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Star version of selective menger spaces

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Abstract
In this paper, we discuss the star version of selective Menger spaces namely R-star-Menger spaces which has been illustrated by suitable examples and investigates some of its topological properties.

Keywords
R-star Menger Space, R-separable, R-2-star Menger, Star- separable, Countable fan tightness.

AMS Subject Classification
54A08, 54A10.

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Contents

1. Introduction

Karl Menger introduced the concept of Menger spaces in 1924 [22]. A space X is *Menger* if for each sequence $(\{\mathscr{U}_n : n \in \mathbb{N}\})$ of open covers of X, there exists a sequence $(\{\mathscr{V}_n : n \in \mathbb{N}\})$ such that for every $n \in \mathbb{N}$, $\mathscr{V}_n$ is a finite subset of $\mathscr{U}_n$ and $\bigcup_{n\in\mathbb{N}} \mathscr{V}_n = X$. As generalization of Menger spaces, Kocinac in [16] and then studied in [17] almost Menger spaces that is if for each sequence $(\mathscr{U}_n : n \in \mathbb{N})$ of open covers of X, there exists a sequence $(\{\mathscr{V}_n : n \in \mathbb{N}\})$ such that for every $n \in \mathbb{N}, \mathscr{V}_n$ is a finite subset of $\mathscr{U}_n$ and $\bigcup\{\mathscr{V}_n' : n \in \mathbb{N}\}$ is a cover of X, where $\mathscr{V}_n' = \{\overline{V} : V \in \mathscr{V}_n\}$.

In this path, we introduce a star version of selective Menger spaces namely R-star-Menger spaces and study some of its topological properties. Scheepers introduced a number of combinatorial properties of a topological space weaker than separability in [25]. Selective separability and R-separability are two of them which have many interesting properties.Recently kočinac studied a lot of selection properties by combining the concepts of Scheepers and Van Dowen and got many interesting results [20]. For any topological space $X, \tau(X)$ will denote its topology. If $A \subseteq X$ and $\mathscr{U}$ is a collection of subsets of X, then the star of A with respect to $\mathscr{U}$ is denoted by $St(A,\mathscr{U})$ and defined as $St(A,\mathscr{U}) = \bigcup\{U \in \mathscr{U} : U \cap A \neq \emptyset\}$. We assume $St^1(A,\mathscr{U}) = St(A,\mathscr{U})$ and for each $k \in \mathbb{N}$ we define $St^{k+1}(A,\mathscr{U}) = St(St^k(A,\mathscr{U}),\mathscr{U})$.

Throughout this paper, ω denote the first infinite cardinal and ω_1 the first uncountable cardinal. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. Every ordinal is often viewed as a space with the usual order topology.

Definition 1.1. *A topological space X is a Menger space if for every sequence $(\mathscr{U}_n : n \in \mathbb{N})$ of open covers of X, there exists a sequence $(\mathscr{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathscr{V}_n$ is a finite subset of $\mathscr{U}_n$ and $\bigcup_{n\in\mathbb{N}} \mathscr{V}_n = X$.*

Example 1.2. *There exists a compact complement topology which has the Menger property. Consider $(\mathbb{R},\tau)$ the Euclidean space of real numbers, we define a new topology by letting $\tau^* = \{X \subseteq R / X = \phi$ or $R - X$ is compact in $(\mathbb{R},\tau)\}$. The compact sets in $(\mathbb{R},\tau)$ are closed under arbitrary intersection and finite unions, we have τ^* is a topology. If $\{O_\alpha\}$ is an open covering of R, then $R - O_{\alpha_0}$ is compact in the Euclidean topology, for any $O_{\alpha_0} \in \{O_\alpha\}$. Therefore $(\mathbb{R},\tau^*)$ is compact. Since each O_α is open in the Euclidean topology, a finite number of them must be covering $R - O_{\alpha_0}$. Therefore, $(\mathbb{R},\tau^*)$ is Menger.*

Example 1.3. *A Urysohn, first countable space which does not have the Menger property. Let $\mathbb{R}$ be the set of real numbers with the Euclidean topology τ and let $\mathbb{Q}$ be the set of rational numbers. We define τ', the pointed rational extension of $\mathbb{R}$, to be the topology generated by $\{x\} \cup (Q \cap U)$ where $x \in U$. Now $(\mathbb{R},\tau')$ is Urysohn, because $(\mathbb{R},\tau)$ is Urysohn and closures* *of open sets in τ and τ' is equal. Also, $(\mathbb{R},\tau')$ is not lindelof and it is not Menger. Hence a Urysohn, first countable space which does not have Menger property.*

Definition 1.4. *A topological space X is said to be a σ-compact space if it is the union of countable many compact sets.*

Notation 1.5. *We use the following notations:*
(a) ω- first infinite cardinal.
(b) ω_1-first uncountable cardinal.
(c) $\mathscr{S}$-Sorgenfrey line.
(d) $\mathscr{O}$- the collections of open cover of X.
(e) St-star, S^*_{fin}- star finite.
(f) SS^*_{fin}- strongly star finite.
(g) $\mathfrak{c}$-cardinality of the set of all real numbers.
(h) $\alpha = ord(A,<_1), \beta = ord(B,<_2); \alpha,\beta$-ordinal.
(i) ω_{omega}-set of all function from ω to itself.
(j) b- unbounding number.

2. Main Results

Definition 2.1. *A topological space X is called an R-star Menger space if for every sequence $\{\mathscr{D}_n : n\in\mathbb{N}\}$ of dense subsets of X and for every sequence of open cover $\{\mathscr{U}_n : n\in\mathbb{N}\}$ of X, there are points $x_n\in\mathscr{D}_n$ for each $n\in\mathbb{N}$ such that $St(\bigcup_{n\in\mathbb{N}}\{x_n\},\mathscr{U}_n) = X$.*

Definition 2.2. *A topological space X is said to be R-separable if for any sequence $\{\mathscr{D}_n : n\in\mathbb{N}\}$ of dense subsets of X, there are points $x_n\in\mathscr{D}_n$ for each $n\in\mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}}\{x_n\}$ is dense in X.*

Example 2.3. *There exists a nested interval topology which is an R-star-Menger space. Since on the open interval $X = (0,1)$ we define a topology τ by declaring open all sets of the form $\mathrm{U}_n = (0,-1/n)$, for $n = 2,3,4,...$ together with $\emptyset$ and X, thus τ is countable, first countable and separable. Then there exists a nested interval topology which is R-separable and hence which is a R-star-Menger space.*

Theorem 2.4. *A topological space X is an R-star Menger space if and only if for every sequence $\{\mathscr{D}_n : n\in\mathbb{N}\}$ of dense subsets of X and basic open cover $\mathscr{U}_B$ there are points $x_n\in\mathscr{D}_n$ for each $n\in\mathbb{N}$ such that $St(\bigcup_{n\in\mathbb{N}}\{x_n\},\mathscr{U}_B) = X$.*

Proof. If X is an R-star Menger space, then the condition is trivial. Conversely, let $\{\mathscr{D}_n : n\in\mathbb{N}\}$ be a sequence of dense subsets of X and $\{\mathscr{U}_n : n\in\mathbb{N}\}$ any sequence of open cover of X. Let $\mathscr{B}$ be an open base for $\tau(X)$. Let $\{\mathscr{U}_B = \{B\in\mathscr{B} : B\subseteq U$, for some $U\in\mathscr{U}_n\}\}$. So $\mathscr{U}_B$ is a basic open cover of X, there are points $x_n\in\mathscr{D}_n$ Such that $St(\bigcup_{n\in\mathbb{N}}\{x_n\},\mathscr{U}_B) = X$. $St(\bigcup_{n\in\mathbb{N}}\{x_n\}\mathscr{U}_n = X$. Hence X is an R-star Menger space. □

Definition 2.5. *A topological space X is said to be a selectively star Menger or M-star Menger if for every sequence $\{\mathscr{D}_n : n\in\mathbb{N}\}$ of dense subsets of and for every sequence of open cover $\{\mathscr{U}_n : n\in\mathbb{N}\}$ of X there exists a family $\{\mathscr{F}_n : n\in\mathbb{N}\}$ of finite subsets of X such that $\mathscr{F}_n\subseteq\mathscr{D}_n$ for each $n\in\mathbb{N}$ such that $St(\bigcup_{n\in\mathbb{N}} mathcal F_n,\mathscr{U}_n) = X$.*

Definition 2.6. *A subset Y of a topological space X is said to be an R-star Menger with respect to X if for every sequence $\{\mathscr{D}_n : n\in\mathbb{N}\}$ of subsets of X for each $n\in\mathbb{N}$ and for every sequence of open cover $\{\mathscr{U}_n : n\in\mathbb{N}\}$ of Y by sets open in X, there are points $x_n\in\mathscr{D}_n$ for each $n\in\mathbb{N}$ such that $Y\subseteq St(\bigcup_{n\in\mathbb{N}}\{x_n\},\mathscr{U}_n)$.*

Theorem 2.7. *If there exist two open R-star Menger subspaces A and B of a space X such that $A\cup B = X$, then X is a selectively Star-Menger space.*

Proof. Let A and B be two open R-star-Menger subspaces of X such that $X = A\cup B$. Let $\{\mathscr{U}_n : n\in\mathbb{N}\}$ be a sequence of open cover of X and $\{\mathscr{D}_n : n\in\mathbb{N}\}$ be any sequence of dense subsets of X. $\mathscr{U}^A$ and $\mathscr{U}^B$ be basic open covers of A and B respectively. Clearly, $\{(\mathscr{D}_n\cap A) : n\in\mathbb{N}\}$ is a sequence of dense subsets in A and $\{(\mathscr{D}_n\cap B) : n\in\mathbb{N}\}$ is a sequence of dense subsets in B. By R-star-Menger spaces A and B, for every $n\in\mathbb{N}$, there are points $x_n'\in(\mathscr{D}_n\cap A)$ and $x_n''\in(\mathscr{D}_n\cap B)$ such that $St(\bigcup_{n\in\mathbb{N}}\{x_n'\},\mathscr{U}^A) = A$ and $St(\bigcup_{n\in\mathbb{N}}\{x_n''\},\mathscr{U}^B) = B$. Thus, for each $n\in\mathbb{N}$, $x_n'\in\mathscr{D}_n$ and $x_n''\in\mathscr{D}_n$, that is for every $n\in\mathbb{N}$, $\mathscr{F}_n = \{x_n', x_n''\}\subseteq\mathscr{D}_n$ and $St(\bigcup_{n\in\mathbb{N}}\mathscr{F}_n,\mathscr{U}^A\cup\mathscr{U}^B) = A\cup B = X$. Let $U\in\mathscr{U}_n$. Then either $U\subseteq A$ or $U\subseteq B$ or $U\cap A\neq\phi\neq U\cap B$. If $U\subseteq A$, then U can be expressed as the union of some members of $\mathscr{U}^A$. If $U\subseteq B$, then U can be expressed as the union of some members of $\mathscr{U}^B$. Let $U\nsubseteq A, U\nsubseteq B$ and $U\subseteq A\cup B$. Then $U\cap A$ can be expressed as the union of some members of $\mathscr{U}^A$ and $U\cap B$ can be expressed as the union of some members of $\mathscr{U}^B$. Thus $U = U\cap X = U\cap(A\cup B) = (U\cap A)\cup(U\cap B)$. Every element of $\mathscr{U}_n$ contains some members of $(\mathscr{U}^A\cup\mathscr{U}^B)$. $St(\bigcup_{n\in\mathbb{N}}\mathscr{F}_n,\mathscr{U}_n) = X$. Hence X is a selectively star-Menger space. □

Corollary 2.8. *If there exist finite number of open R-star Menger subspaces $A_1, A_2, A_3, ..., A_k$ such that $\bigcup_{i=1}^k A_k = X$, then X is a selectively star-Menger space.*

Theorem 2.9. *Every clopen subspace of an R-star-Menger space is an R-star Menger space.*

Proof. Let X be a R-star-Menger space and Y be an clopen subspace of X. Let $\{D_n : n\in\mathbb{N}\}$ be a sequence of dense subsets of Y and $\{\mathscr{U}_n : n\in\mathbb{N}\}$ be a sequence of open cover of Y in the subspace Y. Now, $\{\mathscr{D}_n\cup(X-Y) : n\in\mathbb{N}\}$ is a sequence of dense subsets in X and $\mathscr{U}_n\cup\{X-Y\}$ is an open cover of X. By definition of R-star-Menger spaces of X, there are points $x_n\in\mathscr{D}_n\cup\{X-Y\}$ where $n\in\mathbb{N}$ Such that $St(\bigcup_{n\in\mathbb{N}}\{x_n\},\mathscr{U}_n\cup\{X-Y\}) = X$. Now, we choose those x_n which belong to $\mathscr{D}_n$ for each $n\in\mathbb{N}$. Then $St(\bigcup_{n\in\mathbb{N}}\{x_n\},\mathscr{U}_n) = Y$. Hence Y is an R-star Menger space. □

Theorem 2.10. *Every Menger space is an R-star-Menger space.*

Proof. Let X be a Menger space. Let $\{\mathscr{D}_n : n \in \mathbb{N}\}$ be a sequence of dense subsets of X and $\mathscr{U}_n = \{U_\alpha : \alpha \in \omega\}$ be a sequence of open cover of X. So, there exists a finite sub cover $\mathscr{U}_n^{'} = \{U_n : n \in \mathbb{N}\}$ of $\mathscr{U}_n$. With Out loss of generality, we can suppose each U_n is non-empty. For each $n \in \mathbb{N}$, we choose a $x_n \in \mathscr{D}_n \cap U_n$. And $St(\bigcup_{n\in\mathbb{N}}\{x_n\}, \mathscr{U}_n^{'}) = X$ implies $St(\bigcup_{n\in\mathbb{N}}\{x_n\}, \mathscr{U}_n) = X$. Therefore X is an R-star Menger space. □

Corollary 2.11. *Every σ-compact space is an R-star-Menger space and hence every compact space is an R-star-Menger space.*

Theorem 2.12. *Every R-Separable space is an R-star-Menger space.*

Proof. Let $\{\mathscr{D}_n : n \in \mathbb{N}\}$ be a sequence of dense subsets of X and $\{\mathscr{U}_n : n \in \mathbb{N}\}$ be a sequence of open cover of X. Since X is R-separable, there are points $x_n \in \mathscr{D}_n$ for each $n \in \mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}}\{x_n\}$ is dense in X. Hence $St(\bigcup_{n\in\mathbb{N}}\{x_n\}, \mathscr{U}_n) = X, X$ is an R-star-Menger space. □

Example 2.13. *There exists a R-star-Menger space is not an R-Separable space. Let $|X| \geq \omega_1$ and X is equipped with the co-countable topology. Then X has no countable dense subset, hence it cannot be R-Separable. Let $\{\mathscr{D}_n : n \in \mathbb{N}\}$ be a sequence of dense subsets of X and $\{\mathscr{U}_n : n \in \mathbb{N}\}$ be a sequence of open cover of X. First we take $U_0 \in \mathscr{U}_n$, clearly U_0 is of the form $U_0 = X\ C$ where $C = \{y_i : i \in \mathbb{N}\}$ is a finite subset of X. Now, $U_0 \cap \mathscr{D}_0 \neq \phi$, we select $x_0 \in U_0 \cap \mathscr{D}_0$. Since $\{\mathscr{U}_n : n \in \mathbb{N}\}$ is an sequence of open cover of X, there exists $\mathscr{V}_n \in \mathscr{U}_n$ such that $y_n \in \mathscr{V}_n \in \mathscr{U}_n$, for each $n \in \mathbb{N}$. Since each $\mathscr{D}_n$ is dense in X, we have $\mathscr{V}_n \cap \mathscr{D}_{n+1} \neq \emptyset$, for each $n \in \mathbb{N}$. We select $x_n^{'} \in \mathscr{V}_n \cap \mathscr{D}_{n+1}$ and $x_n^{'} \in \mathscr{D}_{n+1}$ for each $n \in \mathbb{N}$. Let $U_0 \subseteq St(x_0, \mathscr{U}_n), y_0 \in V_0 \subseteq St(x_0^{'}, \mathscr{U}_n), y_1 \in V_1 \subseteq St(x_1^{'}, \mathscr{U}_n)$, and so on $y_n \in V_n \subseteq St(x_n^{'}, \mathscr{U}_n)$ and $U_0 \cup \{y_1, y_2, y_3, ..., y_n, ...\} \subseteq U_0 \cup V_0 \cup V_1 \cup ... \cup V_n \subseteq St(\{x_0, x_0^{'}, x_1^{'}, ..., x_n^{'}, ...\}, \mathscr{U}_n)$. Hence $X St(\{x_0, x_0^{'}, x_1^{'}, ..., x_n^{'}, ...\}, \mathscr{U}_n)$. Hence X is an R-star Menger space.*

Theorem 2.14. *Let $f : X \to Y$ be an open continuous surjection map. If X is an R-star-Menger space, then so is also Y.*

Proof. Let $f : X \to Y$ be an open continuous surjection map. Let $\{\mathscr{E}_n : n \in \mathbb{N}\}$ be a sequence of dense subsets of Y and $\{\mathscr{V}_n : n \in \mathbb{N}\}$ be a sequence of open cover of Y. Then $\{\mathscr{D}_n = f^{-1}(\mathscr{E}_n : n \in \mathbb{N}\}$ is a sequence of dense subset of X and $\mathscr{U}_n = \{f^{-1}(V) : V \in \mathscr{V}_n\}$ is a sequence of open cover of X. Now, by the property of R-star-Menger space of X there are points $x_n \in \mathscr{D}_n$ for each $n \in \mathbb{N}$ such that $St(\bigcup_{n\in\mathbb{N}}\{x_n\}, \mathscr{U}_n) = X$. Let $y_n = f(x_n$ where $n \in \mathbb{N}$. Clearly, $y_n \in \mathscr{E}_n$ for each $n \in \mathbb{N}$. Let $y \in Y$. So there exists $x \in X$ such that $f(x) = y$. Also there exists an $n \in \mathbb{N}$ such that $x \in St(\{x_n\}, \mathscr{U}_n)$, that is, there exists $U = f^{-1}(V)$ for some $V \in \mathscr{V}_n$ such that $x \in U$ and $\{x_n\} \cap \mathscr{U}_n \neq \emptyset$, $x_n \in U$. Thus, $y \in V$ and $y_n \in V$. If $y \in St(\{y_n\}, \mathscr{V}_n)$, then $St(\bigcup_{n\in\mathbb{N}}\{y_n\}, \mathscr{V}_n) = Y$. This proves that Y is an R-star-Menger space. □

Hereafter we look at the iterative star version of R-star-Menger spaces and we assume that $n = \mathbb{N} - \{0\}$.

Definition 2.15. *For each $k \in \mathbb{N}$, a topological space X is said to be R-k-star-Menger if for every sequence $\{\mathscr{D}_n : n \in \mathbb{N}\}$ of dense subsets of X and for every sequence of open cover $\{\mathscr{U}_n : n \in \mathbb{N}\}$ of X there are points $x_n \in \mathscr{D}_n$ such that for each $n \in \mathbb{N}, St^k(\bigcup_{n\in\mathbb{N}}\{x_n\}, \mathscr{U}_n) = X$.*

Theorem 2.16. *Every R-k-star-Menger space is an R-$(k+1)$-star-Menger space.*

Proof. Directly follows from the definition. □

Definition 2.17. *Given a class (or a property) $\mathscr{P}$ of topological spaces, A space X is star-$\mathscr{P}$ if for any sequence of open cover $\{\mathscr{U}_n : n \in \mathbb{N}\}$ Of the space X, there is a subspace $Y \subseteq X$ such that $Y \in \mathscr{P}$ and for each $n \in \mathbb{N}, St(Y, \mathscr{U}_n) = X$.*

Definition 2.18. *A topological space X is said to be Star-separable if for every sequence of open cover $\{\mathscr{U}_n : n \in \mathbb{N}\}$ of X, there exists a separable subspaces Y of X such that for each $n \in \mathbb{N}, St(Y, \mathscr{U}_n) = X$.*

Theorem 2.19. *If X is star-Separable, then X is R-2-star-Menger.*

Proof. Let $\{\mathscr{U}_n : n \in \mathbb{N}\}$ be an sequence of open cover of X and $\{\mathscr{D}_n : n \in \mathbb{N}\}$ be a sequence of dense subsets of X. Since X is star-separable, there exists a separable subspace Y of X such that for each $n \in \mathbb{N}, St(Y, \mathscr{U}_n) = X$. Now, there exists a finite subset $B = \{x_n : n \in \mathbb{N}\}$ of Y such that $\overline{B}^{\tau(Y)} = Y$, hence $Y \subseteq \overline{B}^{\tau(Y)} \subseteq \overline{B}^{\tau(X)} = \overline{B}$. For each $n \in \mathbb{N}$, there exists $U_n \in \mathscr{U}_n$ such that $U_n \cap \mathscr{D}_n = \emptyset$. We take $x_n \in U_n \cap \mathscr{D}_n$. Let $x \in X$. Since for each $n \in \mathbb{N}, St(Y, \mathscr{U}_n) = X$, there exists $U \in \mathscr{U}_n$ such that $x \in U$ and $U \cap Y \neq \emptyset$, and so $U \cap Y \cap B \neq \emptyset$. Let $k \in \mathbb{N}$ be such that $x_k \in U \cap Y \cap B$. Then $U \cap U_k \neq \emptyset$, hence $U \cap St(\bigcup_{n\in\mathbb{N}}\{x_n\}, \mathscr{U}_n) \neq \emptyset$ and so $x \in St^2(\bigcup_{n\in\mathbb{N}}\{x_n\}, \mathscr{U}_n)$. Therefore, $St^2(\bigcup_{n\in\mathbb{N}}\{x_n\}, \mathscr{U}_n) = X$ and thus X is R-2-star-Menger. □

Theorem 2.20. *If X is an R-star-Menger space and Y is a compact space, then $X \times Y$ is an R-2-star-Menger space.*

Proof. Let $\{\mathscr{U}_n : n \in \mathbb{N}\}$ be a sequence of open cover of $X \times Y$ by basic open sets of $X \times Y$. For each $x \in X$, there exists a open neighborhood W_x of $xinX$ such that $W_x \times Y$ is covered by finite number of elements of $\mathscr{U}_n$, say $W_x \times Y \subseteq \bigcup\{U_k(x) \times V_k(x) : 1 \leq k \leq n_x\}$, where $W_x = \cap_{1\leq k\leq n_x} U_k(x)$. Now, $\mathscr{U}_X = \{W_x : x \in X\}$ is an open cover of X. Let $\{D_n : n \in \mathbb{N}\}$ be a sequence of dense subsets of $X \times Y, \pi_X : X \times Y \to X$ be the natural projection from $X \times Y$ to X. Then $\{\pi_X(D_n) : n \in \mathbb{N}\}$ is a sequence of dense subsets of X. By the R-star-Menger space, there are points $x_n \in \pi_X(D_n)$ for each $n \in \mathbb{N}$ and $St(\bigcup_{n\in\mathbb{N}}\{x_n\}, \mathscr{U}_X) = X$. For each x_n, we choose $(x^k, y^k) \in D_n \cap (U_k(x) \times V_k(x)), 1 \leq k \leq n_x$. Let $(x, y) \in X \times Y$. So, there exists W_{x_o} such that $x \in W_{x_o}$ and $W_{x_0} \cap (\bigcup_{n\in\mathbb{N}}\{x_n\}) \neq \emptyset$. Let $x_p \in W_{x_0} \cap (\bigcup_{n\in\mathbb{N}}\{x_n\})$ for some $p \in$

$\mathbb{R}$. So there exists $(x^{k'}, y^{k'}) \in D_n \cap (U_{k'}(x_p) \times V_{k'}(x_p)), 1 \leq k' \leq n_x$. This implies $(x^{k'}, y^{k'}) \in (U_{k'}(x_p) \times V_{k'}(x_p)), 1 \leq k' \leq n_x$ and hence $W_{x_p} \times Y \subseteq St(\bigcup_{k\in\mathbb{N}}\{(x^k, y^k)\}, \bigcup_{k=1}^{n_x}(U_k(x_p) \times V_k(x_p)) \subseteq St(\bigcup_{k\in\mathbb{N}}\{(x^k, y^k)\}, \mathscr{U}_n)$. Also, $W_{x_0} \times Y \cap W_{x_p} \times Y \neq \emptyset, W_{x_0} \times Y \subseteq St^2(\bigcup_{k\in\mathbb{N}}\{(x^k, y^k)\}, \mathscr{U}_n)$
$\Rightarrow (x,y) \in St^2(\bigcup_{k\in\mathbb{N}}\{(x^k, y^k)\}, \mathscr{U}_n)$
so, $X \times Y = St^2(\bigcup_{k\in\mathbb{N}}\{(x^k, y^k)\}, \mathscr{U}_n)$. Hence $X \times Y$ is an R-2-star-Menger space. □

Applying theorem 2.20, by mathematical induction we get the following corollary,

Corollary 2.21. *If X is a R-star-Menger space and $Y_1, Y_2, Y_3, \ldots .Y_n$ are compact spaces, then $X \times Y_1 \times Y_2 \times Y_3 \times \ldots .Y_n$ is a R-$(n+1)$-star-Menger space.*

Theorem 2.22. *If X is a star-Menger space and has the property that for any $x \in X$ for any sequence $\{U_n : n \in \mathbb{N}\}$ of subsets of X such that $x \in \bigcap_{n\in\mathbb{N}} \overline{U}_n$ and for any sequence of open cover $\{\mathscr{U}_n : n \in \mathbb{N}\}$ of X we can choose points $x_n \in U_n$ with $x \in St(\bigcup_{n\in\mathbb{N}}\{x_n\}, \mathscr{U}_n)$, then X is R-2-star-Menger.*

Proof. Let $\{\mathscr{D}_n : n \in \mathbb{N}\}$ be a sequence of dense subsets of X and $\{\mathscr{U}_n : n \in \mathbb{N}\}$ be a sequence of open cover of X. Since X is star-Menger, there exists a finite subset $F = \{x_n : n \in \mathbb{N}\}$ of X such that for each $n \in \mathbb{N}, St(F, \mathscr{U}_n) = X$. Let $L = \{L_n : n \in \mathbb{N}\}$ be a sequence of disjoint infinite subsets of $\mathbb{N}$ such that $\mathbb{N} = \bigcup_{k\in\mathbb{N}} L_n$. Now, $x_n \in \bigcap\{\overline{D}_k : k \in L_n\}$. So there are points $x_k \in D_k$ for each $k \in L_n$ such that $x_n \in St(\bigcup_{k\in L_n}\{x_k\}, \mathscr{U}_n)$ where $n \in \mathbb{N}$. Hence we have points $x_n \in D_n$ for each $n \in \mathbb{N}$. Let $x \in X$. There exists $U \in \mathscr{U}_n$ such that $x \in U$. Since $F \cap U \neq \emptyset$, there exists $x_n \in U$ for some $n \in \mathbb{N}$. Since $x_n \in St(x_k, \mathscr{U}_n)$ for some $k \in L_n$, we can choose $V \in \mathscr{U}_n$ such that $x_n \in V$ and $x_k \in V$. Then $V \subseteq St(\bigcup_{n\in\mathbb{N}}\{x_n\}, \mathscr{U}_n)$ and $U \cap V \neq \emptyset$, so $St(\bigcup_{n\in\mathbb{N}}\{x_n\}, \mathscr{U}_n) \cap U \neq \emptyset$. Therefore, $x \in St^2(\bigcup_{n\in\mathbb{N}}\{x_n\}, \mathscr{U}_n), St^2(\bigcup_{n\in\mathbb{N}}\{x_n\}, \mathscr{U}_n) = X$. Hence X is R-2-star-Menger. □

Definition 2.23. *Let $k \in \mathbb{N}$. A subset Y of topological space X is said to be R-k-star-Menger with respect to X (or Y is a R-k-star-Menger subset of X) if for every sequence $\{\mathscr{D}_n : n \in \mathbb{N}\}$ of subsets of X such that $Y \subseteq \overline{D}_n$ for each $n \in \mathbb{N}$ and for every sequence of open cover $\{\mathscr{U}_n : n \in \mathbb{N}\}$ of Y by the open sets in X there are $x_n \in D_n$ for each $n \in \mathbb{N}$ such that $Y \subseteq St^k(\bigcup_{n\in\mathbb{N}}\{x_n\}, \mathscr{U}_n)$.*

Theorem 2.24. *If A is an R-star-Menger subset of a topological space X, and $A \subseteq B \subseteq \overline{A}$, then B is an R-2-star-Menger subset of X.*

Proof. Let $\{\mathscr{D}_n : n \in \mathbb{N}\}$ be a sequence of subsets of X such that $B \subseteq \overline{D}_n$ for each $n \in \mathbb{N}$ so $A \subseteq \overline{D}_n$ for each $n \in \mathbb{N}$. Let $\{\mathscr{U}_n : n \in \mathbb{N}\}$ be a sequence of open cover of B, so also an open cover of A. Then there are points $x_n \in D_n$ for each $n \in \mathbb{N}$ such that $A \subseteq St(\bigcup_{n\in\mathbb{N}}\{x_n\}, \mathscr{U}_n)$. Now, let $x \in B$. So $x \in \overline{A}$ so that there exists a $U \in \mathscr{U}_n$ such that $x \in U$. Let $y \in U \cap A$. Then there exists $V \in \mathscr{U}_n$ where $n \in \mathbb{N}$ with $x_n \in V$. We get $U \cap V \neq \emptyset$. Thus $U \cap St(\bigcup_{n\in\mathbb{N}}\{x_n\}, \mathscr{U}_n) \neq \emptyset$. Hence $x \in St^2(\bigcup_{n\in\mathbb{N}}\{x_n\}, \mathscr{U}_n)$ so $B \subseteq St^2(\bigcup_{n\in\mathbb{N}}\{x_n\}, \mathscr{U}_n)$. Therefore, B is an R-2-star-Menger of subset of X. □

3. Conclusion

We redefine the definition of strongly star-Menger spaces and investigate their basic properties. Then we extend our study on R-star-Menger spaces, R-separable spaces and R-k-star Menger spaces. In future, based on these results we could apply Menger spaces to other types of topological structures topological groups which may be useful to characterize the classical groups.

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