicontractive and α-demicontractive for some $\alpha > 1$, then the line segment $(1-t)p + t\alpha p$, $t \in [0,1]$ is contained in $F(T)$ for all $p \in F(T)$ such that αp remains in the domain $D(T)$ of T.*

Recall that a mapping T is demiclosed at zero if for any sequence $\{x_n\}$ such that x_n converges weakly to p and Tx_n converges to zero strongly then $Tp = 0$.

L. Maruster and S. Maruster [6] proved the following theorem.

Theorem 2.7. *[6] Let $T : K \to K$ be nonlinear mapping, where K is a closed convex subset of a real Hilbert space H. Suppose the following conditions are satisfied:*

(i) $I - T$ is demiclosed at zero;

(ii) T is demicontractive with constant k with $F(T) \neq \phi$.

Suppose also that T is α-demicontractive for some $\alpha > 1$. Then the sequence $\{x_n\}$ defined by $x_{n+1} = (1-t_n)x_n + t_n Tx_n$, $x_0 \in K$, where $\{t_n\}$ is a real sequence in [0,1] with $0 < a \leq t_n \leq b < 1-k$, converges strogly to a point in $F(T)$.

Above facts inspired us to introduce following one step implicit iteration process. The sequence $\{x_n\}$ is defined by

$$\begin{cases} x_0 \in K, \\ x_n = \gamma_n x_{n-1} + \delta_n Tx_n, \; n \geq 1 \end{cases} \quad (2.3)$$

where $\gamma_n,\ \delta_n \in [0,1]$ such that $\gamma_n + \delta_n = 1$.

The purpose of this paper is to prove the strong convergence theorem using the iteartion process defined by (2.3) for α-demicontractive mappings in Hilbert spaces.

3. Main results

In the sequel, we need the following lemmas.

Lemma 3.1. *[10] Suppose that $\{\rho_n\}$, $\{\sigma_n\}$ are two sequences of nonnegative numbers such that, for some real number $N_0 \geq 1$,*

$$\rho_{n+1} \leq \rho_n + \sigma_n$$

for all $n \geq N_0$. Then we have the following:

(1) If $\sum \sigma_n < \infty$, then $\lim \rho_n$ exists.

(2) If $\sum \sigma_n < \infty$ and $\{\rho_n\}$ has a subsequence converging to zero, then $\lim \rho_n = 0$.

Lemma 3.2. *[3] Let H be a Hilbert space. Then, for all $x, x_i \in H$, $i = 1,2,.....,m$,*

$$\|\gamma x + \sum_{i=1}^{m} \delta^i x_i\|^2 = \gamma\|x\|^2 + \sum_{i=1}^{m} \delta^i \|x_i\|^2 - \sum_{i=1}^{m} \gamma\delta^i \|x_i - x\|^2 - \sum_{\substack{i,j=1 \\ i\neq j}}^{m} \delta^i \delta^j \|x_i - x_j\|^2$$

where $\gamma,\ \delta^i \in [0,1]$, $i = 1,2,\ldots,m$ and $\gamma + \sum_{i=1}^{m} \delta^i = 1$

Theorem 3.3. *Let K be a closed convex subset of a real Hilbert space H, $T : K \to K$ is demicontractive with constant k and $I - T$ is demiclosed at 0 and $F(T) \neq \phi$. Suppose also that T is α-demicontractive for some $\alpha > 1$. Then for suitable x_0, the sequence $\{x_n\}$ defined by (2.3) where $\gamma_n,\ \delta_n \in [0,1]$ such that $\gamma_n + \delta_n = 1$ and satisfying $\gamma_n,\ \delta_n \subset [\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$ converges strongly to a point in $F(T)$.*

Proof. Let $\alpha p \in F(T)$. Using (2.3), (2.2) and Lemma 3.2, we have

$$\begin{aligned} \|x_n - \alpha p\|^2 &= \|\gamma_n x_{n-1} + \delta_n Tx_n - \alpha p\|^2 \\ &= \gamma_n \|x_{n-1} - \alpha p\|^2 + \delta_n \|Tx_n - \alpha p\|^2 \\ &\quad - \gamma_n \delta_n \|x_{n-1} - Tx_n\|^2 \\ &\leq \gamma_n \|x_{n-1} - \alpha p\|^2 + \delta_n [\|x_n - \alpha p\|^2 \\ &\quad + k\|x_n - Tx_n\|^2] - \gamma_n \delta_n \|x_{n-1} - Tx_n\|^2 \end{aligned} \quad (3.1)$$

Also, we have

$$\begin{aligned} \|x_n - Tx_n\|^2 &= \|\gamma_n x_{n-1} + \delta_n Tx_n - Tx_n\|^2 \\ &= \gamma_n^2 \|x_{n-1} - Tx_n\|^2 \end{aligned} \quad (3.2)$$

From (3.1) and (3.2), we have

$$\begin{aligned} \|x_n - \alpha p\|^2 &\leq \gamma_n \|x_{n-1} - \alpha p\|^2 + \delta_n \|x_n - \alpha p\|^2 \\ &\quad + k\delta_n \gamma_n^2 \|x_{n-1} - Tx_n\|^2 \\ &\quad - \gamma_n \delta_n \|x_{n-1} - Tx_n\|^2 \\ &\leq \gamma_n \|x_{n-1} - \alpha p\|^2 + \delta_n \|x_n - \alpha p\|^2 \\ &\quad - \gamma_n \delta_n (1 - k\gamma_n) \|x_{n-1} - Tx_n\|^2 \end{aligned} \quad (3.3)$$

From condition $\gamma_n,\ \delta_n \subset [\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$, we obtain

$$\|x_n - \alpha p\|^2 \leq \|x_{n-1} - \alpha p\|^2 - \varepsilon(1-k\varepsilon)\|x_{n-1} - Tx_n\|^2 \tag{3.4}$$

$$\begin{aligned}\varepsilon(1-k\varepsilon)\sum_{j=1}^{\infty}\|x_{j-1} - Tx_j\|^2 &\leq \sum_{j=1}^{\infty}(\|x_{j-1} - \alpha p\|^2 \\ &\quad - \|x_j - \alpha p\|^2), \\ &= \|x_0 - \alpha p\|^2\end{aligned}$$

$$\sum_{j=1}^{\infty}\|x_{j-1} - Tx_j\|^2 < \infty \tag{3.5}$$

This implies,

$$\lim_{n\to\infty}\|x_{n-1} - Tx_n\| = 0. \tag{3.6}$$

From (3.2),

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0.$$

From (3.4), (3.5) and Lemma 3.1, we get

$$\|x_n - \alpha q\| \to 0 \text{ as } n \to \infty$$

This implies, $x_n \to \alpha q$ as $n \to \infty$. This completes the proof. □

Theorem 3.4. *Let K be a closed convex subset of a real Hilbert space H, $T : K \to K$ is demicontractive with constant k and $I - T$ is demiclosed at 0 and $F(T) \neq \phi$. Suppose also that T is α-demicontractive for some $\alpha > 1$. If $P_K : H \to K$ is the projection opertor of H onto K, then the sequence $\{x_n\}$ defined iteratively by*

$$x_n = P_K(\gamma_n x_{n-1} + \delta_n Tx_n)$$

for each $n \geq 1$ where $\gamma_n,\ \delta_n \in [0,1]$ such that $\gamma_n + \delta_n = 1$ and satisfying $\gamma_n,\ \delta_n \subset [\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$ converges strongly to a point in $F(T)$.

Proof. It follows from the fact that the mapping P_K is nonexpansive (see [1]) and K is a Chebyshev subset of H, therefore P_K is a single-valued mapping. We have,

$$\begin{aligned}\|x_n - \alpha p\|^2 &= \|P_K(\gamma_n x_{n-1} + \delta_n Tx_n) - P_K \alpha p\|^2 \\ &\leq \|\gamma_n x_{n-1} + \delta_n Tx_n - \alpha p\|^2\end{aligned}$$

The rest of the proof are same as in the proof of Theorem 3.3. This completes the proof. □

Remark 3.5. *We can choose the following control parameters: $\gamma_n = \frac{1}{2} - \frac{1}{(n+2)^2}$ and $\delta_n = \frac{1}{2} + \frac{1}{(n+2)^2}$.*

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