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A note on coupled fractional hybrid differential equations involving Banach algebra

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Abstract
In this article, we are concerned with the existence of solution for a nonlinear hybrid differential equations of fractional order in Banach algebra. By using coupled fixed point theorem we establish our main result. Finally an example also provided to show our main result.

Keywords
Riemann-Liouville fractional derivative, hybrid initial value problem, Banach algebra, coupled fixed point theorem.

AMS Subject Classification
26A33, 32A65, 34A38, 47H10

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Contents

1. Introduction

Fractional differential equations (FDEs) have been of great interest today. It is because of its intensive development and applications in the field of fractional calculus. We can refer some of them in [8, 14, 16, 19, 21–23]. FDEs have played an important role in the developments of special functions and integral transforms, control theory, biology, biomedical, bio engineering, variational problems, economics and applied sciences etc., For further details one can refer [1, 2, 7, 15, 20].

The non-linear differential equations and integral equations are used to describe dynamical systems. All dynamical systems do not provide analytical solutions. In such cases the quadratic perturbation of a nonlinear differential equations called hybrid differential equations (HDEs) are used to solve them. Recently HDEs have captured much attention (see [5, 18, 24, 26, 27]).
In 2009, the coupled nonlinear fractional reaction diffusion equations were analyzed by Gafiychuk et.al. [14] and X.Su [25] discussed the two point boundary value problem of the coupled system of FDEs. In recent times, the study on the coupled systems of FDEs are attracted by the mathematical scientists because of its applicable nature. Since variety of problems were discussed. For details one can refer [3, 4, 12, 16, 25].

At first Dhage and Lakshmikantham [13] were studied the existence and uniqueness solutions to the ordinary first order HDEs with perturbation as

$$\frac{d}{dt}\left(\frac{x(t)}{f(t,x(t))}\right) = h(t,x(t)), \quad \text{a.e } t \in \mathscr{J}$$
$$x(t_0) = x_0 \in \mathscr{R}$$

where $f \in \mathfrak{C}(\mathscr{J} \times \mathscr{R}, \mathscr{R} \setminus \{0\})$ and $h \in \mathscr{C}(\mathscr{J} \times \mathscr{R}, \mathscr{R})$ and $\mathscr{J} = [0,1]$ is a bounded interval in $\mathscr{R}$ for some t_0 and $x_0 \in \mathscr{R}$.

In [4], B. Ahmad studied existence and uniqueness results for the following coupled system of boundary value problems for hybrid fractional differential equations.

$$\begin{aligned} {}^cD^{\alpha}\left(\frac{x(t)}{f(t,x(t),y(t))}\right) &= h_1(t,x(t),y(t)), \\ {}^cD^{\beta}\left(\frac{y(t)}{g(t,x(t),y(t))}\right) &= h_2(t,x(t),y(t)), \\ x(0) = x(1) = 0, \quad y(0) &= y(1) = 0 \end{aligned} \tag{1.1}$$

where $\alpha, \beta \in (1,2], 0 < t < 1$, $\mathscr{J} = [0,1]$, ${}^cD$ is the Caputo's fractional derivative, $f, g : \mathscr{J} \times \mathscr{R} \times \mathscr{R} \to \mathscr{R} \setminus \{0\}$ and $h_1, h_2 :$

$\mathscr{J} \times \mathscr{R} \times \mathscr{R} \to \mathscr{R}$ are continuous functions. The system was extended by Baleanu et.al. [9] to multi-point hybrid system and studied sufficient conditions for existence and uniqueness of solutions. A. Ali, et.al. [6] extended the result to the following coupled system

$$\begin{gathered} {}^cD^\alpha\left(\frac{x(t)-f_1(t,x(t),y(t))}{f_2(t,x(t),y(t))}\right)=\phi(t,x(t),y(t)),\\ {}^cD^\beta\left(\frac{y(t)-g_1(t,x(t),y(t))}{g_2(t,x(t),y(t))}\right)=\psi(t,x(t),y(t)),\\ x(0)=a,x(1)=b,y(0)=c,y(1)=d. \end{gathered} \tag{1.2}$$

where $t \in \mathscr{J}=[0,1]$, $\alpha,\beta \in (1,2]$, ${}^cD$ is the Caputo fractional derivative, a,b,c,d are real constants and the nonlinear functions $f_2,g_2 : J\times\mathscr{R}\times\mathscr{R} \to \mathscr{R}^+\setminus\{0\}$, $f_1,g_1,\phi,\psi : \mathscr{J}\times\mathscr{R}\times\mathscr{R}\to\mathscr{R}$ are continuous functions.

In 2016, T. Bashiri et.al. [10] studied the existence results for the following fractional hybrid differential systems in Banach algebra

$$\begin{gathered} D^\sigma\left(\frac{\eta(t)-u(t,\eta(t))}{f(t,\eta(t))}\right)=g(t,\mu(t)),\\ D^\sigma\left(\frac{\mu(t)-u(t,\mu(t))}{f(t,\mu(t))}\right)=g(t,\eta(t)),\ 0<\sigma<1\\ \mu(0)=0,\quad \eta(0)=0,\quad t\in\mathscr{J} \end{gathered}$$

where D^σ denotes the R-L fractional derivative of order σ, $\mathscr{J}=[0,1]$ and the functions $f:\mathscr{J}\times\mathscr{R}\to\mathscr{R}\setminus\{0\}$, $u:\mathscr{J}\times\mathscr{R}\to\mathscr{R}$, $u(0,0)=0$ and $g:\mathscr{J}\times\mathscr{R}\to\mathscr{R}$ satisfy certain conditions.

Motivated by the above inspired works on the coupled system of hybrid fractional differential equations(CHFDEs) we investigate the following CHFDEs

$$\begin{gathered} D^\alpha\left[\frac{x(t)-\omega(t,x(t))}{u(t,x(t))}\right]=v(t,y(t)),\\ D^\alpha\left[\frac{y(t)-\omega(t,y(t))}{u(t,y(t))}\right]=v(t,x(t)),\\ x(0)=a,\quad y(0)=b,\ t\in\mathscr{J}=[0,1] \end{gathered} \tag{1.3}$$

where D^α denotes the R-L fractional derivative of order $0<\alpha<1$, $\mathscr{J}=[0,1]$, $a,b\in\mathscr{R}$ and the functions $u:\mathscr{J}\times\mathscr{R}\to\mathscr{R}\setminus\{0\}$, $\omega:\mathscr{J}\times\mathscr{R}\to\mathscr{R}$, $\omega(0,0)=0$ and $v:\mathscr{J}\times\mathscr{R}\to\mathscr{R}$ satisfy certain conditions. The problem (1.3) is the generalization of the problem mentioned in T. Bashiri [10].

Some basic results and lemma are presented under the section 'Background study' and we establish our main result by a 'coupled fixed point theorem'. Finally we conclude our main result with the aid of a suitable example.

2. Preliminaries

Throughout this section Let E= $\mathbb{C}(\mathscr{J},\mathscr{R})$ be equiped with the supremum norm. Clearly it is a Banach algebra with respect to pointwise operations and the supremum norm.
For any x,y in E and is a Banach algebra under the supremum norm

$$\|x\|=\sup\{|x(t)|:t\in[0,1]\} \tag{2.1}$$

which is again a Banach algebra w.r.t. the multiplication "$\cdot$" defined by

$$(x\cdot y)(t)=x(t)\cdot y(t). \tag{2.2}$$

The product space $E=X\times X$ is a Banach space under the norm

$$\|(x,y)\|=\|x\|+\|y\| \tag{2.3}$$

Define a sum and a scalar multiplication on E $\times$ E as follows.
$(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+y_2)$,
and $k(x,y)=(kx,ky)$, for k $\in\mathscr{R}$
We list out some precise definitions and lemma which are essential to prove our main results latter.

Definition 2.1. *[17] The Riemann-Liouville (R-L shortly) fractional derivative of order $\delta>0$ of a continuous function $\omega:(0,\infty)\to\mathscr{R}$ is*

$$D^\delta w(t)=\frac{1}{\Gamma(n-\delta)}\left(\frac{d}{dt}\right)^n\int_0^t(t-\theta)^{n-\delta-1}w(\theta)d\theta$$

when $n=[\delta]+1$.

Definition 2.2. *[17] The R-L fractional integral of order $\delta>0$ of the function $w\in L^1[\mathscr{R}^+]$ is defined by*

$$I^\delta w(t)=\frac{1}{\Gamma(\delta)}\int_0^t(t-\theta)^{\delta-1}w(\theta)d\theta.$$

Definition 2.3. *[11] An element $(x,y)\in E\times E$ is called a coupled fixed point of mapping $\mathscr{T}:E\times E\to E$ if $\mathscr{T}(x,y)=x$ and $\mathscr{T}(y,x)=y$.*

Let us define ϕ the family of all functions $\Phi:\mathscr{R}^+\to\mathscr{R}^+$ satisfying $\Phi(r)<r$ for $r>0$ and $\Phi(0)=0$.
By a solution of the FHDEs system(1.3) we mean a function (x,y)$\in A\mathbb{C}(\mathscr{J},\mathscr{R}\times\mathscr{R})$ such that:
(i) The function t$\mapsto \dfrac{x(t)-\omega(t,x(t))}{u(t,x(t))}$ is absolutely continuous for every $x\in\mathscr{R}$
(ii) (x,y) satisfies the system of equations in (1.3)
where $A\mathbb{C}(\mathscr{J},\mathscr{R}\times\mathscr{R})$ is the space of absolutely continuous real valued functions defined on $\mathscr{J}$.

Definition 2.4. *[10] An operator : $E\to E$ is called $\sigma-$ nonlinear contraction if there exists a real constant $\sigma\in(0,1)$ and a function $\varphi_{\mathbb{W}}\in\Phi$ such that $\|W\theta-Wv\|=\sigma\varphi_{\mathbb{W}}\|\theta-v\|$ for every θ,v$\in$ E,we call $\varphi_{\mathbb{W}}$ a nonlinear function of W on X.*

Lemma 2.5. *[17, 22] For $\delta\in(0,1)$ and $\omega\in L^1(0,1)$ then $D^\delta I^\delta w(t)=w(t)$ holds*
and the equality

$$I^\delta D^\delta\omega(t)=\omega(t)-\frac{[D^{\delta-1}\omega(t)]_{t=0}}{\Gamma(\delta)}t^{\delta-1}\quad a.e\ on\ \mathscr{J}.$$

As a consequence of Lemma 2.5, we have the following result which is essential for proving the existence result of (1.3).

Lemma 2.6. *Consider $v \in C[0,1], 0 < \alpha < 1,\ a \in \mathscr{R}$ and $u, \omega \in \mathscr{C}([0,1] \times \mathscr{R}, \mathscr{R})$ such that $u(0,a) \neq 0,\ \omega(0,a) = 0$ then the unique solution of the following initial value problem*

$$\left.\begin{array}{c} D^{\alpha}\left[\dfrac{x(t)-\omega(t,x(t))}{u(t,x(t))}\right] = v(t), \quad t \in [0,1] \\ x(0) = a \end{array}\right\} \tag{2.4}$$

is

$$x(t) = \frac{u(t,x(t))}{\Gamma(\alpha)}\Big[D^{\alpha-1}\left(\frac{a}{u(0,a)}\right)t^{\alpha-1} + \int_0^t \frac{v(s)ds}{(t-s)^{1-\alpha}}\Big] + \omega(t,x(t)).$$

Proof. Let $x(t)$ be a solution of (2.4). Apply I^{α} on both sides of (2.4) we have

$$I^{\alpha}D^{\alpha}\left[\frac{x(t)-\omega(t,x(t))}{u(t,x(t))}\right] = I^{\alpha}v(t)$$

from Lemma 2.5, we have

$$\frac{x(t)-\omega(t,x(t))}{u(t,x(t))} - D^{\alpha-1}\frac{\left[\dfrac{x(t)-\omega(t,x(t))}{u(t,x(t))}\right]_{t=0}}{\Gamma(\alpha)}t^{\alpha-1} = I^{\alpha}v(t). \tag{2.5}$$

Consider

$$\left[\frac{x(t)-\omega(t,x(t))}{u(t,x(t))}\right]_{t=0} = \frac{x(0)-\omega(0,x(0))}{u(0,x(0))} = \frac{a-\omega(0,a)}{u(0,a)} = \frac{a}{u(0,a)} \tag{2.6}$$

substitute (2.6) in (2.5), we have

$$\frac{x(t)-\omega(t,x(t))}{u(t,x(t))} - \frac{D^{\alpha-1}\left(\dfrac{a}{u(0,a)}\right)t^{\alpha-1}}{\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{v(s)ds}{(t-s)^{1-\alpha}}$$

$$\frac{x(t)-\omega(t,x(t))}{u(t,x(t))} = \frac{D^{\alpha-1}\left(\dfrac{a}{u(0,a)}\right)t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{v(s)ds}{(t-s)^{1-\alpha}}$$

$$x(t)-\omega(t,x(t)) = \frac{u(t,x(t))}{\Gamma(\alpha)}\Big[D^{\alpha-1}\left(\frac{a}{u(0,a)}\right)t^{\alpha-1} + \int_0^t \frac{v(s)ds}{(t-s)^{1-\alpha}}\Big]$$

$$x(t) = \frac{u(t,x(t))}{\Gamma(\alpha)}\Big[D^{\alpha-1}\left(\frac{a}{u(0,a)}\right)t^{\alpha-1} + \int_0^t \frac{v(s)ds}{(t-s)^{1-\alpha}}\Big] + \omega(t,x(t))$$

□

In a similar way we can get the second part of (1.3) as

$$y(t) = \frac{u(t,y(t))}{\Gamma(\alpha)}\Big[D^{\alpha-1}\left(\frac{b}{u(0,b)}\right)t^{\alpha-1} + \int_0^t \frac{v(s,x(s))ds}{(t-s)^{1-\alpha}}\Big] + \omega(t,y(t))$$

Lemma 2.7. *[10] Let $\tilde{A}$ be a non-empty, closed, convex and bounded subset of the Banach algebra E and $\mathscr{A} = \tilde{A} \times \tilde{A}$. Suppose that $\mathbb{A}, \mathbb{C} : E \to E$ and $\mathbb{B} : \tilde{A} \to E$ be three operators such that*

(A1) The operators $\mathbb{A}$ is $\sigma-$ nonlinear contraction with nonlinear contraction function $\varphi_{\mathbb{A}}$ and $\mathbb{C}$ is $\delta-$ nonlinear contraction with nonlinear contraction function $\varphi_{\mathbb{C}}$.

(A2) $\mathbb{B}$ is completely continuous

(A3) $x = \mathbb{A}x\mathbb{B}y + \mathbb{C}x \implies x \in \tilde{A}$ for all $y \in \tilde{A}$ and

(A4) $4\sigma\left\|\mathbb{B}\tilde{A}\right\| + \delta < 1$ where $\left\|B\tilde{A}\right\| = \sup\{|B(x)| : x \in \tilde{A}\}$ Then the operator $\mathscr{T}(x,y) = \mathbb{A}x\mathbb{B}y + \mathbb{C}x$ has atleast a coupled fixed point in $\mathscr{A}$.

3. Main Results

To obtain our existence result,we consider the following hypothesis.

(H1) Let $u : \mathscr{J} \times \mathscr{R} \to \mathscr{R}\setminus\{0\}$, $\omega : \mathscr{J} \times \mathscr{R} \to \mathscr{R}$, $\omega(0,a) = 0$ are continuous and there exists two functions Φ and Ψ with bounds $\|\Phi\|$ and $\|\Psi\|$ respectively such that for all $x, y \in E$ and $t \in \mathscr{J}$ we have,

$$|u(t,x(t)) - u(t,y(t))| \leq \Phi(t)\,|x(t)-y(t)|$$
$$|\omega(t,x(t)) - \omega(t,y(t))| \leq \Psi(t)\,|x(t)-y(t)|$$

(H2) If $\mathscr{J} = \mathbb{C}(\mathscr{J}, \mathscr{R})$, a $\in \mathscr{R}$ and a subset $\tilde{A}$ of $\mathscr{J}$ then

$$\left|D^{\alpha-1}\left(\frac{a}{u(0,a)}\right)\right| \leq K_1, \quad K_1 \in \mathscr{R}$$

(H3) There exists a continuous function $v \in \mathbb{C}(\mathscr{J}, \mathscr{R})$ such that $v(t,x) \leq v(t)$, $t \in \mathscr{J}$ for all $x \in \mathscr{R}$

(H4) If $E = \mathbb{C}(\mathscr{J}, \mathscr{R})$ and $\tilde{A} \in E, E$ is defined by $\tilde{A} = \{x \in E\,\|x\| \leq \mathbb{N}\}$,
For the sake of convenience, let us take

$$\mathbb{N} = \frac{u_0\varpi + \kappa_0}{1 - \|\Phi\|\,\varpi - \|\Psi\|},$$

$u_0 = \max_{t\in\mathscr{J}}|u(t,0)|$, $\kappa_0 = \max_{t\in\mathscr{J}}|\omega(t,0)|$ and $\Omega = \dfrac{\|v\|}{\Gamma(\alpha+1)}$, $\varpi = \Omega + K_1$, where Ω, $\varpi \in \mathscr{R}$.

Theorem 3.1. *Assuming that the hypothesis $(H1)$ and $(H2)$ hold then (1.3) has atleast one solution on $\mathscr{J}$ provided*

$$4\|\Phi\|\left(\frac{\|v\|}{\Gamma(\alpha+1)}+K_1\right)+\|\Psi\|<1$$

Proof. We shall use coupled fixed point theorem to prove the operator $\mathscr{T}$ has a coupled fixed point on $\mathscr{A}$.

Since $\tilde{A}$ is closed, convex and bounded subset of the Banach algebra $u(0,b)\neq 0$ and $w(0,b)=0$ and by Lemma 2.7. $x(t)$ is the solution of the system of FHDEs (1.3) if and only if $x(t)$ satisfies the system of integral equations as

$$\begin{aligned} x(t) &= \frac{u(t,x(t))}{\Gamma(\alpha)}\Big[D^{\alpha-1}\left(\frac{a}{u(0,a)}\right)t^{\alpha-1} \\ &\quad + \int_0^t \frac{v(s,y(s))ds}{(t-s)^{1-\alpha}}\Big] + w(t,x(t)) \\ y(t) &= \frac{u(t,y(t))}{\Gamma(\alpha)}\Big[D^{\alpha-1}\left(\frac{b}{u(0,b)}\right)t^{\alpha-1} \\ &\quad + \int_0^t \frac{v(s,x(s))ds}{(t-s)^{1-\alpha}}\Big] + w(t,y(t)), \quad t\in\mathscr{J}. \end{aligned} \tag{3.1}$$

Let us consider the operators $\mathbb{A},\mathbb{B}$ and $\mathbb{C}$ such that $\mathbb{A},\mathbb{C}: E\to E$ and $\mathbb{B}:\tilde{A}\to E$

$$\begin{aligned} \mathbb{A}x(t) &= u(t,x(t)) \\ \mathbb{B}x(t) &= \frac{1}{\Gamma(\alpha)}\Big[D^{\alpha-1}\left(\frac{a}{u(0,a)}\right)t^{\alpha-1} \\ &\quad + \int_0^t \frac{v(s,y(s))ds}{(t-s)^{1-\alpha}}\Big] \\ \mathbb{C}x(t) &= \omega(t,x(t)), \quad t\in\mathscr{J}. \end{aligned}$$

Now the system of equations in (3.1) is transformed to the following system of operator equations

$$\begin{aligned} x(t) &= \mathbb{A}x(t)\mathbb{B}y(t)+\mathbb{C}x(t) \\ y(t) &= \mathbb{A}y(t)\mathbb{B}x(t)+\mathbb{C}y(t), \quad t\in\mathscr{J}. \end{aligned}$$

Now, we shall show that the operators $\mathbb{A},\mathbb{B}$ and $\mathbb{C}$ satisfy all the hypothesis of Lemma 2.7 by the following steps:

Step:1 By hypothesis $(H1)$, let $x,y\in E$, we have

$$\begin{aligned} |\mathbb{A}x(t)-\mathbb{A}y(t)| &= |u(t,x(t))-u(t,y(t))| \\ &\leq \Phi(t)\,|x(t)-y(t)| \end{aligned}$$

Taking supremum over t,

$$\|\mathbb{A}x(t)-\mathbb{A}y(t)\| \leq \|\Phi\|\,\|x(t)-y(t)\|.$$

Therefore $\mathbb{A}$ is Lipschitzian on E with Lipschitz constant $\|\Phi\|$. Analogously $\mathbb{C}$ is Lipschitzian on E with Lipschitz constant $\|\Psi\|$.

Step:2 To prove $\mathbb{B}$ is compact and continuous operator on $\tilde{A}$. Let $\{x_n\}$ be a sequence in $\tilde{A}$ converging to a point $x\in\tilde{A}$ then,by Lebesgue dominated convergence theorem,

$$\begin{aligned} &\lim_{n\to\infty}\mathbb{B}x_n(t) \\ &= \lim_{n\to\infty}\frac{1}{\Gamma(\alpha)}\Big[D^{\alpha-1}\left(\frac{a}{u(0,a)}\right)t^{\alpha-1} \\ &\quad + \int_0^t \frac{v(s,y_n(s))ds}{(t-s)^{1-\alpha}}\Big] \\ &= \lim_{n\to\infty}\frac{1}{\Gamma(\alpha)}\left[D^{\alpha-1}\left(\frac{a}{u(0,a)}\right)t^{\alpha-1}\right] \\ &\quad + \lim_{n\to\infty}\frac{1}{\Gamma(\alpha)}\left[\int_0^t \frac{v(s,y_n(s))ds}{(t-s)^{1-\alpha}}\right] \\ &= \frac{1}{\Gamma(\alpha)}\left[D^{\alpha-1}\left(\frac{a}{u(0,a)}\right)t^{\alpha-1}\right] \\ &\quad + \frac{1}{\Gamma(\alpha)}\left[\int_0^t \lim_{n\to\infty}\frac{v(s,y_n(s))ds}{(t-s)^{1-\alpha}}\right] \\ &= \frac{1}{\Gamma(\alpha)}\left[D^{\alpha-1}\left(\frac{a}{u(0,a)}\right)t^{\alpha-1} + \int_0^t \frac{v(s,y(s))ds}{(t-s)^{1-\alpha}}\right] \\ &= \mathbb{B}x(t), \quad \text{for every } t\in\mathscr{J}. \end{aligned}$$

$$\lim_{n\to\infty}\mathbb{B}x_n(t) = \mathbb{B}x(t), \quad \text{for every } t\in\mathscr{J}.$$

hence $\mathbb{B}x_n(t)\to\mathbb{B}x(t)$ as $n\to\infty$ uniformly on $\mathscr{R}^+$ and hence $\mathbb{B}$ is continuous on $\tilde{A}$.
Next we show that $\mathbb{B}(s)$ is uniformly bounded set in $\tilde{A}$.
Let $x\in\tilde{A}$, by assumption $(H2)$ and $(H3)$ for $t\in\mathscr{J}$, we have

$$\begin{aligned} |\mathbb{B}x(t)| &= \frac{1}{\Gamma(\alpha)}\Big|\int_0^t (t-s)^{\alpha-1}v(s,y(s))ds \\ &\quad + D^{\alpha-1}\left(\frac{a}{u(0,a)}\right)\Big| \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|v(s,y(s))|ds \\ &\quad + \left|D^{\alpha-1}\left(\frac{a}{u(0,a)}\right)\right| \\ &\leq \frac{|v|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\ &\quad + K_1, \\ &\leq \frac{|v|}{\Gamma(\alpha+1)}+K_1 \end{aligned}$$

Taking supremum we get

$$\begin{aligned} \|Bx(t)\| &\leq \frac{\|v\|}{\Gamma(\alpha+1)}+K_1, \quad \text{for } K_1\in\mathscr{R} \\ &\leq \Omega+K_1, \\ &\leq \varpi, \quad \text{for} \quad \text{for all } \ x\in\tilde{A} \end{aligned}$$

and hence $\mathbb{B}$ is uniformly bounded on $\tilde{A}$.

Now we will prove that $\mathbb{B}(\tilde{A})$ is equicontinuous.
First we prove $\mathbb{B}$ is uniformly bounded.

Let $\theta_1, \theta_2 \in \mathscr{J}$ for any $x \in \tilde{A}$,

$$\begin{aligned}
&|\mathbb{B}x(\theta_1) - \mathbb{B}x(\theta_2)| \\
&= \frac{1}{\Gamma(\alpha)}\Bigg|D^{\alpha-1}\left(\frac{a}{u(0,a)}\right)t^{\alpha-1} \\
&\quad + \int_0^{\theta_1}(\theta_1 - s)^{\alpha-1}v(s,y(s))ds \\
&\qquad - D^{\alpha-1}\left(\frac{a}{u(0,a)}\right)t^{\alpha-1} \\
&\quad - \int_0^{\theta_2}(\theta_2 - s)^{\alpha-1}v(s,y(s))ds\Bigg| \\
&\le \frac{1}{\Gamma(\alpha)}\Bigg|\int_0^{\theta_1}(\theta_1 - s)^{\alpha-1}v(s,y(s))ds \\
&\quad - \int_0^{\theta_2}(\theta_2 - s)^{\alpha-1}v(s,y(s))ds\Bigg| \\
&\le \frac{|v|}{\Gamma(\alpha)}\left[\left|\int_0^{\theta_1}(\theta_1 - s)^{\alpha-1} - (\theta_2 - s)^{\alpha-1}ds\right| \right. \\
&\quad \left. + \left|\int_{\theta_1}^{\theta_2}(\theta_2 - s)^{\alpha-1}ds\right|\right] \\
&\le \frac{|v|}{\Gamma(\alpha)}\left[\left|\int_0^{\theta_1}(\theta_1 - s)^{\alpha-1}ds \right.\right. \\
&\quad \left.\left. - \int_0^{\theta_1}(\theta_2 - s)^{\alpha-1}ds\right| + \left|\int_{\theta_1}^{\theta_2}(\theta_2 - s)^{\alpha-1}ds\right|\right] \\
&\le \frac{\|v\|}{\Gamma(\alpha+1)}\left(\left|\theta_1^\alpha - \theta_2^\alpha\right| \right. \\
&\quad \left. + \left|(\theta_2 - \theta_1)^\alpha\right|\right) \\
&\le \Omega\left[|\theta_1^\alpha - \theta_2^\alpha| + |(\theta_2 - \theta_1)^\alpha|\right]
\end{aligned}$$

Since θ^α is uniformly continuous on $\mathscr{J}$, for any $\Delta > 0$ there exists $\delta_1 > 0$ such that if $|\theta_1 - \theta_2| < \delta_1$, we have

$$|\theta_1^\alpha - \theta_2^\alpha| < \frac{1}{2\Omega}\Delta.$$

Put

$$\delta = \min\left\{\delta_1, \left(\frac{1}{2\Omega}\Delta\right)^{1/\alpha}\right\},$$

if $|\theta_2 - \theta_1| < \delta$, we have

$$|\mathbb{B}x(\theta_1) - \mathbb{B}x(\theta_2)| < \Omega\left(\frac{1}{2\Omega}\Delta + \frac{1}{2\Omega}\Delta\right) = \Delta.$$

This implies that $B(\tilde{A})$ is equicontinuous. As a result $\mathbb{B}$ is completely continuous operator on $\tilde{A}$.

Step:3 In order to prove $(A3)$ of Lemma 2.6, let $x \in E$ and $y \in \tilde{A}$ such that $x = \mathbb{A}x\mathbb{B}y + \mathbb{C}x$ by assumption $(H1)$ and $(H2)$ we have

$$\begin{aligned}
|x(t)| &= |Ax(t)By(t) + Cx(t)| \\
&\le |u(t,x(t)) - u(t,0) + u(t,0)| \\
&\qquad \frac{1}{\Gamma(\alpha)}\left|D^{\alpha-1}\left(\frac{a}{u(0,a)}\right)t^{\alpha-1}\right. \\
&\qquad\qquad \left. + \int_0^t \frac{v(s,x(s))}{(t-s)^{1-\alpha}}ds\right| \\
&\qquad + |\omega(t,x(t)) + \omega(t,0) - \omega(t,0)| \\
&\le (\|\Phi\|\,\|x(t)\| + u_0)\left(\frac{\|v\|}{\Gamma(\alpha+1)} + K_1\right) \\
&\quad + \|\Psi\|\,\|x(t)\| + \kappa_0 \\
&\le (\|\Phi\|\,\|x(t)\| + u_0)\,\varpi + \|\Psi\|\,\|x(t)\| + \kappa_0
\end{aligned}$$

By taking supremum over t on $\mathscr{J}$ and by $(H4)$ we conclude that

$$\begin{aligned}
\|x(t)\|\,(1 - \|\Phi\|\,\varpi - \|\Psi\|) &\le u_0\varpi + \kappa_0 \\
\|x(t)\| &\le \frac{u_0\varpi + \kappa_0}{1 - \|\Phi\|\,\varpi - \|\Psi\|} = \mathbb{N}
\end{aligned}$$

which imply the conclusion that $x \in \tilde{A}$.

Step:4 Finally we will prove $4\sigma\left\|\mathbb{B}\tilde{A}\right\| + \delta \le 1$,that is, (A4)of Lemma 2.7 holds.

since $\left\|\mathbb{B}\tilde{A}\right\| = \sup\{|B(x)(t)|\} \le \dfrac{\|v\|}{\Gamma(\alpha+1)} + K_1$

and we have

$$\begin{aligned}
&4\sigma\left\|\mathbb{B}\tilde{A}\right\| + \delta \le 4\,\|\Phi\|\,(\frac{\|v\|}{\Gamma(\alpha+1)} + K_1) + \|\Psi\| \\
&\le 4\,\|\Phi\|\,\varpi + \|\Psi\| < 1,
\end{aligned}$$

where $\sigma = \|\Phi\|$, $\varpi = \dfrac{\|v\|}{\Gamma(\alpha+1)} + K_1$ and $\delta = \|\Psi\|$.

So the assumption $(A4)$ of Lemma 2.7 has been proved. Therefore all the conditions of Lemma 2.7 are satisfied. And hence the operator $\mathscr{T}(x,y) = \mathbb{A}x\mathbb{B}y + \mathbb{C}x$ has a coupled fixed point on $\mathscr{A}$. That is the FHDEs in (1.3) has a solution on $\mathscr{J}$. Hence the proof is completed. □

4. Example

In this section we present an example to establish our main result.

Example 4.1. *Consider the following coupled system of nonlinear hybrid fractional differential equations*

$$\begin{aligned}
&D^{\frac{1}{3}}\left(\frac{x(t) - \omega(t,x(t))}{u(t,x(t))}\right) = \frac{t|y(t)|}{(7+e^t)|y(t)+1|}, \\
&D^{\frac{1}{3}}\left(\frac{y(t) - \omega(t,y(t))}{u(t,y(t))}\right) = \frac{t|x(t)|}{(7+e^t)|x(t)+1|}, \\
&\qquad x(0) = 2, \quad \textit{and } y(0) = 3. \qquad (4.1)
\end{aligned}$$

where

$$u(t,x(t)) = \frac{1}{15}t\left(\frac{1}{18}\cos x(t) + \frac{|x(t)|+x^2(t)}{1+|\sin x(t)|}\right)+1$$
$$\omega(t,x(t)) = \frac{1}{18}e^t\left(\frac{1}{4}\sin x(t) + \frac{x^2(t)+2|x(t)|}{2+|\sin x(t)|}\right)$$

for choosing arbitrary $x,y \in E$ *and* $t \in [0,1]$, *we have*

$$|u(t,x(t)) - u(t,y(t))| \leq \frac{t}{18}|x(t)-y(t)|$$

$$|\omega(t,x(t)) - \omega(t,y(t))| \leq \frac{1}{4}e^t|x(t)-y(t)|$$

$$|v(t,x(t)) - v(t,y(t))| \leq \frac{t}{7+e^t}$$

we can conclude that

$$\Phi(t) = \frac{1}{18}t,$$
$$\Psi(t) = \frac{e^t}{18},$$
$$W_0 = 0, u_0 = 1.003703704,$$
$$\Omega \approx 0.115231068,$$
$$\|\Phi\| \approx 0.0555,$$
$$\|\Psi\| \approx 0.15102,$$
$$K_1 = 2.215464337,$$
$$\varpi \approx 2.330695405,$$
$$\mathbb{N} \approx 3.250752883.$$

So we have

$$4\|\Phi\|\varpi + \|\Psi\| \approx 0.668900519 < 1.$$

It follows by theorem (1.3)*, the problem* (4.1) *has a solution on* $[0,1]$.

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