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On maximal and minimal μ-clopen sets in GT spaces

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Abstract

In this paper, we introduce the notions of maximal and minimal μ-clopen sets in a generalized topological space and their some properties. We obtain that maximal and minimal μ-clopen sets are independent of maximal and minimal μ-open and μ-closed sets. We observed that the existence of maximal μ-clopen set in a generalized topological space not only ensure the μ-disconnectedness of a generalized topological space but also the existence of minimal μ-clopen set in that space.

Keywords

μ-open set, μ-closed set, maximal μ-open set, minimal μ-closed set, μ-clopen, maximal μ-clopen set, minimal μ-clopen set.

AMS Subject Classification

54A05, 54D05.

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Contents

1. Introduction

There are several generalizations of open sets of topological spaces. Two well-discussed generalizations of open sets of topological spaces are semi-open [4] and pre-open sets [5]). It is observed that the semi-open, pre-open sets of a topological space possess properties resembling those of open sets of the topological space with some exception e.g. the family of semi-open (or pre-open) sets are not closed even under finite intersections. Starting from this observation, Császár [1] finally introduced and studied the concepts of generalized topology.

Let X be a nonempty set and μ be a subcollection of the power set of X. Then μ is a generalized topology on X if $\emptyset \in \mu$ and μ is closed under unions. We write, GT (resp. GT space) to denote a generalized topology (resp. generalized topological space) for brevity. The members of μ are called the μ-open sets in X and the complement of a μ-open set is called a μ-closed set in X. The generalized interior of a subset A of X is the union of all μ-open sets contained in A and is denoted by $i_\mu(A)$. The generalized closure of a subset A of X is the intersection of all μ-closed sets containing A and is denoted by $c_\mu(A)$. It is easy to see that $i_\mu(A) = X - c_\mu(X-A)$. By a proper μ-open set (resp. μ-closed set) of X, we mean a μ-open set G (resp. μ-closed set E) such that $G \neq \emptyset$ and $G \neq X$ if X is μ-open (resp. $E \neq X$ and $E \neq \emptyset$ if $\emptyset$ is μ-closed). We also write $\mathbb{R}$ to denote the set of real numbers.

We observed that a subset has a double natures in X such as a subset which is both maximal and minimal μ-open, then either X is μ-disconnected or number of subset in X is unique to certain condition, e. g. theorem 2.4 [8]. Due to theorem 3.10 of this paper we obtain that the existence of a set which both maximal and minimal μ-clopen not only assures the μ-disconnectedness of space but also ensure the existence of another set which is also both maximal μ-clopen and minimal μ-clopen and these two are only μ-clopen sets in that space. This notion reveals the difference among of maximal and minimal μ-open sets which the maximal and minimal μ-clopen and also beauty of maximal and minimal μ-clopen sets.

2. Maximal, minimal μ-open set(resp. μ-closed set) and μ-clopen set

We recall the following known definitions and results to make the article self sufficient as far as practical.

Definition 2.1 (Roy and Sen [6]). *A proper μ-open set A of a GT space X is called a maximal μ-open set if there is no*

μ-open set $U(\neq A, X)$ such that $A \subset U \subset X$.

Definition 2.2 (Roy and Sen [6]). *A proper μ-closed set E of a GT space X is called a minimal μ-closed set if there is no μ-closed set $F(\neq \emptyset, E)$ such that $\emptyset \subset F \subset E$.*

Definition 2.3 (S. Al Ghour et al. [7]). *A proper μ-open set U of X is said to be a minimal μ-open set if the only proper μ-open set which is contained in U is U.*

Definition 2.4 (Mukharjee [8]). *A proper μ-closed set E in a GT space X is called a maximal μ-closed set if any μ-closed set which contains E is X or E.*

Definition 2.5 (Mukharjee [8]). *A proper subset of a GT space X is said to be μ-clopen if it is both μ-open and μ-closed.*

Definition 2.6 (Császár [3]). *A GT space X is μ-connected if X can not be expressed as $G \cup H = X$ where G, H are disjoined μ-open sets. As usual, if X is not μ-connected then X is called μ-disconnected. So X is μ-disconnected if there exists two disjoint μ-open sets G, H such that $G \cup H = X$.*

Theorem 2.7 (Roy and Sen [6]). *If A is a maximal μ-open set and B is a μ-open set in a GT space X, then either $A \cup B = X$ or $B \subset A$. If B is also a maximal μ-open set distinct from A, then $A \cup B = X$.*

Theorem 2.8 (Roy and Sen [6]). *If F is a minimal μ-closed set and E is a μ-closed set in a GT space X, then either $E \cap F = \emptyset$ or $F \subset E$. If E is also a minimal μ-closed set distinct from F, then $F \cap E = \emptyset$.*

Theorem 2.9 (Mukharjee [8]). *If U is a minimal μ-open set and W is a μ-open set such that $U \cap W$ is a μ-open set, then either $U \cap W = \emptyset$ or $U \subset W$. If W is also a minimal μ-open set distinct from U, then $U \cap W = \emptyset$.*

Theorem 2.10 (Mukharjee [8]). *If G is a maximal μ-open set and H is a minimal μ-open set in a GT space X such that $G \cap H$ is a μ-open set then either $H \subset G$ or the space is μ-disconnected.*

Theorem 2.11 (Mukharjee [8]). *If E is a maximal μ-closed set and F is any μ-closed set in a GT space X such that $E \cup F$ is a μ-closed set, then either $E \cup F = X$ or $F \subset E$.*

Theorem 2.12 (Roy and Sen [6]). *A proper μ-open set A in a GT space X is maximal μ-open iff $X - A$ is minimal μ-closed in X.*

Similarly, we see that a proper μ-closed set A in a GT space X is a maximal μ-closed iff $X - A$ is minimal μ-open in X.

Theorem 2.13 (Mukharjee [8]). *If a GT space X has a set G which is both maximal and minimal μ-open and $G \cap H$ is a μ-open for a proper μ-open set H, then either G=H or the GT space μ-disconnected.*

3. Maximal and minimal μ-clopen sets

We now introduce some new notions and obtain their following properties .

Definition 3.1. *A proper μ-clopen set U of a GT space X is said to be minimal μ-clopen if V is a proper μ-clopen set such that $V \subset U$, then $V = U$ or $V = \emptyset$.*

Example 3.2. *Let $X = (-1, 1)$ and $\mu = \{\emptyset, X, \{0\}, (-1, 0), (-1, 0], (0, 1), [0, 1), X - \{0\}\}$.*

In the GT space (X, μ), $[0, 1)$ is a minimal μ-clopen set but it is neither minimal μ-open nor minimal μ-closed.
We observed that a minimal μ-open or a minimal μ-closed set may not be minimal μ-clopen. So the notion of minimal μ-clopen set is independent to the notion of minimal μ-open as well as minimal μ-closed sets. It is clearly to see that if a set A is both minimal μ-open and minimal μ-closed, then A is minimal μ-clopen. In fact, a μ-clopen set is minimal μ-clopen if it is either minimal μ-open or minimal μ-closed.

Definition 3.3. *A proper μ-clopen set U of a GT space X is said to be maximal μ-clopen if V is a proper μ-clopen set such that $V \subset U$, then $V = U$ or $B = X$.*

In the GT space (X, μ), Example 3.2, $(-1, 0)$ is a maximal μ-clopen set but it is neither maximal μ-open nor maximal μ-closed.
We observed that a maximal μ-open or a maximal μ-closed set may not be maximal μ-clopen. So, the notion of maximal μ-clopen set is independent to the notion of maximal μ-open as well as maximal μ-closed sets. It is clearly to see that if a set A is both maximal μ-open and maximal μ-closed, then A is maximal μ-clopen. In fact, a μ-clopen set is maximal μ-clopen if it is either maximal μ-open or maximal μ-closed.

In [8] we analyzed that if a GT space has only one proper μ-open set, then it is both maximal and minimal μ-open. But we note that it is neither maximal μ-clopen nor minimal μ-clopen. If a GT space X has only two proper μ-open sets such that one is not contain in other, then both are maximal μ-clopen and minimal μ-clopen. Also maximal μ-clopen or minimal μ-clopen sets can exists in a μ-disconnected GT space.
The result of following theorem are almost immediate and hence the proofs of them are omitted.

Theorem 3.4. *In a GT space X, if A is a minimal μ-clopen set and B is a μ-clopen set in X, then $A \cap B = \emptyset$ or $A \subset B$.*

Corollary 3.5. *In a GT space X, if A and B are distinct minimal μ-clopen set in X, then $A \cap B = \emptyset$.*

Theorem 3.6. *In a GT space X, if A is a maximal μ-clopen set and B is a μ-clopen set in X, then $A \cap B = X$ or $B \subset A$.*

Corollary 3.7. *In a GT space X, if A and B are distinct maximal μ-clopen set in X, then $A \cup B = X$.*

Lemma 3.8. *In a GT space X, If A is minimal μ-clopen set, then $X-A$ is maximal μ-clopen in X and conversely.*

Proof. Let ther exists a μ-clopen set B such that $X-B \subset A$. Then we obtain $X-A \subset B$. Since A is minimal μ-clopen, we get $X-B=A$ or $X-B=\emptyset$. Which implies that $X-A=B$ or $B=X$. So $X-A$ is maximal μ-clopen set. The converse is similarly follows. □

Theorem 3.9. *In a GT space X, if A is a minimal μ-clopen set and B is a maximal μ-clopen set in X then either $A \subset B$ or $A=X-B$.*

Proof. From the Theorem 2.10 , A and B are μ-clopen, then $A \cap B=\emptyset$ and $A \cup B=X$ implies $A=X-B$. Hence the theorem is proof. □

Theorem 3.10. *In a GT space X contain a set A which is both maximal and minimal μ-clopen, then (i) A and $X-A$ are the only sets in the space are both maximal and minimal μ-clopen and (ii) A and $X-A$ are the only proper μ-clopen sets in the space.*

Proof. (i) Since A is both maximal and minimal μ-clopen by Lemma 3.8 , $X-A$ is also both maximal and minimal μ-clopen. If a another set $B \neq A$ in X which is both maximal and minimal μ-clopen, then by Lemma 3.8, $X-A$ is also both maximal and minimal μ-clopen. Since A, B are both maximal and minimal μ-clopen by Corollary 3.5 and corollary 3.7. We have $A \cap B=\emptyset$ and $A \cup B=X$, which implies that $B=X-A$ i.e., if $A \neq B$ B and $X-B$ are identical to $X-A$ and A respectively. Considering all probable combination of A, B, $X-A$ and $X-B$, we get same result.
(ii) Let E is a proper μ-clopen set in X. By maximal and minimal μ-clopenness of A we get $A \cup E=X$ or $E \subset A$ and $A \cap E=\emptyset$ or $A \subset E$. $A \cup E=X$ together with $A \cap E=\emptyset$ imply that $E=X-A$. $A \cup E=X$ and $A \subset E$ imply that $E=X$. $E \subset A$ and $A \cap E=\emptyset$ imply that $E=\emptyset$. □

Theorem 3.11. *In a GT space X, maximal μ-clopen and minimal μ-clopen sets appear in pairs.*

Proof. By the theorem 3.10 if a GT space X has a set A which is both maximal and minimal μ-clopen, then $X-A$ is also both maximal and minimal μ-clopen, and the space cannot have more than a pair of such sets. Due to lemma 3.8 if A is a maximal (resp. minimal) μ-clopen set in X then $X-A$ is minimal (resp. maximal) μ-clopen in X.

□

Theorem 3.12. *In a GT space X if A is a maximal μ-open set and B is a minimal μ-open set with A is not a subset of B, then A is maximal μ-clopen and B is minimal μ-clopen.*

Proof. By the theorem 3.9, we have $A=X-B$, So A, B both are μ-clopen. Since A (resp. B) is both μ-clopen and maximal (minimal) μ-open, it is easy to see that A (resp. B) is maximal (resp. minimal) μ-clopen. □

Theorem 3.13. *In a GT space X, if E is a maximal μ-clopen set, then $E \cup U$ is not a proper μ-clopen set distinct from E for any proper μ-open or μ-closed U in X.*

Proof. If possible, let $E \cup U$ be a proper μ-clopen set. Since $E \subset E \cup U$ and E is a maximal μ-clopen set we have $E \cup U=X$ or $E=E \cup U$. Now $E=E \cup U$ implies that $U \subset E$. □

Theorem 3.14. *In a GT space X, if E is a minimal μ-clopen set, then $E \cap U$ is not a proper μ-clopen set distinct from E for any proper μ-open or μ-closed U in X.*

Proof. Dual role of maximal and minimal μ-clopen set so, similar proof is omitted. □

Theorem 3.13 and Theorem 3.14 gives some ideas about μ-clopen set in GT space consists of a maximal and minimal μ-clopen set.

Theorem 3.15. *In a GT space X, if $\mathscr{A}$ is a collection of distinct maximal μ-clopen sets and $A \in \mathscr{A}$, then $\bigcap_{B \in \mathscr{A}-\{A\}} B \neq \emptyset$. If $\mathscr{A}$ is a finite collection, then $\bigcap_{B \in \mathscr{A}-\{A\}} B$ is minimal μ-clopen iff $X-A=\bigcap_{B \in \mathscr{A}-\{A\}} B$.*

Proof. While $B \in \mathscr{A}-A$ is a maximal μ-clopen set, $X-B$ is a minimal μ-clopen set. By Theorem 3.9 we ge $X-B \subset A$. So, we have $X-\bigcap_{B \in \mathscr{A}-\{A\}} B \subset A$ which implies that $A=X$ if $\bigcap_{B \in \mathscr{A}-\{A\}} B=\emptyset$. Since A being a maximal μ-clopen set, then $A=X$ is not possible. So we have $\bigcap_{B \in \mathscr{A}-\{A\}} B \neq \emptyset$. Obviously, $\bigcap_{B \in \mathscr{A}-\{A\}} B$ is a minimal μ-clopen set if $X-A=\bigcap_{B \in \mathscr{A}-\{A\}} B$. For converse, let $\bigcap_{B \in \mathscr{A}-\{A\}} B$ is a minimal μ-clopen set. If $\mathscr{A}$ is a finite collection, then $\bigcap_{B \in \mathscr{A}-\{A\}} B$ is a μ-clopen set. Since $X-A \subset \bigcap_{B \in \mathscr{A}-\{A\}} B$, we have $X-A \subset \bigcap_{B \in \mathscr{A}-\{A\}} B$. A being a maximal μ-clopen, $X-A$ is minimal μ-clopen. If $\bigcap_{B \in \mathscr{A}-\{A\}}$ is minimal μ-clopen set distinct from $X-A$, then by Corrolary 3.5, we have $(\bigcap_{B \in \mathscr{A}-\{A\}} B) \cap (X-A)=\emptyset$ which implies that $\bigcap_{B \in \mathscr{A}-\{A\}} B \subset A$. Thus we obtain $X-A \subset \bigcap_{B \in \mathscr{A}-\{A\}} B \subset A$, this ia an absurd result. So we have $X-A=\bigcap_{B \in \mathscr{A}-\{A\}} B$. □

Theorem 3.16. *In a GT space X, if $\mathscr{A}$ is a collection of distinct minimal μ-clopen sets and $A \in \mathscr{A}$, then $\bigcup_{B \in \mathscr{A}-\{A\}} B \neq X$. If $\mathscr{A}$ is a finite collection, then $\bigcup_{B \in \mathscr{A}-\{A\}} B$ is maximal μ-clopen iff $X-A=\bigcup_{B \in \mathscr{A}-\{A\}} B$.*

Proof. Similar proof of the above Theorem. □

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