Malaya Journal of Matematik, Vol. 06, No. 04, 866-872, 2018

https://doi.org/10.26637/MJM0604/0026

γ-Generalized directable fuzzy automata

V. Karthikeyan[1*]

Abstract
The purpose of this paper is to study the structural characterizations of γ-generalized directable fuzzy automata. We introduce γ-necks, γ-local necks, γ-directable, γ-trap-directable, γ-monogenically directable, γ- monogenically strongly directable,γ-monogenically trap directable, γ- uniformly monogenically directable, γ-uniformly monogenically strongly directable, γ-uniformly monogenically trap-directable fuzzy automata. We prove γ-generalized directable fuzzy automaton is an extension of a γ-uniformly monogenically strongly directable fuzzy automaton by a γ-uniformly monogenically trap-directable fuzzy automaton. We obtain equivalent conditions of γ-generalized directable fuzzy automaton.

Keywords
γ-monogenically directable, γ-monogenically trap-directable, γ-uniformly monogenically trap-directable, γ-generalized directable.

AMS Subject Classification
03D05, 20M35, 18B20, 68Q45, 68Q70, 94A45.

[1] *Department of Mathematics, Government College of Engineering, Dharmapuri,Tamilnadu, India.*
***Corresponding author**: [1] vkarthikau@gmail.com
Article History: Received **03** September **2018**; Accepted **22** December **2018**

Contents

1. Introduction

Automata theory is the study of abstract computing devices or machines. Before there were computers, in 1930's Alen Turing presented an abstract machine that has all the capabilities of today's computers. In 1940's and 1950's simpler kinds of machines, which we today call finite automata, were studied by a number of researchers. These automata originally proposed to model brain function, turned out to be extremely useful for a variety of other purposes. Also in the late 1950's, the linguist Naum Chomsky began the study of formal grammars, while not strictly machines, these grammars have close relationships to abstract automata and serve today on the basis of some important software components including parts of compilers. In 1969, S. Cook extend Turing's study of what could and what could not be computed. Finite automata are useful models for many kind of software. The following are some of them. Software for designing digital circuits, Pattern matching, File searching program and so on. A finite automaton consists of finite set of states and set of transitions from state to state that occur on input symbols chosen from a finite set of elements called alphabet. Any system that is at each moment in one of finite number of discrete states and moves among the states in response to individual input signals can be modeled by a finite automaton. Automata are basically language acceptors. The family of languages accepted by any finite automata is called the family of regular languages.

Fuzzy concept is introduced whenever uncertainty occurs. Fuzzy sets are sets whose elements have degrees of membership. Fuzzy sets were introduced by Lotfi A. Zadeh in 1965 [26] as an extension of the classical notion of set. In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition - an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval $[0,1]$. Fuzzy sets generalize classical sets, since the indicator functions of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1. In fuzzy set theory, classical bivalent sets are usually called crisp sets [5]. The fuzzy set theory can be used in a wide range of domains in which information is incomplete or imprecise.

Automata are the prime example of general computational systems over discrete spaces. The incorporation of fuzzy logic into automata theory resulted in fuzzy automata which can handle continuous spaces. Moreover, they are able to model uncertainty which is inherent in many applications. L. A. Zadeh (1965) [26] introduced the notion of fuzzy subset of a set as a tool for representing uncertainty. His ideas have been applied to wide range of scientific areas. W. Z. Wee (1967) [25] applied the ideas of Zadeh in automata theory and language theory. E. S. Santos (1968) [22] proposed fuzzy automata as a model of pattern recognition and control systems. Many other researchers like W. Wechler (1978) have worked in these areas. K. S. Fu and R. W. McLaren (1965) worked in applications of stochastic automata as a model of learning systems [7]. The syntactic approach to pattern recognition was examined by K. S. Fu (1982) using formal deterministic and stochastic languages [8]. Friedrich Steimann and Klaus-Peter Adlassnig (1994) dealt with applications of fuzzy automata in the field of Clinical Monitoring [23]. J. N. Mordeson and D. S. Malik gave a detailed account of fuzzy automata and languages in their book (2002) [17].

Directable automata, known also as synchronizable, cofinal and reset automata, are a significant type of automata with very interesting algebraic properties and important applications in various branches of Computer science [synchronization in binary messages, verification software, etc.]. They have been a subject of interest of many eminent authors since 1964, when they were introduced by J. Cerny in [4], although some of their special types were investigated even several years earlier. Various other specializations and generalizations of directable automata have appeared recently. In the papers by Petkovic, Ciric and Bogdanovic, and Bogdanovic, Ciric, Imreh, Petkovic and Steinby, and in the Ph.D thesis by Petkovic, trap-directable, trapped, generalized directable, locally directable, uniformly locally directable and other related kinds of automata have been introduced and studied. Certain generalizations of directability and definiteness have also appeared in theories of nondeterministic automata and tree automata and languages (cf. [6], [9, 10], [11], and [18] for example). In some origins various other names for directable automata and directing words are used. For example, J. E. Pin used in [19, 20] the names "synchronizable automata" and "synchronizing words", M. Ito and J. Duske in [12] used the name "'cofinal automata", whereas the names "reset automata" and "'reset words" were used by I. Rystsov. In [16] definite automata were studied under the name "local automata". Rosa Montalbano investigated the problem of a completing a finite automata preserving its properties in the case of deterministic local automata.

T. Petkovic et al. [1] discussed directable automata, monogenically directable, generalized directable using necks. T. Petkovic et al.[3] introduce and studied trap-directable, trapped automata and other related automata. Also, we refer the survey paper Directable automata and their generalizations were investigated by S. Bogdanovic et al [2]. The notion of the generalized directable automaton was introduced by T. Petkovic et al.[3]. Consequently T. Petkovic et al.[21] were studied structural characterizations of generalized directable automata. Subsequently, the necks and local necks, generalized directable of fuzzy automata were studied and discussed in [13–15]. In this paper we introduce γ-necks, γ-local necks, γ-monogenically directable, γ- monogenically strongly directable, γ-monogenically trap directable, γ- uniformly monogenically directable, γ-uniformly monogenically strongly directable, γ-uniformly monogenically trap-directable,γ-generalized directable fuzzy automata and study their structural characterizations. We prove γ-generalized directable fuzzy automaton is an extension of a γ-uniformly monogenically strongly directable fuzzy automaton by a γ-uniformly monogenically trap-directable fuzzy automaton. We obtain equivalent conditions for a γ-generalized directable fuzzy automaton.

2. Preliminaries

Definition 2.1. *[17] A fuzzy automaton $S = (D, I, \psi)$, where,*

D - set of states $\{d_0, d_1, d_2, \dots, d_n\}$,
I - alphabets (or) input symbols,
ψ - function from $D \times I \times D \to [0,1]$,

The set of all words of I is denoted by I^*. The empty word is denoted by λ, and the length of each $t \in I^*$ is denoted by $|t|$.

Definition 2.2. *[17] Let $S = (D, I, \psi)$ be a fuzzy automaton. The extended transition function is defined by $\psi^* : D \times I^* \times D \to [0,1]$ and is given by*

$$\psi^*(d_i, \lambda, d_j) = \begin{cases} 1 & \text{if } d_i = d_j \\ 0 & \text{if } d_i \neq d_j \end{cases}$$

$\psi^*(d_i, tt', d_j) = \vee_{q_r \in D}\{\psi^*(d_i, t, d_r) \wedge \psi(d_r, t', d_j)\}, t \in I^*, t' \in I$.

Definition 2.3. *[13] Let $S = (D, I, \psi)$ be a fuzzy automaton. Let $D' \subseteq D$. Let ψ' is the restriction of ψ and let $S' = (D', I, \psi')$. The fuzzy automaton S' is called a subautomaton of S if*

(i) $\psi' : D' \times I \times D' \to [0,1]$ and
(ii) For any $d_i \in D'$ and $\psi'(d_i, t, d_j) > 0$ for some $t \in I^*$, then $d_j \in D'$.

Definition 2.4. *[17] Let $S = (D, I, \psi)$ be a fuzzy automaton. S is said to be strongly connected if for every $d_i, d_j \in D$, there exists $t \in I^*$ such that $\psi^*(d_i, t, d_j) > 0$. Equivalently, S is strongly connected if it has no proper subautomaton.*

Definition 2.5. *[14]* *A relation R on a set D is said to be equivalence relation if it is reflexive, symmetric and transitive.*

Definition 2.6. *[14]* *Let $S = (D,\ I,\ \psi)$ be a fuzzy automaton. An equivalence relation R on D in S is called congruence relation if $\forall d_i, d_j \in D$ and $t \in I, d_i R d_j$ implies that, then there exists $d_l, d_k \in D$ such that $\psi(d_i, a, d_l) > 0, \psi(d_j, a, d_k) > 0$ and $d_l R d_k$.*

Definition 2.7. *[13]* *Let $S = (D,\ I,\ \psi)$ be a fuzzy automaton. Let $S' = (D',\ I,\ \psi')$ be a subautomaton of S. A relation $R_{S'}$ on S is defined as follows. For any $d_i, d_j \in D$, we say that $(d_i, d_j) \in R_{S'}$ if and only if either $d_i = d_j$ or $d_i, d_j \in D'$.*
This relation is clearly an equivalence relation and it is also congruence. This relation is called Rees congruence relation on D in S determined by S'. A fuzzy automaton S/S' is called Rees factor fuzzy automaton determined by the relation $R_{S'}$ and it is defined as $S/S' = (\ \overline{D},\ I,\ \psi_{S/S'})$, where $\overline{D} = \{\ [d_i]\ /\ d_i \in D\}$ and $\psi_{S/S'} : \overline{D} \times I \times \overline{D} \to [0,1]$.

Definition 2.8. *[13]* *Let $S = (D,\ I,\ \psi)$ be a fuzzy automaton. A state $d_j \in D$ is called a neck of S if there exists $t \in I^*$ such that $\psi^*(d_i,\ t,\ d_j) > 0$ for every $d_i \in D$.*
In that case d_j is also called t-neck of S and the word t is called a directing word of S.
If S has a directing word, then we say that S is a directable fuzzy automaton.

Remark 2.9. *In this paper we consider only determinstic fuzzy automaton.*

3. γ-Necks, γ-Local Necks, γ-Generalized Directable of Fuzzy Automata

Definition 3.1. *Let $S = (D,\ I,\ \psi)$ be a fuzzy automaton and let $d_i \in D$. The γ-subautomaton of S generated by d_i is denoted by $\langle d_i \rangle$. It is given by*
$\langle d_i \rangle = \{\ d_j \mid \psi^*(d_i,\ t,\ d_j) \geq \gamma > 0,\ t \in I^*,\ \gamma \in [0,1]\}$. If it exists, then it is called the γ- least subautomaton of S containing d_i.

Definition 3.2. *Let $S = (D,\ I,\ \psi)$ be a fuzzy automaton. For any non-empty $D' \subseteq D$, the γ-subautomaton of S generated by D' is given by $\langle D' \rangle$ and is given by*
$\langle D' \rangle = \{\ d_j \mid \psi^*(d_i,\ t,\ d_j) \geq \gamma > 0,\ d_i \in D',\ t \in I^*\}$. It is called the γ-least subautomaton of S containing D'. The γ-least subautomaton of a fuzzy automaton S if it exists is called the γ-kernel of S.

Definition 3.3. *Let $S = (D,\ I,\ \psi)$ be a fuzzy automaton. A state $d_j \in D$ is called a γ-neck of S if there exists $t \in I^*$ such that $\psi^*(d_i,\ t,\ d_j) \geq \gamma,\ \gamma \in [0,1]$ for every $d_i \in D$.*
In that case d_j is also called t-neck of S and the word t is called a γ-directing word of S.
If S has a γ- directing word, then we say that S is a γ-directable fuzzy automaton.

Remark 3.4. *1) The set of all γ-necks of a fuzzy automaton S is denoted by $\gamma N(S)$.*
2) The set of all γ-directing words of a fuzzy automaton S is denoted by $\gamma DW(S)$.
3) A fuzzy automaton S is called γ-strongly directable if $D = \gamma N(S)$.

Definition 3.5. *Let $S = (D,\ I,\ \psi)$ be a fuzzy automaton. A state $d_j \in D$ is called a γ-trap of S if $\psi^*(d_j,\ t,\ d_j) \geq \gamma$, $\forall t \in I^*$.*
If S has exactly one γ-trap, then S is called one γ-trap fuzzy automaton. The set of all γ-traps of a fuzzy automaton S is denoted by $\gamma TR(S)$.
A fuzzy automaton S is called a γ-trapped fuzzy automaton, for each $d_i \in D$,if there exists a word $t \in I^*$ such that $\psi^*(d_i,\ t,\ d_j) \geq \gamma,\ \ d_j \in \gamma TR(S)$.

Definition 3.6. *Let $S = (D,\ I,\ \psi)$ be a fuzzy automaton. If S has a single γ-neck, then S is called a γ-trap-directable fuzzy automaton.*

Definition 3.7. *Let $S = (D,\ I,\ \psi)$ be a fuzzy automaton. A state $d_i \in D$ is called γ-local neck of S if it is γ-neck of some γ- directable subautomaton of S. The set of all γ-local necks of S is denoted by $\gamma LN(S)$.*

Definition 3.8. *Let $S = (D,\ I,\ \psi)$ be a fuzzy automaton. S is called γ-monogenically directable if every monogenic subautomaton of S is γ-directable.*

Definition 3.9. *Let $S = (D,\ I,\ \psi)$ be a fuzzy automaton. S is called γ-monogenically strongly directable if every monogenic subautomaton of S is γ-strongly directable.*

Definition 3.10. *Let $S = (D,\ I,\ \psi)$ be a fuzzy automaton. S is called γ-monogenically trap-directable if every monogenic subautomaton of S has a single γ-neck.*

Definition 3.11. *Let $S = (D,\ I,\ \psi)$ be a fuzzy automaton. If $t\ \in\ I^*$ is γ-common directing word of S if t is a γ-directing word of every monogenic subautomaton of S. The set all γ-common directing words of S will be denoted by $\gamma CDW(S)$. In other words, $\gamma CDW(S) = \cap_{d_i\ \in\ D}\gamma DW(\langle d_i \rangle)$.*

Definition 3.12. *Let $S = (D,\ I,\ \psi)$ be a fuzzy automaton. S is called γ- uniformly monogenically directable fuzzy automaton if every monogenic subautomaton of S is γ-directable and have atleast one γ-common directing word.*

Definition 3.13. *Let $S = (D,\ I,\ \psi)$ be a fuzzy automaton. S is called γ- uniformly monogenically strongly directable fuzzy automaton if every monogenic subautomaton of S is strongly γ-directable and have atleast one γ-common directing word.*

Definition 3.14. *Let $S = (D,\ I,\ \psi)$ be a fuzzy automaton. S is called γ- uniformly monogenically trap directable fuzzy automaton if every monogenic subautomaton of S has a single γ-neck and have atleast one γ-common directing word.*

Definition 3.15. *Let $S = (D,\ I,\ \psi)$ be a fuzzy automaton. S is called a γ-generalized directable fuzzy automaton if for every $t' \in I^*$ and $d_i \in D$, there exists a word $t \in I^*$ and $d_j \in D$ such that $\psi(d_i, tt't, d_j) \geq \gamma > 0 \Leftrightarrow \psi(d_i, t, d_j) \geq \gamma > 0$ and the word t is called γ-generalized directing word of a fuzzy automaton S.*

4. Properties of γ-Necks, γ-Local Necks, and γ-Generalized Directable Fuzzy Automata

Theorem 4.1. *Let $S = (D,\ I,\ \psi)$ be a fuzzy automaton. If $\gamma N(S) \neq \phi$, then $\gamma N(S)$ is a γ-subautomaton of S.*

Proof. Let $S = (D,\ I,\ \psi)$ be a fuzzy automaton. Let $d_j \in \gamma N(S)$ and $t' \in I^*$. Assume that d_j is a t-γ-neck of S, for some $t \in I^*$. Then for each $d_i \in D$ we have
$\psi^*(d_i,\ tt',\ d_k) = \wedge_{d_j\ \in\ D}\{\{\psi^*(d_i,\ t,\ d_j), \psi^*(d_j,\ t',\ d_k)\}\} \geq \gamma$, it means that d_k is a tt'-γ-neck of S and hence, $d_k \in \gamma N(S)$. Therefore, $\gamma N(S)$ is a γ-subautomaton of S. □

Theorem 4.2. *A fuzzy automaton $S = (D,\ I,\ \psi)$ is γ-strongly directable fuzzy automaton if and only if it is strongly connected and γ-directable.*

Proof. Let $S = (D,\ I,\ \psi)$ be a γ-strongly directable fuzzy automaton. It is clearly γ-directable. Now we will prove it is strongly connected. It is enough to show that for any $d_i, d_j \in D$, there exists $t \in I^*$ such that $\psi^*(d_i, t, d_j) \geq \gamma > 0$. Since $d_j \in \gamma N(S)$ $[\gamma N(S) = D]$, $\psi^*(d_k, t, d_j) \geq \gamma > 0$, for every $d_k \in S$.
Therefore, $\psi^*(d_i, t, d_j) \geq \gamma > 0$. Thus, S is strongly connected.
Conversely, let S be strongly connected and γ-directable. Then $\gamma N(S) \neq \phi$ and by Theorem 4.1, $\gamma N(S)$ is γ-subautomaton of S. Since S is strongly connected, there is no proper subautomaton. Hence, $D = \gamma N(S)$. Thus, S is strongly γ-directable fuzzy automaton. □

Theorem 4.3. *A fuzzy automaton $S = (D,\ I,\ \psi)$ is γ-directable if and only if it is an extension of a γ-strongly directable fuzzy automaton S' by a γ-trap-directable fuzzy automaton S''.*
(i) $\gamma DW(S'').\gamma DW(S') \subseteq \gamma DW(S) \subseteq \gamma DW(S'') \cap \gamma DW(S')$;
(ii) $\gamma N(S) = S'$.

Proof. Let S be γ-directable fuzzy automaton. Then $\gamma N(S)$ is non-empty and by Theorem 4.1, $\gamma N(S)$ is a γ-subautomaton of S.
The Rees factor fuzzy automaton $S/\gamma N(S)$ is also γ-directable. Further, by Rees factor, $S/\gamma N(S)$ is a γ- trap-directable fuzzy automaton and hence, S is an extension of a γ-strongly directable fuzzy automaton $\gamma N(S)$ by a γ- trap-directable fuzzy automaton $S/\gamma N(S)$.
Conversely, let S be an extension of γ-strongly directable fuzzy automaton S' by a γ-trap-directable fuzzy automaton S''. Let $t \in \alpha DW(S'')$ and $t' \in \gamma DW(S')$. Then for all $d_i,\ d_j \in D$ we have that $\psi^*(d_i,\ t,\ d_k) \geq \gamma$, $\psi^*(d_j,\ t,\ d_k) \geq \gamma$, where $d_k \in S'$. Hence,
$\psi^*(d_i,\ tt',\ d_m) = \wedge\{\{\psi^*(d_i,\ t,\ d_k), \psi^*(d_k,\ t',\ d_m)\}\} \geq \gamma$
Thus, $tt' \in \gamma DW(S)$ and hence, S is a γ-directable fuzzy automaton.
If $t \in \gamma DW(S'')$ and $t' \in \gamma DW(S')$, then $tt' \in \gamma DW(S)$.
Therefore, $\gamma DW(S'').\gamma DW(S') \subseteq \gamma DW(S)$.
Let $t \in \gamma DW(S)$. Since S is an extension of a γ-strongly directable fuzzy automaton S' by a γ-trap-directable fuzzy automaton S''.
Therefore, t is a γ-directing word of S' and S''.
Hence, $\gamma DW(S) \subseteq \gamma DW(S') \cap \gamma DW(S'')$.
Thus, (i) holds and which implies that $\gamma N(S)$ is the γ- kernel of S, so $\gamma N(S) \subseteq S'$.
Conversely, assume that $d_j \in S'$. Since S' is γ-strongly directable, we conclude that there exists $t' \in \gamma DW(S')$ such that $\psi^*(d_i,\ t',\ d_j) \geq \gamma$, for every $d_i \in S'$. Hence, for every $d_i \in D$ and $t \in \gamma DW(S'')$, $\psi^*(d_i,\ t,\ d_l) \geq \gamma$, where $d_l \in S'$.
Now, $\psi^*(d_i,\ tt',\ d_j) = \wedge_{d_l\ \in\ S'}\{\{\psi(d_i,\ t,\ q_l), \psi(d_l,\ t',\ d_j)\}\} \geq \gamma$. Therefore, $d_j \in \gamma N(S)$ and hence, $\gamma N(S) = S'$. □

Theorem 4.4. *Let $S = (D,\ I,\ \psi)$ be a fuzzy automaton and $d_i \in D$. Then the following conditions are equivalent:*
(i) d_i is a γ-local neck;
(ii) $\langle d_i \rangle$ is a γ-strongly directable fuzzy automaton;
(iii) for every $t' \in I^*$, there exists $t \in I^*$ such that $\psi^*(d_i,\ t't,\ d_i) \geq \gamma > 0$.

Proof. $(i) \Rightarrow (ii)$
Let d_i be a γ-local neck of S. Then there exists a γ-directable subautomaton S' of S such that $d_i \in \gamma N(S')$. Thus $\gamma N(S')$ is a γ-strongly directable fuzzy automaton. Also, $\langle d_i \rangle \subseteq \gamma N(S')$, and $\gamma N(S')$ is strongly connected, then $\langle d_i \rangle = \gamma N(S')$. Therefore, $\langle d_i \rangle$ is a γ-strongly directable fuzzy automaton.
$(ii) \Rightarrow (iii)$
Let $\langle d_i \rangle$ be a γ-strongly directable fuzzy automaton. Then d_i is a t-γ-neck of $\langle d_i \rangle$ for some $t \in I^*$. Since $\langle d_i \rangle$ is γ-strongly directable, for every $t' \in I^*$, there exists some $d_l \in \langle d_i \rangle$ such that $\psi^*(d_i,\ t',\ d_l) \geq \gamma > 0$. Now,
$\psi^*(d_i,\ t't,\ d_i) = \{\wedge_{d_l\ \in\ D}\{\psi^*(d_i,\ t',\ d_l), \psi^*(d_l,\ t,\ d_i)\}\} \geq \gamma > 0$.
$(iii) \Rightarrow (i)$
(iii) clearly shows that d_i is a $t-$ γ-neck of $\langle d_i \rangle$, and hence, it is a γ-local neck of S. □

Theorem 4.5. *Let $S = (D,\ I,\ \psi)$ be a fuzzy automaton. If $\gamma LN(S) \neq \phi$, then $\gamma LN(S)$ is a γ-subautomaton of S.*

Proof. Let $d_i \in \gamma LN(S)$ and $t \in I$. Then, the γ-monogenic subautomaton $\langle d_i \rangle$ of S is γ-strongly directable. Now, $\langle d_i \rangle \subseteq \langle d_l \rangle$, for some $d_l \in \langle d_i \rangle$. Since $\langle d_i \rangle$ is strongly connected, $\langle d_i \rangle = \langle d_l \rangle$. Therefore, d_l is also a γ-local neck of S, i.e., $d_l \in \gamma LN(S)$. Hence, $\gamma LN(S)$ is a γ-subautomaton of S. □

Theorem 4.6. *A fuzzy automaton $S = (D,\ I,\ \psi)$ is γ-generalized directable if and only if it is an extension of a*

γ-uniformly monogenically strongly directable fuzzy automaton S' by a γ-uniformly monogenically trap-directable fuzzy automaton S''. In that case:
(i) $\gamma DW(S'').\gamma CDW(S') \subseteq \gamma GDW(S) \subseteq \gamma DW(S'') \cap \gamma CDW(S)$;
(ii) $\gamma LN(S) = S'$.

Proof. Let S be a γ-generalized directable fuzzy automaton.
Let $S' = \{d_j \mid \psi^*(d_i,t,d_j) \geq \gamma > 0, \ d_i \in D, \ t \in \gamma GDW(S)\}$ be a subautomaton of S. Now we have to show that S' is a γ uniformly monogenically strongly directable fuzzy automaton.
Let $d_j \in S'$. That is, $\psi^*(d_i,t,d_j) \geq \gamma > 0$, for some $d_i \in D$ and $t \in \gamma GDW(S)$.
Then for every $t' \in I^*$ we have that
$\psi^*(d_i,t,d_j) \geq \gamma > 0 \Leftrightarrow \psi^*(d_i,tt't,d_j) \geq \gamma > 0$
$\Leftrightarrow \{\wedge_{d_j \in Q}\{\psi^*(d_i,t,d_j), \psi^*(d_j,t't,d_j)\}\} \geq \gamma > 0$
$\Rightarrow \psi^*(d_j,t't,d_j) \geq \gamma > 0$.
d_j is a *gamma*-local neck and $\langle d_j \rangle$ is a γ-strongly directable fuzzy automaton. Further $t \in \gamma CDW(\langle d_j \rangle)$.
Therefore, S' is a γ-uniformly monogenically strongly directable fuzzy automaton and $\gamma GDW(S) \subseteq \gamma CDW(S')$—(1)
Define Rees congruence on S. Then there exists a γ-factor fuzzy automaton
$S'' = S/S'$ which is a γ-uniformly monogenically trap-directable fuzzy automaton and $\gamma GDW(S) \subseteq DW(S'')$ ——(2)
From (1) and (2), $\gamma GDW(S) \subseteq \gamma CDW(S') \cap \gamma DW(S'')$ —(3)
Conversely, let S be an extension of a γ-uniformly monogenically strongly directable fuzzy automaton S' by a γ-uniformly monogenically trap-directable fuzzy automaton S''. Consider an arbitrary $d_i \in D$, $t_1 \in \gamma DW(S'')$,
$t_2 \in \gamma CDW(S')$ and $t' \in I^*$.
Now, let $t = t_1t_2 \in I^*$. Then
$\psi^*(d_i,t_1,d_k) \geq \gamma > 0$, $\psi^*(d_i,t_1t_2t't_1,d_k) \geq \gamma > 0$ where $d_k \in \langle d_i \rangle$ for some γ-strongly directable subautomaton $\langle d_i \rangle$ of S'.
Now,
$\psi^*(d_i,tt't,d_j) = \psi^*(d_i,t_1t_2t't_1t_2,d_j) \geq \gamma > 0$
$\Leftrightarrow \{\wedge_{d_k \in D}\{\psi^*(d_i,t_1t_2t't_1,d_k), \psi^*(d_k,t_2,d_j)\}\} \geq \gamma > 0$
$\Rightarrow \psi^*(d_k,t_2,d_j) \geq \gamma > 0$. Now,
$\psi^*(d_i,t_1t_2,d_j) = \{\wedge_{d_k \in D}\{\psi^*(d_i,t_1,d_k), \psi^*(d_k,t_2,d_j)\}\} \geq \gamma > 0$.
$\psi(d_i,tt't,d_j) \geq \gamma > 0 \Leftrightarrow \psi(d_i,t,d_j) \geq \gamma > 0$.
Therefore, S is a γ-generalized directable fuzzy automaton and $t \in \gamma GDW(S)$.
Hence, $\gamma DW(S'').\gamma CDW(S') \subseteq \gamma GDW(S)$ ———(4)
From (3) and (4),
$\gamma DW(S'').\gamma CDW(S') \subseteq \gamma GDW(S) \subseteq \gamma DW(S'') \cap \gamma CDW(S')$.
Now let us prove that $\gamma LN(S) = S'$.
Clearly, $S' \subseteq \gamma LN(S)$. Conversely, let $d_i \in \gamma LN(S)$. Then, $\forall t' \in I^*$, $\exists t \in I^*$ such that $\psi^*(d_i,t't,d_i) \geq \gamma > 0$.
If we assume that $t' \in \gamma DW(S'')$, then $\psi^*(d_i,t',d_k) \geq \gamma > 0$, for some $d_k \in S'$. Now,
$\psi(d_i,t't,d_i) \geq \gamma > 0 \Leftrightarrow$
$\{\wedge_{d_k \in D}\{\psi^*(d_i,t',d_k), \psi^*(d_k,t,d_i)\}\} \geq \gamma > 0$
$\Rightarrow \psi^*(d_k,t,d_i) \geq \gamma > 0$
$\Rightarrow d_i \in S'$[Since S' is strongly connected].
Therefore, $\gamma LN(S) \subseteq S'$. Hence, $\gamma LN(S) = S'$.

□

Theorem 4.7. *Let $S = (D, I, \psi)$ be a fuzzy automaton. Then the following conditions are equivalent:*
(i) S is a γ-generalized directable fuzzy automaton;
(ii) every strongly connected subautomaton of S is γ-directable;
(iii) every subautomaton of S contains a γ-direcable subautomaton;
(iv) $(\forall d_i \in D)(\exists t \in I^*)(\forall t' \in I^*)(\exists t_1 \in I^*)$ such that $\psi^*(d_i,tt't_1,d_l) \geq \gamma > 0 \Leftrightarrow \psi^*(d_i,tt',d_l) \geq \gamma > 0$, for some $d_l \in D$.

Proof. $(i) \Rightarrow (ii)$
Let S be a γ-generalized directable fuzzy automaton.
Let $S' = \{d_j \mid \psi^*(d_i,t,d_j) \geq \gamma > 0, \ d_i \in D, \ t \in \gamma GDW(S)\}$ be a subautomaton of S. Now we have to show that S' is a γ-strongly directable fuzzy automaton.
Let $d_j \in S'$. That is, $\psi^*(d_i,t,d_j) \geq \gamma > 0$, for some $d_i \in D$ and $t \in \gamma GDW(S)$.
Then for every $t' \in I^*$ we have that
$\psi^8(d_i,t,d_j) \geq \gamma > 0 \Leftrightarrow \psi^*(d_i,tt't,d_j) \geq \gamma > 0$
$\Leftrightarrow \wedge_{d_j \in D}\{\psi^*(d_i,t,d_j), \psi^*(d_j,t't,d_j)\} \geq \gamma > 0$
$\Rightarrow \psi^*(d_j,t't,d_j) \geq \gamma > 0$.
Thus d_j is a γ-local neck and $\langle d_j \rangle$ is a γ-strongly directable fuzzy automaton.
$(ii) \Rightarrow (i)$
It is clear that S is an extension of a fuzzy automaton S' by a trap-directable fuzzy automaton S'', where S' is a direct sum strongly connected of fuzzy automata S'_i, $i \in [1,n]$.
By the hypothesis it follows that S'_i is a γ-directable fuzzy automaton, for every $i \in [1,n]$.
Since $\gamma DW(S'_i)$ is an ideal of I^*, for each $i \in [1,n]$ and the intersection of any finite family of ideals is non-empty, then there exists $t \in \cap_{i=1}^{n} \gamma DW(S'_i)$.
Thus S is a γ generalized directable fuzzy automaton.
$(ii) \Rightarrow (iii)$
Let S'' be any γ-strongly connected directable subautomaton of S.
Since S'' is strongly connected and γ-directable, $\gamma N(S'') = S''$ which is a γ-least subautomaton of S.
If S' is any other γ-subautomaton of S, then $S'' \subseteq S'$. Hence, S' is the γ-subautomaton of S that contains a γ-directable subautomaton S''.
$(iii) \Rightarrow (i)$
Consider an arbitrary $d_i \in D$. By the hypothesis, theγ- monogenic subautomaton $\langle d_i \rangle$ contains a γ-directable subautomaton S'.
Therefore, there exists a $t_1 \in I^*$ such that $\psi^*(d_i,t_1,d_k) \geq \gamma > 0$, for some $d_k \in S'$.
Let $t = t_1t_2$, where $t_2 \in \gamma DW(S')$ and let $t' \in I^*$.
Now,$\psi^*(d_i,t,d_j) = \psi^*(d_i,t_1t_2,d_j)$
$= \wedge_{d_k \in S'}\{\psi^*(d_i,t_1,d_k), \psi^*(d_k,t_2,d_j)\}$.
Since $\psi^*(d_i,t_1,d_k) \geq \gamma > 0$ and S is a deterministic fuzzy automaton, we have $\psi^*(d_k,t_2,d_j) \geq \gamma > 0$.
Therefore, $\psi^*(d_i,u,d_j) \geq \gamma > 0$ —(1)

Now, $\psi^*(d_i,tt't,d_j) = \{\wedge_{d_j \in D}\{\psi^*(d_i,t,d_j), \psi^*(d_j,t't,d_j)\}\}$.
Since from (1), $\psi^*(d_i,t,d_j) \geq \gamma > 0$ and S is a deterministic fuzzy automaton,
we have $\psi^*(d_j,t't,d_j) \geq \gamma > 0$.
Therefore, $\psi^*(d_i,tt't,d_j) \geq \gamma > 0$ ——(2)
From (1) and(2), we have
$(\forall d_i \in D)(\exists t \in I^*)(\forall t' \in I^*)$ such that $\psi^*(d_i,tt't,d_j) \geq \gamma > 0 \Leftrightarrow \psi^*(d_i,t,d_j) \geq \gamma > 0$.
$(i) \Rightarrow (iv)$
By the hypothesis, $(\forall\, d_i \in D)(\exists\, t \in I^*)(\forall\, t' \in I^*)$ such that
$\psi^*(d_i,tt't,d_j) \geq \gamma > 0 \Leftrightarrow \psi^*(d_i,t,d_j) \geq \gamma > 0$. Let $t_1 = tt'_1$ for some $t'_1 \in I^*$.
Now,$\psi^*(d_i,tt'tt'_1,d_l) = \psi^*(d_i,tt'tt'_1,d_l)$
$= \wedge_{d_j \in D}\{\psi(d_i,tt't,d_j), \psi(d_j,t_1,d_l)\}$
By the hypothesis, $\psi^8(d_i,tt't,d_j) \geq \gamma > 0$ and since S is a deterministic fuzzy automaton $\psi^*(d_j,t_1,d_l) \geq \gamma > 0$. Therefore,
$\psi^*(d_i,tt't_1,d_l) \geq \gamma > 0$ —(1)
$\psi^*(d_i,tt_1,d_l) = \psi^*(d_i,ttt'_1,d_l)$
$\Rightarrow \wedge_{d_j \in D}\{\psi^*(d_i,tt,d_j), \psi^*(d_j,t_1,d_l)\}$.
Since $\psi^*(d_i,t,d_j) \geq \gamma > 0$, we have $\psi^*(d_i,tt,d_j) \geq \gamma > 0$ and therefore, $\psi^*(d_j,t_1,t_l) \geq \gamma > 0$.
Hence, $\psi^*(d_i,tt_1,d_l) \geq \gamma > 0$—(2)
From (1)and (2), $\psi^*(d_i,tt't_1,d_l) \geq \gamma > 0 \Leftrightarrow \psi^*(d_i,tt_1,d_l) \geq \gamma > 0$.
Therefore, $(\forall\, d_i \in D)(\exists\, t \in I^*)(\forall\, t' \in I^*)(\exists\, t_1 \in I^*)$ such that
$\psi^*(d_i,tt't_1,d_l) \geq \gamma > 0 \Leftrightarrow \psi^*(d_i,tt_1,d_l) \geq \gamma > 0$.

$(iv) \Rightarrow (ii)$
By the hypothesis, $(\forall\, d_i \in D)(\exists\, t \in I^*)(\forall\, t' \in I^*)(\exists\, t_1 \in I^*)$ such that
$\psi^*(d_i,tt't_1,d_l) \geq \gamma > 0 \Leftrightarrow \psi^*(d_i,tt_1,d_l) \geq \gamma > 0$ for some $d_l \in D$. Take an arbitrary γ- strongly connected subautomaton S' of S and $d_i, d_k \in S'$.
Now, $\psi^*(d_i,t,d_j) \geq \gamma > 0$ and $\psi^*(d_k,t,d_l) \geq \gamma > 0$, for some $t \in I^*$ and $d_j, d_l \in S'$. Since S' is γ-strongly connected, there exists $t'_1 \in I^*$ such that
$\psi^*(d_i,tt'_1,d_l) \geq \gamma > 0$. ——(1)
For that t'_1, there exists $t_2 \in I^*$ such that
$\psi^*(d_i,tt'_1t_2,d_m) \geq \gamma > 0 \Leftrightarrow \psi^*(d_i,tt_2,d_m) \geq \gamma > 0$, for some $d_m \in S'$—(2)
$\psi^*(d_i,tt_2,d_m) \geq \gamma > 0 \Leftrightarrow \psi^*(d_i,tt'_1t_2,d_m) \geq \gamma > 0$.
$\Leftrightarrow \wedge_{d_l \in S'}\{\psi^*(d_i,tt'_1,d_l), \psi^*(d_l,t_2,d_m)\} \geq \gamma > 0$
$\Rightarrow \psi^*(d_l,t_2,d_m) \geq \gamma > 0$.
Now $\psi^*(d_k,tt_2,d_m) = \{\wedge_{d_l \in S'}\{\psi^*(d_k,t,d_l), \psi^*(d_l,t_2,d_m)\}\} \geq \gamma > 0$
Therefore, we have proved that d_i and d_k are γ-mergeable. Hence,S' is a γ-directable fuzzy automaton. □

5. Conclusion

In this paper we introduce γ-necks, γ-local necks, γ-monogenically directable, γ- monogenically strongly directable, γ-monogenically trap directable, γ- uniformly monogenically directable, γ-uniformly monogenically strongly directable, γ-uniformly monogenically trap-directable, γ-generalized directable fuzzy automata and study their structural characterizations. We prove γ-generalized directable fuzzy automaton is an extension of a γ-uniformly monogenically strongly directable fuzzy automaton by a γ-uniformly monogenically trap-directable fuzzy automaton. We obtain equivalent conditions for a γ-generalized directable fuzzy automaton.

References

[1] M. Bogdanovic, S. Bogdanovic, M. Ciric, and T. Petkovic, Necks of automata, *Novi Sad J. Math.* 34(2) (2004), 5 - 15.
[2] M. Bogdanovic, B. Imreh, M. Ciric, and T. Petkovic, Directable automata and their Generalization (A Survey), *Novi Sad J. Math.*, 29(2) (1999), 31-74.
[3] S. Bogdanovic, M. Ciric, and T. Petkovic, Directable automata and Transition Semigroups, *Acta Cybernetica(Szeged),* 13 (1998), 385-403.
[4] J. Cerny, Poznamka k homogenym experimentom s konecinymi automatami, it Mat-Fyz. Cas. SAV, 14 (1964), 208-215.
[5] D. Dubois, and H. Prade, Fuzzy sets and systems, *Academic Press,* New York, (1988).
[6] Z. Esik, Definite tree automata and their cascade compositions, *Automata and Formal Languages, VII (Salgotarjan, 1993), Publ. Math. Debrecen,* 48 (3-4) (1996), 243-261.
[7] K. S. Fu, and R. W. McLaren, An application of stochastic automata to the synthesis of learning systems, *School of Elec. Eng., Purdue University, Tech. Rept. TR-EE65-17* (1965).
[8] K. S. Fu, Syntactic pattern recognition and applications, *Prentice-Hall, Englewood Cliffs, NJ,* (1982).
[9] U. Heuter, Definite tree languages, *Bull. EATCS* 35 (1988), 137-144.
[10] U. Heuter, Generalized definite tree languages, *Mathem. Found. Comput. Sci. (Proc. Symp., Porabka-Kozubnik, Poland 1989). Lect. Notes in Comput. Sci. 379, Springer-Verlag, Berlin* (1989), 270-280.
[11] B.Imreh, and M. Steinby, Directable non-deterministic automata, *Acta Cybernetica,* 14 (1999), 105-115.
[12] M. Ito, and J. Duske, On cofinal and definite automata, *Acta Cybernetica,* 6 (1983), 181 - 189.
[13] V. Karthikeyan, and M. Rajasekar, Necks of fuzzy automata , *Proceedings of International Conference on Mathematical Modeling and Applied Soft Computing, Shanga Verlag,* July 11-13, (2012), 15-20.
[14] V. Karthikeyan, and M. Rajasekar, Local necks of fuzzy automata , Advances in Theoretical and Applied Mathematics,7(4), (2012), 393-402.
[15] V. Karthikeyan, and M. Rajasekar, Generalized directable fuzzy automata , International Journal of Computer Applications, 131(12), (2015), 1-5.
[16] R. Montalbano, Local automata and completion, *STACS'93 10th Annual. Symp. on Ther. Comp. Sci.,*

Wurzburg, Germany, Lecture Notes in Computer Science, 665, Springer-Verlag, (1993), 332-342.

[17] J. N. Mordeson, and D. S. Malik, Fuzzy automata and languages-theory and applications, *Chapman & Hall/ CRC Press,* (2002).

[18] M. Nivat, and A. Podelski, Definite tree automata (cont'd), *Bull. EATCS,* 38 (1989), 186-190.

[19] J. E. Pin, Sur les mots synchronisants dans un automata fini, *Elektron. Inform. Verarb. U. Kybernetik, EIK,* 14 (1978), 297-303.

[20] J. E. Pin, Sur un cas particular de la conjecture de cerny, *Automata, Languages and Programming, ICALP'79(Proc. Coll., Udine 1979), LNCS 62, Springer-Verlag, Berlin*, (1979), 345-352.

[21] Z. Popovic, S. Bogdanovic, T. Petkovic,and M. Ciric., Generalized directable automata, Words, Languages and Combinatories.III. Proceedings of the Third International Colloquium in Kyoto, Japan, (M.Ito and T. Imaka, eds.), World Scientific, (2003), 378- 395.

[22] E. S. Santos, General formulation sequential machines, *Information and Control,* 12 (1968), 5-10.

[23] F. Steimann, and K.P. Adlassnig, Clinical monitoring with fuzzy automata, *Fuzzy Sets and Systems,* 61 (1994), 37-42.

[24] M. Steinby, On definite automata and related systems, *Ann. Acad. Sci. Fenn., Ser. A, I Math. 444, Helsinki* (1969).

[25] W. G. Wee, On generalizations of adaptive algorithms and application of the fuzzy sets concepts to pattern classification Ph.D. Thesis, Purdue University, (1967).

[26] L. A. Zadeh, Fuzzy sets, Information and Control, 8 (3) (1965),338-353.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆