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On some topological sets

Pratap Kumar Saha[1]*

Abstract

In this paper, we study semi-open, pre-open, α-open and β-open sets, and obtain some relations between them.

Keywords

Semi-open, pre-open, α-open, β-open.

AMS Subject Classification

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[1] *Department of Mathematics, Behala College, Kolkata-700060, West Bengal, India.*

***Corresponding author**: [1]pratapsaha2@gmail.com

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Contents

1. Introduction

As usual, we write X to denote the topological space $(X, \mathscr{P})$. For a subset A of the topological space X, $Int(A)$ (resp. $Cl(A)$) stands for the interior (resp. closure) of A in X. Throughout the paper, N denotes the set of natural numbers and R, the set of real numbers.

Levine [8] introduced the notion of semi-open sets in 1963: a subset A of a topological space X is called semi-open if there exists an open set G such that $G \subset A \subset Cl(G)$. It is easy to see that a subset A of a topological space X is semi-open in X iff $A \subset Cl(Int(A))$ [8]. The closure and interior are two very basic and perhaps the most important operators in areas of study involving topology. Both of them when operated on a subset of a topological space give rise to a set whose nature depends upon the order of their operation e.g. semi-open sets [8], locally dense sets [3]. The locally dense sets (also called pre-open sets by Mashhour et al. [10]) introduced by Corson and Michael [3]: a subset A of a topological space X is called locally dense if there exists an open set U such that $A \subset U \subset Cl(A)$. Instead of using both pre-open and locally dense randomly in the paper, we agree to write pre-open sets to symbolize locally dense sets also. We observe that a subset A of X is pre-open iff $A \subset Int(Cl(A))$ [10]. We see that in pre-open sets of a topological space the positions of the interior and closure operators in semi-open sets are just interchanged. Though the study of semi-open and pre-open sets in topological spaces initiated about six decades back, they are still in the focus of research works e.g. see [7, 11, 12]. Andrijević [1] introduced and studied the notion of semi-pre-open sets: a subset A of a topological space X is called semi-pre-open if there exists a pre-open set U such that $U \subset A \subset Cl(U)$. Equivalently, a subset A of a topological space X is semi-pre-open iff $A \subset (Cl(Int(Cl(A)))$. Njåstad [9] and El-Monsef et al. [2] independently introduced and studied β-sets and β-open sets respectively which are same as semi-pre-open sets. Changing the role of interior and closure operators in semi-pre-open sets, we get α-sets, i.e. a subset A of X is an α-set [9] iff $A \subset Int(Cl(Int(A)))$. In tune with recent trend of topologists, we agree to call α-sets as α-open sets. Throughout the paper, we use the term β-open sets to represent β-sets or semi-pre-open sets as well.

We list all of the sets described above as a gist for better understanding and for effective use of them in the sequel. A subset A of a topological space $(X, \mathscr{P})$ is

(i) semi-open iff $A \subset Cl(Int(A))$ [8],

(ii) pre-open iff $A \subset Int(Cl(A))$ [10],

(iii) α-open iff $A \subset Int(Cl(Int(A)))$ [9],

(iv) β-open iff $A \subset Cl(Int(Cl(A)))$ [1, 2, 9].

Obviously, open sets are included in either of above four categories of sets. The following diagram epitomizes the implication relations among them. The implications are not reversible.

As usual, a subset of X is called semi-closed [4], pre-closed [10], α-closed and β-closed if its complement is semi-open, pre-open, α-open and β-open respectively. So a subset B of a topological space X is

$$\begin{tikzcd}[column sep=small, row sep=small] & & \text{semi-open} \arrow[dr] & \\ \text{open} \arrow[r] & \alpha\text{-open} \arrow[ur] \arrow[dr] & & \beta\text{-open} \\ & & \text{pre-open} \arrow[ur] & \end{tikzcd}$$

Note: An arrow between two notions in above diagram stands to mean implies that.

(i) semi-closed iff $Int(Cl(B)) \subset B$ [4],

(ii) pre-closed iff $Cl(Int(B)) \subset B$ [10],

(iii) α-closed iff $Cl(Int(Cl(B))) \subset B$,

(iv) β-closed iff $Int(Cl(Int(B))) \subset B$.

2. Properties of Some Topological Sets

Recall that a subset A of a topological space is called regularly open if $A = Int(Cl(A))$ which means that $A \subset Int(Cl(A))$ and $Int(Cl(A)) \subset A$. That is, a subset A of X is regularly open iff A is both pre-open and semi-closed on X [5, p. 8]. A subset of X is regularly closed if its complement is regularly open. So a subset B of X is regularly closed iff $B = Cl(Int(B))$ which means $B \subset Cl(Int(B))$ and $Cl(Int(B)) \subset B$. That is, a subset B of X is regularly closed iff B is both semi-open and pre-closed. Also for any subset A of X, $Int(Cl(A))$ (resp. $Cl(Int(A))$) is regularly open (resp. regularly closed). So if a subset A of X is semi-open (resp. α-open, pre-closed, β-closed), then there exists a regularly closed set E such that $A \subset E$ (resp. $A \subset Int(E)$, $E \subset A$, $Int(E) \subset A$). But converses are not true.

Example 2.1 (Mukharjee et al. [11]). *For $a \in R$, we define*

$$\mathscr{T} = \{\emptyset, R, (-\infty, a-1), (-\infty, a), [a, \infty), (-\infty, a-1) \cup [a, \infty)\}.$$

In the topological space $(R, \mathscr{T})$, $(-\infty, a)$ is regularly closed and $(-\infty, b) \subset (-\infty, a) = Int((-\infty, a))$ for $-\infty < b < a-1$ but $(-\infty, b)$ is neither semi-open nor α-open. Also $[a, \infty)$ is regularly closed and $[a, \infty) = Int([a, \infty)) \subset (-\infty, a-1) \cup [a, \infty)$ but $(-\infty, a-1) \cup [a, \infty)$ is neither pre-closed nor β-closed.

We also note that if a subset A of X is pre-open (resp. β-open, semi-closed, α-closed), then there exists a regularly open set G such that $A \subset G$ (resp. $A \subset Cl(G)$, $G \subset A$, $Cl(G) \subset A$). But converses are not true. In the topological space of Example 2.1, $(-\infty, a)$ is regularly open. For any subset D of $[a-1, a)$, we see that $D \subset (-\infty, a) = Cl((-\infty, a))$ but D is neither pre-open nor β-open. Also $[a, \infty)$ in the topological space of Example 2.1 is regularly open and $[a, \infty) \subset (-\infty, a-1) \cup [a, \infty)$ but $(-\infty, a-1) \cup [a, \infty)$ is neither semi-closed nor α-closed.

Definition 2.2 (Mukharjee et al. [11]). *Let A be a semi-open set in X. A is said to be covered if there exist open sets G and $H (\neq X)$ such that $G \subset A \subset H \subset Cl(G)$. A is said to be uncovered if $G \subset A \subset Cl(G)$ for some open set G of X, then $G \subset Int(Cl(G)) \subset A \subset Cl(G)$.*

If A is uncovered semi-open, then there does not exist an open set $H (\neq A)$ such that $G \subset A \subset H \subset Cl(G)$. If A is uncovered semi-open but not open, then A can not be covered semi-open. Also if A is covered semi-open but not open, then A can not be uncovered semi-open. Any nontrivial open set of a topological space is obviously a covered semi-open set. In Example 2.1, $(-\infty, a-1)$ is an open set but not an uncovered semi-open set which means that an open set may not be an uncovered semi-open set.

As usual, the complement of a covered (resp. uncovered) semi-open set is said to be a covered (resp. uncovered) semi-closed set.

We see that $A \subset X$ is a covered semi-closed iff there exist closed sets E and $F (\neq \emptyset)$ such that $Int(E) \subset F \subset A \subset E$ and A is uncovered semi-closed iff there exists a closed set E such that $Int(E) \subset A \subset Cl(Int(E)) \subset E$. Thus if A is uncovered, then there does not exist a closed set $F (\neq A)$ such that $Int(E) \subset F \subset A \subset Cl(E)$. If A is uncovered semi-closed but not closed, then A can not be covered semi-closed. Also if A is covered semi-closed but not closed, then A can not be uncovered semi-closed. Any nontrivial closed set of a topological space is obviously a covered semi-closed set. In Example 2.1, $[a-1, \infty)$ is a closed set but not an uncovered semi-closed set which means that a closed set may not be an uncovered semi-closed set. In Example 2.1, $[a - \frac{1}{n}, \infty)$ is covered semi-closed for each $n \in N$.

Example 2.3. *For $a, b \in R$ with $b > a+1$, we define*

$$\mathscr{T} = \{\emptyset, R, (-\infty, a), (b, \infty), (-\infty, a) \cup (b, \infty)\}.$$

In the topological space $(R, \mathscr{T})$, for each $n \in N$, $[a + \frac{1}{n}, \infty)$ is a semi-closed set which is uncovered.

Theorem 2.4 (Mukharjee et al. [11]). *Covered semi-open sets of a topological space are pre-open.*

It can be seen that a subset A of a topological space X is semi-closed if there exists a closed set E such that $Int(E) \subset A \subset E$.

Theorem 2.5. *Covered semi-closed sets of a topological space are pre-closed.*

Proof. Similar to that of Theorem 2.4.

Alternatively, if A is a covered semi-closed set, then $X - A$ is a covered semi-open set. By Theorem 2.4, $X - A$ is pre-open and so A is pre-closed. □

Theorem 2.6. *The following assertions hold good in a topological space X:*

(a) $A \subset X$ is both semi-closed and β-open, iff there exists a regularly open set G such that $G \subset A \subset Cl(G)$.

(b) $A \subset X$ *is both semi-open and* β*-closed, iff there exists a regularly closed set* E *such that* $Int(E) \subset A \subset E$.

Proof. (a) Firstly, let A be both semi-closed and β-open. We put $G = Int(Cl(A))$. Then G is regularly open. By semi-closedness and β-openness of A, we have $G \subset A$ and $A \subset Cl(G)$ respectively i.e., $G \subset A \subset Cl(G)$.

Conversely, let there exist a regularly open set G such that $G \subset A \subset Cl(G)$. Then $Cl(A) = Cl(G)$ which means that $Int(Cl(A)) = Int(Cl(G)) = G$ and $Cl(G) = Cl(Int(Cl(A)))$. From $G \subset A \subset Cl(G)$, we have $G \subset A$ which implies $Int(Cl(A)) \subset A$ and $A \subset Cl(G)$ which implies that $A \subset Cl(Int(Cl(A)))$.
(b): Just like that of (a) above. □

Remark 2.7. *In (a) of Theorem 2.6,* A *is also semi-open. But a semi-open set may not be both semi-closed and* β*-open. In Example 2.1, we see that* $(-\infty, a-1) \subset (-\infty, b) \subset Cl((-\infty, a-1))$ *where* $a-1 < b < a$*. So* $(-\infty, b)$ *is semi-open but not semi-closed. Note that* $(-\infty, a-1)$ *is not regularly open. Then it follows that regularity of* G *in Theorem 2.6 (a) is very much essential to be* A *both semi-closed and* β*-open.*

Theorem 2.8. $A \subset X$ *is regularly open as well as regularly closed if*

(a) *A is both pre-closed and α-open.*

(b) *A is both pre-open and α-closed.*

Proof. (a) Let A be both pre-closed and α-open. We put $E = Cl(Int(A))$. Then E is regularly closed. By pre-closedness and α-openness of A, we have $E \subset A$ and $A \subset Int(E)$ respectively i.e., $E \subset A \subset Int(E)$. But $Int(E) \subset E$ implies that $A = E = Int(E)$. Since E is closed, $Int(E)$ is regularly open. So A is both regularly open and regularly closed.

(b) Let A be both pre-open and α-closed. We put $G = Int(Cl(A))$. Then G is regularly open. By pre-openness and α-closedness of A, we have $A \subset G$ and $Cl(G) \subset A$ respectively i.e., $Cl(G) \subset A \subset G$. But $G \subset Cl(G)$ which implies that $A = G = Cl(G)$. Since G is open, $Cl(G)$ is regularly closed. So A is both regularly open and regularly closed. □

The converses of Theorem 2.8 are also true as open (resp. closed) sets are α-open (resp. α-closed) and so pre-open (resp. pre-closed).

We note that the notions of semi-open and pre-open sets are independent. However, we have the following results on semi-open and pre-open sets.

Theorem 2.9. *If* A *is semi-open and* B *is pre-open on* X *with* $Cl(A) = Cl(B)$*, then there exist two open sets* G *and* H *such that* $Cl(G) = Cl(H) = Cl(A) = Cl(B)$.

Proof. Due to semi-openness of A, we have an open set G such that $G \subset A \subset Cl(G)$ which implies that $Cl(G) = Cl(A)$. As B is pre-open, we obtain an open set H such that $B \subset H \subset Cl(B)$ which means that $Cl(B) = Cl(H)$. So it follows that $Cl(G) = Cl(A) = Cl(B) = Cl(H)$. □

Theorem 2.10. *Let* A *and* B *be semi-open and pre-open respectively on* X *such that* $A \subset B$ *and* $Cl(A) = Cl(B)$*. Then* A *is also pre-open and* B *is also semi-open on* X.

Proof. Since A is semi-open, there exists an open set G such that $G \subset A \subset Cl(G)$ which implies that $Cl(A) = Cl(G)$. Since B is pre-open, we have an open set H such that $B \subset H \subset Cl(B)$. Since $A \subset B$, $G \subset Int(A) \subset Int(B) \subset B \subset H \subset Cl(B) = Cl(A) = Cl(G)$. So we have $G \subset B \subset Cl(G)$, i.e. B is semi-open. Again we have $A \subset B \subset H \subset Cl(B) = Cl(A)$. Hence $A \subset H \subset Cl(A)$ which means that A is pre-open. □

Theorem 2.11. *Let* B *be pre-open and* H *be open on* X *such that* $B \subset H \subset Cl(B)$*. If* A *is semi-open on* X *such that* $B \subset A$ *and* $Int(A) = Int(B)$*, then* $A \cap H$ *is both semi-open and pre-open on* X.

Proof. For A, we obtain an open set G such that $G \subset A \subset Cl(G)$ which implies that $Cl(A) = Cl(G)$. Since $Int(A) = Int(B)$, $G \subset Int(A) = Int(B) \subset B \subset H \subset Cl(B) \subset Cl(A) = Cl(G)$. Also $B = B \cap H \subset A \cap H \subset H$ which in turn implies that $G \subset B \subset A \cap H \subset H \subset Cl(B) \subset Cl(A) = Cl(G)$. $B \subset A \cap H \subset H \subset Cl(B)$ implies that $Cl(B) = Cl(A \cap H)$. Thus we get $A \cap H \subset H \subset Cl(A \cap H)$. So it follows that $A \cap H$ is pre-open on X. Also we have $G \subset A \cap H \subset Cl(B) \subset Cl(A) = Cl(G)$ which implies that $A \cap H$ is semi-open. □

Theorem 2.12. *Let* A *be semi-open on* X *and* B *be a subset of* X *such that* $B \subset A$ *and* $Int(A) = Int(B)$*. Then* B *is semi-open on* X.

Proof. For A, we obtain an open set G such that $G \subset A \subset Cl(G)$ which implies that $Cl(A) = Cl(G)$. Now $G \subset Int(A) = Int(B) \subset B \subset Cl(B) \subset Cl(A) = Cl(G)$. So we have $G \subset B \subset Cl(G)$, i.e. B is semi-open. □

Dualizing results from Theorem 2.9 to Theorem 2.12, we have the results from Theorem 2.13 to Theorem 2.16. The proofs of these results are omitted as the proofs are similar to the proofs of corresponding results already established.

Theorem 2.13. *If* A *is semi-closed and* B *is pre-closed on* X *with* $Int(A) = Int(B)$*, then there exist two closed sets* E *and* F *such that* $Int(E) = Int(F) = Int(A) = Int(B)$.

Theorem 2.14. *Let* A *and* B *be semi-closed and pre-closed respectively on* X *such that* $B \subset A$ *and* $Int(A) = Int(B)$*. Then* A *is also pre-closed and* B *is also semi-closed on* X.

Theorem 2.15. *Let* B *be pre-closed and* E *be closed on* X *such that* $Int(B) \subset E \subset B$*. If* A *is semi-closed on* X *such that* $A \subset B$ *and* $Cl(A) = Cl(B)$*, then* $A \cup E$ *is both semi-closed and pre-closed on* X.

Theorem 2.16. *Let* A *be semi-closed on* X *and* B *be a subset of* X *such that* $A \subset B$ *and* $Cl(A) = Cl(B)$*. Then* B *is semi-closed on* X.

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