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Fixed point theorems based on well-posedness and periodic point property in ordered bicomplex valued metric spaces

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Abstract

In this paper, we aim to establish and prove several results on common fixed point for pair of mappings satisfying more general contraction conditions portrayed by rational expressions in bicomplex valued metric spaces. Also, we introduced well-posedness and periodic point property of mappings satisfying a rational inequality in ordered bicomplex valued metric spaces.

Keywords

Bicomplex number, Bicomplex valued metric space, Weakly increasing mappings, Well-posedness, Periodic point, Common fixed point. **AMS Subject Classification**

47H10, 54H25.

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Contents

1. Introduction

Fixed point theory is a widely studied field in mathematical analysis and has many applications in different branches of pure and applied mathematics. There has been a lot of research works on fixed point theory in different types of metric spaces. In this theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Very well known Banach contraction principle which ensures the existence and uniqueness of fixed point of contraction self mappings in complete metric spaces.

There has been a number of generalizations of metric spaces such as rectangular metric spaces, fuzzy metric spaces, ordered metric spaces, partially ordered metric spaces, G-metric spaces and cone metric spaces etc (see [4],[5],[6],[8], [13],[15]).

In 2011, Azam *et al*. [2] introduced the notion of complex valued metric spaces and established some fixed point results for a pair of mappings for contraction condition satisfying a rational expression. Since then several authors studied the existence and uniqueness of fixed points of self-mappings and established fixed point results on complex valued metric spaces (see [1],[3],[12],[14],[16]).

Inspired from the works on complex valued metric space, Choi *et al*. [7] introduced the notion of bicomplex valued metric space which is a generalization of complex valued metric space and established sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a contractive condition.

The purpose of this paper is to establish some common fixed point theorems for pair of weakly increasing mappings under some conditions in ordered bicomplex valued metric space and we also introduced well-posedness and periodic point property of mappings satisfying a rational inequality in ordered bicomplex valued metric spaces.

2. Preliminaries

We recall some notations and definitions which will be required in the subsequent sections.

Let $\mathbb{R}$, $\mathbb{R}_0^+$, $\mathbb{C}$ and $\mathbb{N}$ be the sets of real numbers, non negative real numbers, complex numbers and positive integers respectively. The set $\mathbb{C}$ is given as

$$\mathbb{C} := \left\{ z = x + iy \,|\, x, y \in \mathbb{R} \text{ and } i^2 = -1 \right\}.$$

Define a partial order relation $\precsim$ on $\mathbb{C}$ as follows (see [2]): For any $z_1, z_2 \in \mathbb{C}$,

$$z_1 \precsim z_2 \text{ if and only if } \Re(z_1) \le \Re(z_2) \text{ and } \Im(z_1) \le \Im(z_2), \tag{2.1}$$

where $\Re(z)$ and $\Im(z)$ are respectively the real and imaginary parts of the complex number z.

Thus $z_1 \precsim z_2$ if any one of the following statements holds:

(o_1) $\Re(z_1) = \Re(z_2)$ and $\Im(z_1) = \Im(z_2)$;

(o_2) $\Re(z_1) < \Re(z_2)$ and $\Im(z_1) = \Im(z_2)$;

(o_3) $\Re(z_1) = \Re(z_2)$ and $\Im(z_1) < \Im(z_2)$;

(o_4) $\Re(z_1) < \Re(z_2)$ and $\Im(z_1) < \Im(z_2)$.

We write $z_1 \precnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$, i.e., any one of (o_2), (o_3) and (o_4) is satisfied and we write $z_1 \prec z_2$ if only (o_4) is satisfied. Considering (o_1)-(o_4), the following properties for the partial order $\precsim$ on $\mathbb{C}$ hold:

(p_1) $0 \precsim z_1 \precsim z_2 \Longrightarrow |z_1| \le |z_2|$;

(p_2) $z_1 \precsim z_2$ and $z_2 \precsim z_3 \Longrightarrow z_1 \precsim z_3$;

(p_3) $z_1 \precsim z_2$ and $\lambda > 0$ $(\lambda \in \mathbb{R}) \Longrightarrow \lambda z_1 \precsim \lambda z_2$.

We recall the set of bicomplex numbers $\mathbb{C}_2$ (see [9, 11]):

$$\mathbb{C}_2 = \left\{ w = p_0 + i_1 p_1 + i_2 p_2 + i_1 i_2 p_3 \,\middle|\, p_k \in \mathbb{R}\, ; k = 0, \ldots, 3 \right\},$$

where i_1, i_2 are independent imaginary units such that $i_1^2 = -1 = i_2^2$. The product of i_1 and i_2 defines a hyperbolic unit j such that $j^2 = 1$. The products of all units are commutative and satisfy

$$i_1 i_2 = j, \quad i_1 j = -i_2, \quad i_2 j = -i_1.$$

We can also express $\mathbb{C}_2$ as

$$\mathbb{C}_2 = \{ w = z_1 + i_2 z_2 \mid z_1, z_2 \in \mathbb{C} \},$$

where $z_1 = p_0 + i_1 p_1$, $z_2 = p_2 + i_1 p_3$.

The inverse of $w = z_1 + i_2 z_2$ exists if $z_1^2 + z_2^2 \neq 0$ (see [11]). Indeed, if $\left| z_1^2 + z_2^2 \right| \neq 0$, then the inverse w^{-1} of w is defined as

$$w^{-1} = \frac{1}{w} = \frac{z_1 - i_2 z_2}{z_1^2 + z_2^2}.$$

A bicomplex number $w = p_0 + i_1 p_1 + i_2 p_2 + i_1 i_2 p_3$, $(p_k \in \mathbb{R}; k = 0, 1, 2, 3)$ is said to be degenerated (see [11]) if the matirx $\begin{pmatrix} p_0 & p_1 \\ p_2 & p_3 \end{pmatrix}$ is degenerated.

One can easily verify that if w is degenerated and $0 < \min(p_0, p_1, p_2, p_3)$, then w^{-1} exists and is also degenerated.

Let $u = u_1 + i_2 u_2$, $v = v_1 + i_2 v_2 \in \mathbb{C}_2$. Define a partial order relation $\precsim_{i_2}$ on $\mathbb{C}_2$ as follows (see [7]):

$$u \precsim_{i_2} v \text{ if and only if } u_1 \precsim v_1 \text{ and } u_2 \precsim v_2, \tag{2.2}$$

where the partial order $\precsim$ in (2.2) is given as in (2.1). We find that $u \precsim_{i_2} v$ if any one of the following properties hold:

(bco_1) $u_1 = v_1$ and $u_2 = v_2$;

(bco_2) $u_1 \prec v_1$ and $u_2 = v_2$;

(bco_3) $u_1 = v_1$ and $u_2 \prec v_2$;

(bco_4) $u_1 \prec v_1$ and $u_2 \prec v_2$.

We write $u \precnsim_{i_2} v$ if $u \precsim_{i_2} v$ and $u \neq v$, i.e., one of (bco_2), (bco_3) and (bco_4) is satisfied and we write $u \prec_{i_2} v$ if only (bco_4) is satisfied.

Norm of a bicomplex number $w = z_1 + i_2 z_2$, denoted by $||w||$, is defined as

$$||w|| = \|z_1 + i_2 z_2\| = \left(|z_1|^2 + |z_2|^2 \right)^{\frac{1}{2}}.$$

If $w = p_0 + i_1 p_1 + i_2 p_2 + i_1 i_2 p_3$, where $p_k \in \mathbb{R}; k = 0, 1, 2, 3$, then

$$\|w\| = \left(p_0^2 + p_1^2 + p_2^2 + p_3^2 \right)^{\frac{1}{2}}.$$

For any two bicomplex numbers $u, v \in \mathbb{C}_2$ one can easily verify that

(i) $0 \precsim_{i_2} u \precsim_{i_2} v \Rightarrow \|u\| \le \|v\|$,

(ii) $\|u + v\| \le \|u\| + \|v\|$,

(iii) $\|\alpha u\| = \alpha \|u\|$,

where α is non-negative real number.

Further, for any two bicomplex numbers $u, v \in \mathbb{C}_2$, $\|uv\| \le \sqrt{2}\,\|u\|\,\|v\|$ holds. Also $\|uv\| = \|u\|\,\|v\|$ whenever at least one of u and v is degenerated (see [11]). One can easily deduce that $\left\|u^{-1}\right\| = \|u\|^{-1}$ holds for any degenerated bicomplex number u with $0 \precnsim_{i_2} u$.

Choi *et al.* [7] define a bicomplex valued metric as follows: Let X be a nonempty set. A function $d : X \times X \to \mathbb{C}_2$ is a bicomplex valued metric on X if it satisfies the following properties: For $x, y, z \in X$,

(bcm_1) $0 \precsim_{i_2} d(x,y)$ for all $x, y \in X$;

(bcm_2) $d(x,y) = 0$ if and only if $x = y$;

(bcm_3) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(bcm_4) $d(x,y) \precsim_{i_2} d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then (X, d) is called a bicomplex valued metric space.

For example, let $X = \mathbb{R}$ and a mapping $d : X \times X \longrightarrow \mathbb{C}_2$ be defined by

$$d(x,y) := (1 + 2i_1 + i_2 + 2i_1 i_2)\,|x - y|\,;\ x, y \in X,$$

where $|\ |$ is the usual real modulus. One can easily check that (X, d) is a bicomplex valued metric on $\mathbb{C}$. Further, for all $x, y \in X$, the corresponding determinant for $d(x,y)$ is

$$\begin{vmatrix} p_0 & p_1 \\ p_2 & p_3 \end{vmatrix} = \begin{vmatrix} |x-y| & 2\,|x-y| \\ |x-y| & 2\,|x-y| \end{vmatrix} = 0.$$

Therefore (X,d) is a bicomplex-valued metric space such that $d(x,y)$, for all $x,y \in X$, is degenerated.

A bicomplex valued metric space (X,d) together with a partially order relation $\precsim$ on X is called ordered bicomplex valued metric space.

A sequence in a nonempty set X is a function $x : \mathbb{N} \to X$, which is expressed by its range set $\{x_n\}$, where $x(n) := x_n$ $(n \in \mathbb{N})$. Let $\{x_n\}$ be a sequence in a bicomplex valued metric space (X,d). The sequence $\{x_n\}$ is said to converge to $x \in X$ if and only if for any $0 \prec_{i_2} \varepsilon \in \mathbb{C}_2$, there exists $N \in \mathbb{N}$ depending on ε such that $d(x_n,x) \prec_{i_2} \varepsilon$ for all $n > N$. It is denoted by $x_n \to x$ as $n \to \infty$ or by $\lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ in a bicomplex valued metric space (X,d) is said to be a Cauchy sequence if and only if for any $0 \prec_{i_2} \varepsilon \in \mathbb{C}_2$, there exists $N \in \mathbb{N}$ depending on ε such that $d(x_m,x_n) \prec_{i_2} \varepsilon$ for all $m,n > N$. A bicomplex valued metric space (X,d) is said to be complete if and only if every Cauchy sequence in X converges in X.

Let $\precsim$ be any partially order relation on a set X. A pair (f,g) of self-maps on X is said to be weakly increasing if $fx \precsim gfx$ and $gx \precsim fgx$, for all $x \in X$. If $f = g$, then we have $fx \precsim f^2x$, for all $x \in X$ and in this case, we say that f is a weakly increasing map (see [6]). For example, let $X = [0,\infty)$ be together with usual ordering $\leq$ on $\mathbb{R}$. Let $f : X \to X$ be defined by

$$\begin{aligned} fx &= x^{\frac{1}{3}},\ if\ 0 \leq x \leq 1 \\ &= x,\ if\ 1 < x \leq 2 \\ &= 0,\ if\ 2 < x < \infty. \end{aligned}$$

Note that if $x \in [0,1]$, then $fx = x^{1/3} \leq x^{1/9} = f^2x$. Also when $x \in (1,2]$, then $fx = x = f^2x$ and if $x \in (2,\infty)$, then $fx = 0 = f^2x$. Thus $fx \leq f^2x$, for all $x \in X$ and so f is a weakly increasing map. Note that f is not increasing since $2 < 3$ and $f(2) = 2 \nleq 0 = f(3)$.

A point x in X said to be a fixed point of a self-map f on X if $fx = x$. A fixed point problem is to find some x in X such that $fx = x$ and we denote it by $FP(f\ ;X)$. A point $x \in X$ is called a common fixed point of a pair (f,g) of self-maps on X if $fx = gx = x$. A common fixed point problem is to find some x in X such that $fx = gx = x$ and we denote it by $CFP(f,g;X)$.

A nonempty subset W of a partially ordered set X is said to be totally ordered if every two elements of W are comparable.

M. Abbas *et al.*[3] define well-posedness of fixed point and common fixed point problems for order contractive mappings. A fixed point problem $FP(S;X)$ is called well-posed if $F(S)$, the set of fixed points of S, is singleton and for any sequence $\{x_n\}$ in X whose every term is comparable with $x^* \in F(S)$ and $\lim_{n\to 0} d(Sx_n,x_n) = 0$ implies $x^* = \lim_{n\to 0} x_n$.

A common fixed point problem $CFP(S,T;X)$ is called well-posed if $CF(S,T)$, the set of common fixed points of S and T, is singleton and for any sequence $\{x_n\}$ in X whose every term is comparable with $x^* \in CF(S,T)$ and $\lim_{n\to 0} d(Sx_n,x_n) = 0$ or $\lim_{n\to 0} d(Tx_n,x_n) = 0$ implies $x^* = \lim_{n\to 0} x_n$.

If a map T satisfies $F(T) = F(T^n)$ for each $n \in N$, where $F(T)$ denotes the set of fixed points of T, then it is said to have property P (see [10]). The set $O(x,\infty) = \{x, Tx, T^2x, ...\}$ is called the orbit of x.

Here we present two assertions which will be required in the sequel.

Lemma 2.1. *[7] Let (X,d) be a bicomplex valued metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to $x \in X$ if and only if $\|d(x_n,x)\| \to 0$ as $n \to \infty$.*

Lemma 2.2. *[7] Let (X,d) be a bicomplex valued metric space and $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} x_n = x$. Then for any $a \in X$, $\lim_{n\to\infty} \|d(x_n,a)\| = \|d(x,a)\|$.*

3. Main Results

We begin with a common fixed point theorem for weakly increasing maps on an ordered bicomplex valued metric space.

Theorem 3.1. *Let $(X,\precsim)$ be a partially ordered set such that there exists a complete bicomplex valued metric d on X with degenerated $d(x,y)+d(u,v)$ for all $x,y,u,v \in X$ and let the pair (S,T) be weakly increasing self-maps on X. Also, for every comparable $x,y \in X$, we have*

$$\begin{aligned} (i)\ d(Sx,Ty) \precsim_{i_2} \alpha \frac{\{d(x,Ty)\}^2+\{d(y,Sx)\}^2}{d(x,Ty)+d(y,Sx)} \\ +\beta[d(x,Sx)+d(y,Ty)] \\ +\gamma[d(x,Ty)+d(y,Sx)] \\ +\delta d(x,y); \end{aligned}$$

$$\begin{aligned} if\ d(x,Ty)+d(y,Sx) \neq 0; \alpha,\beta,\gamma,\delta \geq 0\ and \\ 2\alpha+2\beta+2\gamma+\delta < 1. \end{aligned}$$

$$(ii)\ d(Sx,Ty) = 0,\ if\ d(x,Ty)+d(y,Sx) = 0.$$

If S or T is continuous or for any non decreasing sequence $\{x_n\}$ with $x_n \to z$ in X we necessarily have $x_n \precsim z$ for all $n \in \mathbb{N}$, then S and T have a common fixed point. Moreover, the set of common fixed points of S and T is totally ordered if and only if S and T have a unique common fixed point.

Proof. First, we shall show that if S or T has a fixed point, then it is a common fixed point of S and T. Let u be a fixed point of S i.e., $Su = u$. The condition (i) of this theorem gives,

$$\begin{aligned} d(Su,Tu) \precsim_{i_2} \alpha \frac{\{d(u,Tu)\}^2+\{d(u,Su)\}^2}{d(u,Tu)+d(u,Su)} \\ +\beta[d(u,Su)+d(u,Tu)] \\ +\gamma[d(u,Tu)+d(u,Su)] \\ +\delta d(u,u). \end{aligned}$$

$$i.e., d(u,Tu) \precsim_{i_2} \alpha \frac{\{d(u,Tu)\}^2}{d(u,Tu)} + \beta d(u,Tu) + \gamma d(u,Tu).$$

$$i.e., (1-\alpha-\beta-\gamma)d(u,Tu) \precsim_{i_2} 0.$$

Since $\alpha+\beta+\gamma<1$, then $d(u,Tu)\precsim_{i_2} 0$ implies $d(u,Tu)=0$, i.e., $Tu=u$ and so u is a common fixed point of S and T. Similarly, if u is a fixed point of T, then we can easily get that u is also a fixed point of S.

Now, let x_0 be an arbitrary point of X. If $Sx_0=x_0$, there is nothing to proof. So we consider $Sx_0\neq x_0$. Let us define a sequence $\{x_n\}$ in X as follows:

$$Sx_n=x_{n+1} \text{ and } Tx_{n+1}=x_{n+2}, \text{for } n=0,1,2,3...$$

Now, since (S,T) is weakly increasing,

$$x_1=Sx_0\lesssim TSx_0=Tx_1=x_2,$$

$$x_2=Sx_1\lesssim TSx_1=Tx_2=x_3,$$

$$x_3=Sx_2\lesssim TSx_2=Tx_3=x_4,$$

Continuing this process, we have

$$x_1\lesssim x_2\lesssim x_3\lesssim ....\lesssim x_n\lesssim x_{n+1}\lesssim ...$$

Assume that $d(x_{2n},x_{2n+1})>0$ for every $n\in\mathbb{N}$. If not, then $x_{2n}=x_{2n+1}$ for some $n\in\mathbb{N}$. For all those n, $x_{2n}=x_{2n+1}=Sx_{2n}$ and the proof is obvious. Now, since x_{2n} and x_{2n+1} are comparable, then taking $d(x_{2n},x_{2n+1})>0$ for $n=0,1,2,3,...$ Using the condition (i) of this theorem,

$$d(x_{2n+1},x_{2n+2})=d(Sx_{2n},Tx_{2n+1})$$

$$\begin{aligned} i.e., d(x_{2n+1},x_{2n+2})\precsim_{i_2}\ & \alpha\frac{\left[\begin{array}{c}\{d(x_{2n},Tx_{2n+1})\}^2\\ +\{d(x_{2n+1},Sx_{2n})\}^2\end{array}\right]}{d(x_{2n},Tx_{2n+1})+d(x_{2n+1},Sx_{2n})}\\ &+\beta[d(x_{2n},Sx_{2n})+d(x_{2n+1},Tx_{2n+1})]\\ &+\gamma[d(x_{2n},Tx_{2n+1})+d(x_{2n+1},Sx_{2n})]\\ &+\delta d(x_{2n},x_{2n+1}).\end{aligned}$$

$$\begin{aligned} i.e., d(x_{2n+1},x_{2n+2})\precsim_{i_2}\ & \alpha\frac{\left[\begin{array}{c}\{d(x_{2n},x_{2n+2})\}^2\\ +\{d(x_{2n+1},x_{2n+1})\}^2\end{array}\right]}{d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+1})}\\ &+\beta[d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})]\\ &+\gamma[d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+1})]\\ &+\delta d(x_{2n},x_{2n+1}).\end{aligned}$$

$$\begin{aligned} i.e., d(x_{2n+1},x_{2n+2})\precsim_{i_2}\ & \alpha d(x_{2n},x_{2n+2})\\ &+\beta[d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})]\\ &+\gamma d(x_{2n},x_{2n+2})+\delta d(x_{2n},x_{2n+1}).\end{aligned}$$

$$\begin{aligned} i.e., d(x_{2n+1},x_{2n+2})\precsim_{i_2}\ & \alpha d(x_{2n},x_{2n+1})+\alpha d(x_{2n+1},x_{2n+2})\\ &+\beta d(x_{2n},x_{2n+1})+\beta d(x_{2n+1},x_{2n+2})\\ &+\gamma d(x_{2n},x_{2n+1})+\gamma d(x_{2n+1},x_{2n+2})\\ &+\delta d(x_{2n},x_{2n+1}).\end{aligned}$$

$$i.e., d(x_{2n+1},x_{2n+2})\precsim_{i_2}\frac{(\alpha+\beta+\gamma+\delta)}{(1-\alpha-\beta-\gamma)}d(x_{2n},x_{2n+1}).$$

Setting $0\leq h=\frac{(\alpha+\beta+\gamma+\delta)}{(1-\alpha-\beta-\gamma)}<1$, we have $d(x_{2n+1},x_{2n+2})\precsim_{i_2} hd(x_{2n},x_{2n+1})$, for all $n\geq 0$.

In similar way, we have $d(x_{2n},x_{2n+1})\precsim_{i_2} hd(x_{2n-1},x_{2n})$, for all $n\geq 0$. Hence for all $n\geq 0$,

$$d(x_{n+1},x_{n+2})\precsim_{i_2} hd(x_n,x_{n+1}).$$

and consequently, for all $n\geq 0$,

$$\begin{aligned} d(x_{n+1},x_{n+2})&\precsim_{i_2} hd(x_n,x_{n+1})\\ &\precsim_{i_2} h^2d(x_{n-1},x_n)\\ &\quad ...\\ &\precsim_{i_2} h^{n+1}d(x_0,x_1).\end{aligned}$$

Now, for $m>n$ we have

$$\begin{aligned} d(x_n,x_m)&\precsim_{i_2} d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+...+d(x_{m-1},x_m)\\ &\precsim_{i_2}(h^n+h^{n+1}+h^{n+2}+...+h^{m-1})d(x_0,x_1)\\ &\precsim_{i_2}\frac{h^n}{1-h}d(x_0,x_1).\end{aligned}$$

This implies that

$$\|d(x_n,x_m)\|\leq\frac{h^n}{1-h}\|d(x_0,x_1)\|.$$

Taking limit as $n\to\infty$, we have $\|d(x_n,x_m)\|\to 0$,

i.e., $d(x_n,x_m)\to 0$ as $n\to\infty$. Therefore $\{x_n\}$ is a cauchy sequence in X and since X is complete, the sequence $\{x_n\}$ converges to a point v (say) in X. If S or T is continuous, then it is clear that $Sv=v=Tv$. If neither S nor T is continuous, then $x_n\lesssim v$ for all $n\in\mathbb{N}$. We claim that v is a fixed point of S.

From the condition (i) of this theorem

$$\begin{aligned} d(v,Sv)&\precsim_{i_2} d(v,x_{n+2})+d(x_{n+2},Sv)\\ &=d(v,x_{n+2})+d(Tx_{n+1},Sv)\\ &=d(v,x_{n+2})+d(Sv,Tx_{n+1}).\end{aligned}$$

$$\begin{aligned} i.e., d(v,Sv)\precsim_{i_2} d(v,x_{n+2})+\ & \alpha\frac{\left[\begin{array}{c}\{d(v,Tx_{n+1})\}^2\\ +\{d(x_{n+1},Sv)\}^2\end{array}\right]}{d(v,Tx_{n+1})+d(x_{n+1},Sv)}\\ &+\beta[d(v,Sv)+d(x_{n+1},Tx_{n+1})]\\ &+\gamma[d(v,Tx_{n+1})+d(x_{n+1},Sv)]\\ &+\delta d(v,x_{n+1}).\end{aligned}$$

$$\begin{aligned} i.e., d(v,Sv)\precsim_{i_2} d(v,x_{n+2})+\ & \alpha\frac{\left[\begin{array}{c}\{d(v,x_{n+2})\}^2\\ +\{d(x_{n+1},Sv)\}^2\end{array}\right]}{d(v,x_{n+2})+d(x_{n+1},Sv)}\\ &+\beta[d(v,Sv)+d(x_{n+1},x_{n+2})]\\ &+\gamma[d(v,x_{n+2})+d(x_{n+1},Sv)]\\ &+\delta d(v,x_{n+1}).\end{aligned}$$

$$
\begin{aligned}
i.e., \|d(v,Sv)\| \quad \leq \quad & \|d(v,x_{n+2})\| \\
& +\sqrt{2}\alpha \frac{\|d(v,x_{n+2})\|^2+\|d(x_{n+1},Sv)\|^2}{\|d(v,x_{n+2})+d(x_{n+1},Sv)\|} \\
& +\beta[\|d(v,Sv)\|+\|d(x_{n+1},x_{n+2})\|] \\
& +\gamma[\|d(v,x_{n+2})\|+\|d(x_{n+1},Sv)\|] \\
& +\delta\,\|d(v,x_{n+1})\|.
\end{aligned}
$$

Taking limit as $n\to\infty$ and in view of Lemma (2.2), we get

$$
\begin{aligned}
\|d(v,Sv)\| \quad \leq \quad & \|d(v,v)\| \\
& +\sqrt{2}\alpha \frac{\|d(v,v)\|^2+\|d(v,Sv)\|^2}{\|d(v,v)+d(v,Sv)\|} \\
& +\beta[\|d(v,Sv)\|+\|d(v,v)\|] \\
& +\gamma[\|d(v,v)\|+\|d(v,Sv)\|] \\
& +\delta\,\|d(v,v)\|.
\end{aligned}
$$

$$
i.e., (1-\sqrt{2}\alpha-\beta-\gamma)\,\|d(v,Sv)\|\leq 0.
$$

Since $\sqrt{2}\alpha+\beta+\gamma<1$, then $\|d(v,Sv)\|\leq 0, i.e., d(v,Sv)=0$. This implies that v is a fixed point of S and consequently, v is a common fixed point of S and T. Hence S and T have common fixed point in X.

Now, suppose that set of common fixed points of S and T is totally ordered. If possible, let z be another common fixed point of S and T. From the condition(i) of this theorem

$$
\begin{aligned}
d(v,z)=d(Sv,Tz) \precsim_{i_2} \; & \alpha\frac{\{d(v,Tz)\}^2+\{d(z,Sv)\}^2}{d(v,Tz)+d(z,Sv)} \\
& +\beta[d(v,Sv)+d(z,Tz)] \\
& +\gamma[d(v,Tz)+d(z,Sv)] \\
& +\delta d(v,z).
\end{aligned}
$$

$$
\begin{aligned}
i.e., d(v,z) \precsim_{i_2} \; & \alpha\frac{\{d(v,z)\}^2+\{d(z,v)\}^2}{d(v,z)+d(z,v)} \\
& +\beta[d(v,v)+d(z,z)] \\
& +\gamma[d(v,z)+d(z,v)] \\
& +\delta d(v,z).
\end{aligned}
$$

$$
i.e., (1-\alpha-2\gamma-\delta)d(v,z)\precsim_{i_2} 0.
$$

Since $\alpha+2\gamma+\delta<1$, then $d(v,z)\precsim_{i_2} 0, i.e., v=z$. Hence S and T have a unique common fixed point in X. Conversely, if S and T have only one common fixed point, then the set of common fixed point of S and T being singleton, is totally ordered. □

The supporting example to the Theorem 3.1 is given as follows:

Example 3.2. *Let $X=[0,1]$. We consider the partial order on X as: $x\precsim y$ if and only if $y\leq x$, where $\leq$ is the usual order on $\mathbb{R}$. Let us define $d: X\times X\to\mathbb{C}_2$ as*

$$
d(x,y)=(1+i_1-i_2-i_1i_2)\,|x-y|\,,\ \textit{for all } x,y\in X.
$$

Then one can check that (X,d) is a bicomplex valued complete metric sapce with degenerated $d(x,y)+d(u,v)$ for all $x,y,u,v\in X$. Now, we define the self maps S,T on X by

$$
Sx=\frac{x}{5} \textit{ and } Tx=\frac{x}{10}, \textit{ for all } x\in X.
$$

Clearly S and T are both continuous on X. Now

$$
\begin{aligned}
Sx &= \frac{x}{5}\precsim\frac{x}{50}=T\frac{x}{5}=TSx \textit{ and} \\
Tx &= \frac{x}{10}\precsim\frac{x}{50}=S\frac{x}{10}=STx.
\end{aligned}
$$

Therefore the pair (S,T) is weakly increasing. Again, we calculate the followings:

Case I: If $x\precsim\frac{y}{2}$, then $x\geq\frac{y}{2}$. Now, if we set $k=(1+i_1-i_2-i_1i_2)\in\mathbb{C}_2$,

$$
d(Sx,Ty)=k\,|Sx-Ty|=\frac{k}{5}\left|x-\frac{y}{2}\right|=\frac{k}{5}\left(x-\frac{y}{2}\right)\precsim_{i_2}\frac{k}{5}x.
$$

$$
\begin{aligned}
i.e., d(Sx,Ty) \precsim_{i_2} \; & \frac{2}{5}k\left(\frac{4x}{5}+\frac{9y}{10}\right) \\
\precsim_{i_2} \; & \frac{1}{30}k\frac{\left|x-\frac{y}{10}\right|^2+\left|y-\frac{x}{5}\right|^2}{\left|x-\frac{y}{10}\right|+\left|y-\frac{x}{5}\right|} \\
& +\frac{2}{5}k\left(\left|x-\frac{x}{5}\right|+\left|y-\frac{y}{10}\right|\right) \\
& +\frac{1}{30}k\left(\left|x-\frac{y}{10}\right|+\left|y-\frac{x}{5}\right|\right) \\
& +\frac{1}{30}k\,|x-y|\,.
\end{aligned}
$$

Case II: If $\frac{y}{2}\precsim x$, then $x\leq\frac{y}{2}$. Now

$$
d(Sx,Ty)=k\,|Sx-Ty|=\frac{k}{5}\left|x-\frac{y}{2}\right|=\frac{k}{5}\left(\frac{y}{2}-x\right)\precsim_{i_2}\frac{k}{10}y.
$$

$$
\begin{aligned}
i.e., d(Sx,Ty) \precsim_{i_2} \; & \frac{2}{5}k\left(\frac{4x}{5}+\frac{9y}{10}\right) \\
\precsim_{i_2} \; & \frac{1}{30}k\frac{\left|x-\frac{y}{10}\right|^2+\left|y-\frac{x}{5}\right|^2}{\left|x-\frac{y}{10}\right|+\left|y-\frac{x}{5}\right|} \\
& +\frac{2}{5}k\left(\left|x-\frac{x}{5}\right|+\left|y-\frac{y}{10}\right|\right) \\
& +\frac{1}{30}k\left(\left|x-\frac{y}{10}\right|+\left|y-\frac{x}{5}\right|\right) \\
& +\frac{1}{30}k\,|x-y|\,.
\end{aligned}
$$

In both cases,

$$
\begin{aligned}
d(Sx,Ty) \precsim_{i_2} \; & \frac{1}{30}k\frac{\left|x-\frac{y}{10}\right|^2+\left|y-\frac{x}{5}\right|^2}{\left|x-\frac{y}{10}\right|+\left|y-\frac{x}{5}\right|} \\
& +\frac{2}{5}k\left(\left|x-\frac{x}{5}\right|+\left|y-\frac{y}{10}\right|\right) \\
& +\frac{1}{30}k\left(\left|x-\frac{y}{10}\right|+\left|y-\frac{x}{5}\right|\right) \\
& +\frac{1}{30}k\,|x-y|\,.
\end{aligned}
$$

$$i.e., d(Sx,Ty) \precsim_{i_2} \alpha \frac{\{d(x,Ty)\}^2+\{d(y,Sx)\}^2}{d(x,Ty)+d(y,Sx)} +\beta[d(x,Sx)+d(y,Ty)] +\gamma[d(x,Ty)+d(y,Sx)] +\delta\,|x-y|,$$

where $\alpha=\frac{1}{30},\beta=\frac{2}{5},\gamma=\frac{1}{30},\delta=\frac{1}{30}>0$ *with* $2\alpha+2\beta+2\gamma+\delta=\frac{29}{30}<1$. *Thus all the conditions of Theorem 3.1 are satisfied and here* 0 *is the unique common fixed point of* S *and* T *in* X.

Corollary 3.3. *Let* $(X,\lesssim)$ *be a partially ordered set such that there exists a complete bicomplex valued metric* d *on* X *with degenerated* $d(x,y)+d(u,v)$ *for all* $x,y,u,v\in X$ *and let* T *be weakly increasing self-map on* X. *Also, for every comparable* $x,y\in X$, *we have*

$$(i)\ d(Tx,Ty) \precsim_{i_2} \frac{\alpha[\{d(x,Ty)\}^2+\{d(y,Tx)\}^2]}{d(x,Ty)+d(y,Tx)} +\beta[d(x,Tx)+d(y,Ty)] +\gamma[d(x,Ty)+d(y,Tx)] +\delta d(x,y);$$

$$if\ d(x,Ty)+d(y,Tx) \neq 0;\alpha,\beta,\gamma,\delta\geq 0 \text{ and } 2\alpha+2\beta+2\gamma+\delta < 1.$$

$$(ii)\ d(Tx,Ty)=0,\ if\ d(x,Ty)+d(y,Tx)=0.$$

If T *is continuous or for any non decreasing sequence* $\{x_n\}$ *with* $x_n\to z$ *in* X *we necessarily have* $x_n\lesssim z$ *for all* $n\in\mathbb{N}$, *then* T *has a fixed point. Moreover, the set of fixed points of* T *is totally ordered if and only if* T *have a unique common fixed point.*

Proof. Putting $S=T$ in Theorem 3.1, the result is obvious. □

Theorem 3.4. *Let* $(X,\lesssim)$ *be a partially ordered set such that there exists a complete bicomplex valued metric* d *on* X *with degenerated* $d(x,y)+d(u,v)$, *for all* $x,y,u,v\in X$ *and let* S,T *be weakly increasing self-maps on* X. *Also, for every comparable* $x,y\in X$, *we have*

$$(i)\ d(Sx,Ty) \precsim_{i_2} \alpha\frac{d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)}{d(x,Ty)+d(y,Sx)} +\frac{d(x,Ty)d(y,Sx)}{d(x,Sx)+d(y,Ty)};$$

$$if\ d(x,Ty)+d(y,Sx)\neq 0 \text{ and } d(x,Sx)+d(y,Ty)\neq 0;\ 0\leq\alpha<1.$$

$$(ii)\ d(Sx,Ty)=0,\ if\ d(x,Ty)+d(y,Sx)=0 \text{ or } d(x,Sx)+d(y,Ty)=0.$$

If S *or* T *is continuous or for any non decreasing sequence* $\{x_n\}$ *with* $x_n\to z$ *in* X *we necessarily have* $x_n\lesssim z$ *for all* $n\in\mathbb{N}$, *then* S *and* T *have a common fixed point. Moreover, the set of common fixed points of* S *and* T *is totally ordered if and only if* S *and* T *have a unique common fixed point.*

Proof. We want to claim that if S or T has a fixed point, then it is a common fixed point of S and T. So for this, let u be a fixed point of S i.e., $Su=u$. Using the condition (i) of this theorem,

$$d(u,Tu)=d(Su,Tu) \precsim_{i_2} \alpha\frac{\begin{bmatrix} d(u,Su)d(u,Tu)\\ +d(u,Tu)d(u,Su)\end{bmatrix}}{d(u,Tu)+d(u,Su)} +\frac{d(u,Tu)d(u,Su)}{d(u,Su)+d(u,Tu)}.$$

$$i.e., d(u,Tu) \precsim_{i_2} \alpha\frac{d(u,u)d(u,Tu)+d(u,Tu)d(u,u)}{d(u,Tu)+d(u,u)} +\frac{d(u,Tu)d(u,u)}{d(u,u)+d(u,Tu)}.$$

$$i.e., d(u,Tu) \precsim_{i_2} 0.$$

Hence $d(u,Tu)=0, i.e., Tu=u$ and therefore u is a common fixed point of S and T. Similarly, if u is a fixed point of T, then it is also a fixed point of S.

Now, let x_0 be an arbitrary point of X. If $Sx_0=x_0$, then there is nothing to proof. So assume that $Sx_0\neq x_0$. Let us define a sequence $\{x_n\}$ in X as follows:

$$Sx_n=x_{n+1} \text{ and } Tx_{n+1}=x_{n+2}, \text{ for } n=0,1,2,3...$$

Now

$$x_1=Sx_0\lesssim TSx_0=Tx_1=x_2,$$

$$x_2=Sx_1\lesssim TSx_1=Tx_2=x_3,$$

$$x_3=Sx_2\lesssim TSx_2=Tx_3=x_4,$$

Continuing this process, we have

$$x_1\lesssim x_2\lesssim x_3\lesssim ....\lesssim x_n\lesssim x_{n+1}\lesssim ...$$

Assume that $d(x_{2n},x_{2n+1})>0$ for every $n\in\mathbb{N}$. If not, then $x_{2n}=x_{2n+1}$ for some $n\in\mathbb{N}$. For all those n, $x_{2n}=x_{2n+1}=Sx_{2n}$ and the proof is obvious. Now, since x_{2n} and x_{2n+1} are comparable, then taking $d(x_{2n},x_{2n+1})>0$ for $n=0,1,2,3,...$ Using the condition (i) of this theorem,

$$\begin{aligned} d(x_{2n+1},x_{2n+2}) &= d(Sx_{2n},Tx_{2n+1})\\ &\precsim_{i_2} \alpha\frac{\begin{bmatrix} d(x_{2n},Sx_{2n})d(x_{2n},Tx_{2n+1})\\ +d(x_{2n+1},Tx_{2n+1})d(x_{2n+1},Sx_{2n})\end{bmatrix}}{d(x_{2n},Tx_{2n+1})+d(x_{2n+1},Sx_{2n})} +\frac{d(x_{2n},Tx_{2n+1})d(x_{2n+1},Sx_{2n})}{d(x_{2n},Sx_{2n})+d(x_{2n+1},Tx_{2n+1})}\\ &= \alpha\frac{\begin{bmatrix} d(x_{2n},x_{2n+1})d(x_{2n},x_{2n+2})\\ +d(x_{2n+1},x_{2n+2})d(x_{2n+1},x_{2n+1})\end{bmatrix}}{d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+1})} +\frac{d(x_{2n},x_{2n+2})d(x_{2n+1},x_{2n+1})}{d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})}.\end{aligned}$$

$$i.e., d(x_{2n+1},x_{2n+2}) \precsim_{i_2} \alpha d(x_{2n},x_{2n+1}).$$

Similarly, we have $d(x_{2n},x_{2n+1}) \precsim_{i_2} \alpha d(x_{2n-1},x_{2n})$, for all $n \geq 0$. Hence for all $n \geq 0$,

$$d(x_{n+1},x_{n+2}) \precsim_{i_2} \alpha d(x_n,x_{n+1})$$

and consequently, for all $n \geq 0$,

$$\begin{aligned} d(x_{n+1},x_{n+2}) &\precsim_{i_2} \alpha d(x_n,x_{n+1}) \\ &\precsim_{i_2} \alpha^2 d(x_{n-1},x_n) \\ &\quad ... \\ &\precsim_{i_2} \alpha^{n+1} d(x_0,x_1). \end{aligned}$$

Now, for $m > n$, we have

$$\begin{aligned} d(x_n,x_m) &\precsim_{i_2} d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + ... \\ &\qquad + d(x_{m-1},x_m) \\ &\precsim_{i_2} (\alpha^n + \alpha^{n+1} + \alpha^{n+2} + ... + \alpha^{m-1}) d(x_0,x_1) \\ &\precsim_{i_2} \frac{\alpha^n}{1-\alpha} d(x_0,x_1). \end{aligned}$$

$$i.e., \|d(x_n,x_m)\| \leq \frac{\alpha^n}{1-\alpha} \|d(x_0,x_1)\|.$$

Since $0 \leq \alpha < 1$, letting $n \to \infty$, we have $\|d(x_n,x_m)\| \to 0, i.e., d(x_n,x_m) \to 0$ as $m,n \to \infty$. Hence $\{x_n\}$ is a cauchy sequence in X and since X is complete, the sequence $\{x_n\}$ converges to a point v (say) in X. If S or T is continuous, then it is clear that $Sv = v = Tv$. If neither S nor T is continuous, $x_n \lesssim v$ for all $n \in \mathbb{N}$. We claim that v is a fixed point of S. From the condition (i) of this theorem,

$$\begin{aligned} d(v,Sv) &\precsim_{i_2} d(v,x_{n+2}) + d(x_{n+2},Sv) \\ &= d(v,x_{n+2}) + d(Tx_{n+1},Sv) \\ &= d(v,x_{n+2}) + d(Sv,Tx_{n+1}). \end{aligned}$$

$$i.e., d(v,Sv) \precsim_{i_2} d(v,x_{n+2}) + \alpha \frac{\begin{array}{c} d(v,Sv)d(v,Tx_{n+1}) \\ +d(x_{n+1},Tx_{n+1})d(x_{n+1},Sv) \end{array}}{d(v,Tx_{n+1}) + d(x_{n+1},Sv)} + \frac{d(v,Tx_{n+1})d(x_{n+1},Sv)}{d(v,Sv) + d(x_{n+1},Tx_{n+1})}.$$

$$i.e., d(v,Sv) \precsim_{i_2} d(v,x_{n+2}) + \alpha \frac{\begin{array}{c} d(v,Sv)d(v,x_{n+2}) \\ +d(x_{n+1},x_{n+2})d(x_{n+1},Sv) \end{array}}{d(v,x_{n+2}) + d(x_{n+1},Sv)} + \frac{d(v,x_{n+2})d(x_{n+1},Sv)}{d(v,Sv) + d(x_{n+1},x_{n+2})}.$$

$$\begin{aligned} i.e., \|d(v,Sv)\| \leq\ & \|d(v,x_{n+2})\| \\ & +\sqrt{2}\alpha \\ & \times \left[\frac{\begin{array}{c} \|d(v,Sv)\| \, \|d(v,x_{n+2})\| \\ + \|d(x_{n+1},x_{n+2})\| \, \|d(x_{n+1},Sv)\| \end{array}}{\|d(v,x_{n+2}) + d(x_{n+1},Sv)\|} \right] \\ & +\sqrt{2} \frac{\|d(v,x_{n+2})\| \, \|d(x_{n+1},Sv)\|}{\|d(v,Sv) + d(x_{n+1},x_{n+2})\|}. \end{aligned}$$

Taking limit as $n \to \infty$ and in view of Lemma 2.2, we get

$$\begin{aligned} i.e., \|d(v,Sv)\| \leq\ & \|d(v,v)\| \\ & +\sqrt{2}\alpha \frac{\left[\begin{array}{c} \|d(v,Sv)\| \, \|d(v,v)\| \\ + \|d(v,v)\| \, \|d(v,Sv)\| \end{array} \right]}{\|d(v,v) + d(v,Sv)\|} \\ & +\sqrt{2} \frac{\|d(v,v)\| \, \|d(v,Sv)\|}{\|d(v,Sv) + d(v,v)\|}. \end{aligned}$$

Thus $\|d(v,Sv)\| \leq 0, i.e., d(v,Sv) = 0, i.e., v = Sv$. Therefore v is a fixed point of S and consequently, v is a common fixed point of S and T. Hence S and T have common fixed point in X. Now, suppose that the set of common fixed points of S and T is totally ordered. If possible let, z be another common fixed point of S and T. Since $d(v,Sv) + d(z,Tz) = d(v,v) + d(z,z) = 0$, the condition (ii) of this theorem follows that $d(v,z) = d(Sv,Tz) = 0$, i.e.,$v = z$. Hence the common fixed point of S and T is unique.

Conversely, if S and T have only one common fixed point then the set of common fixed point of S and T being singleton is totally ordered. □

Corollary 3.5. *Let $(X,\lesssim)$ be a partially ordered set such that there exists a complete bicomplex valued metric d on X with degenerated $d(x,y) + d(u,v)$ for all $x,y,u,v \in X$ and let T be weakly increasing self-map on X. Also, for every comparable $x,y \in X$, we have*

$$(i)\ d(Tx,Ty) \precsim_{i_2} \alpha \frac{d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)}{d(x,Ty) + d(y,Tx)} + \frac{d(x,Ty)d(y,Tx)}{d(x,Tx) + d(y,Ty)};$$

$$\begin{aligned} & if\ d(x,Ty) + d(y,Tx) \neq 0 \\ & and\ d(x,Tx) + d(y,Ty) \neq 0;\ 0 \leq \alpha < 1. \end{aligned}$$

$$\begin{aligned} (ii)\ d(Tx,Ty) = 0;\ & if\ d(x,Ty) + d(y,Tx) = 0 \\ & or\ d(x,Tx) + d(y,Ty) = 0. \end{aligned}$$

If T is continuous or for any non decreasing sequence $\{x_n\}$ with $x_n \to z$ in X we necessarily have $x_n \lesssim z$, for all $n \in \mathbb{N}$, then T have a fixed point. Moreover, the set of fixed points of T is totally ordered if and only if T has a unique fixed point.

Proof. Putting $S = T$ in Theorem 3.4, the result is obvious. □

Theorem 3.6. *Let $(X,\lesssim)$ be a partially ordered set such that there exists a complete bicomplex valued metric d on X with degenerated $d(x,y) + d(u,v)$, for all $x,y,u,v \in X$. Suppose that S and T be self-maps on X as in Theorem 3.1. Then the common fixed point problem of S and T is well-posed.*

Proof. By Theorem 3.1, the mappings S and T have a unique common fixed point, suppose this is $u \in X$. Let $\{x_n\}$ be a sequence in X whose every term is comparable with u and $\lim_{n\to 0} d(Sx_n,x_n)=0$ or $\lim_{n\to 0} d(Tx_n,x_n)=0$. We consider that $\lim_{n\to 0} d(Sx_n,x_n)=0$. If for some n, $d(x_n,u)+d(u,Sx_n)=0$, then for those n, by the condition (ii) of Theorem 3.1, $d(Sx_n,Tu)=0$ and so

$$d(u,x_n)=d(x_n,Tu)\precsim_{i_2} d(x_n,Sx_n)+d(Sx_n,Tu).$$

$$i.e., d(u,x_n)\precsim_{i_2} d(Sx_n,x_n). \tag{3.1}$$

If $d(x_n,u)+d(u,Sx_n)\neq 0$ for each $n\geq 0$, then we have

$$\begin{aligned} d(u,x_n) &= d(Tu,x_n)\\ &= d(Tu,Sx_n)+d(Sx_n,x_n)\\ &= d(Sx_n,Tu)+d(Sx_n,x_n).\end{aligned}$$

$$\begin{aligned} i.e., d(u,x_n)\precsim_{i_2} \alpha&\frac{\{d(x_n,Tu)\}^2+\{d(u,Sx_n)\}^2}{d(x_n,Tu)+d(u,Sx_n)}\\ &+\beta[d(x_n,Sx_n)+d(u,Tu)]\\ &+\gamma[d(x_n,Tu)+d(u,Sx_n)]\\ &+\delta d(x_n,u)+d(Sx_n,x_n).\end{aligned}$$

$$\begin{aligned} i.e., d(u,x_n)\precsim_{i_2} \alpha&\frac{\{d(x_n,u)\}^2+\{d(u,Sx_n)\}^2}{d(x_n,u)+d(u,Sx_n)}\\ &+\beta[d(x_n,Sx_n)+d(u,u)]\\ &+\gamma[d(x_n,u)+d(u,Sx_n)]\\ &+\delta d(x_n,u)+d(Sx_n,x_n).\end{aligned}$$

$$\begin{aligned} i.e., d(u,x_n)\precsim_{i_2} \alpha&\frac{\{d(x_n,u)\}^2+\{d(u,Sx_n)\}^2}{d(x_n,u)+d(u,Sx_n)}\\ &+\beta d(x_n,Sx_n)\\ &+\gamma[d(x_n,u)+d(u,x_n)+d(x_n,Sx_n)]\\ &+\delta d(x_n,u)+d(Sx_n,x_n).\end{aligned}$$

$$\begin{aligned} i.e., d(u,x_n)\precsim_{i_2} &\frac{\alpha[\{d(x_n,u)\}^2+\{d(u,Sx_n)\}^2]}{d(x_n,u)+d(u,Sx_n)}\\ &+(1+\beta+\gamma)d(x_n,Sx_n)\\ &+(2\gamma+\delta)d(x_n,u).\end{aligned}$$

The following three cases may be arise:
Case (i) : $d(x_n,u)\neq 0$ and $d(u,Sx_n)\neq 0$, then

$$\begin{aligned} d(u,x_n)\precsim_{i_2} &\frac{\alpha\{d(x_n,u)\}^2}{d(x_n,u)+d(u,Sx_n)}+\frac{\alpha\{d(u,Sx_n)\}^2}{d(x_n,u)+d(u,Sx_n)}\\ &+(1+\beta+\gamma)d(x_n,Sx_n)+(2\gamma+\delta)d(x_n,u).\end{aligned}$$

$$\begin{aligned} i.e., d(u,x_n)\precsim_{i_2} &\frac{\alpha\{d(x_n,u)\}^2}{d(x_n,u)}+\frac{\alpha\{d(u,Sx_n)\}^2}{d(u,Sx_n)}\\ &+(1+\beta+\gamma)d(x_n,Sx_n)\\ &+(2\gamma+\delta)d(x_n,u).\end{aligned}$$

$$\begin{aligned} i.e., d(u,x_n)\precsim_{i_2} &\alpha d(x_n,u)+\alpha d(u,Sx_n)\\ &+(1+\beta+\gamma)d(x_n,Sx_n)\\ &+(2\gamma+\delta)d(x_n,u).\end{aligned}$$

$$\begin{aligned} i.e., d(u,x_n)\precsim_{i_2} &\alpha d(x_n,u)+\alpha d(u,x_n)+\alpha d(x_n,Sx_n)\\ &+(1+\beta+\gamma)d(x_n,Sx_n)\\ &+(2\gamma+\delta)d(x_n,u).\end{aligned}$$

$$i.e., d(u,x_n)\precsim_{i_2} \frac{(1+\alpha+\beta+\gamma)}{(1-2\alpha-2\gamma-\delta)}d(x_n,Sx_n), \tag{3.2}$$

where the quantity $\frac{(1+\alpha+\beta+\gamma)}{(1-2\alpha-2\gamma-\delta)}$ is positve finite, as $2\alpha+2\beta+2\gamma+\delta<1$.

Case (ii) : $d(x_n,u)\neq 0$ and $d(u,Sx_n)=0$, then

$$\begin{aligned} d(u,x_n)\precsim_{i_2} &\frac{\alpha\{d(x_n,u)\}^2}{d(x_n,u)+d(u,Sx_n)}+\frac{\alpha\{d(u,Sx_n)\}^2}{d(x_n,u)+d(u,Sx_n)}\\ &+(1+\beta+\gamma)d(x_n,Sx_n)+(2\gamma+\delta)d(x_n,u)\\ \precsim_{i_2} &\frac{\alpha\{d(x_n,u)\}^2}{d(x_n,u)}+(1+\beta+\gamma)d(x_n,Sx_n)\\ &+(2\gamma+\delta)d(x_n,u)\\ \precsim_{i_2} &\alpha d(x_n,u)+(1+\beta+\gamma)d(x_n,Sx_n)\\ &+(2\gamma+\delta)d(x_n,u).\end{aligned}$$

$$i.e., d(u,x_n)\precsim_{i_2} \frac{(1+\beta+\gamma)}{(1-\alpha-2\gamma-\delta)}d(x_n,Sx_n), \tag{3.3}$$

where the quantity $\frac{(1+\beta+\gamma)}{(1-\alpha-2\gamma-\delta)}$ is positive finite, as $2\alpha+2\beta+2\gamma+\delta<1$.

Case (iii) : $d(x_n,u)=0$ and $d(u,Sx_n)\neq 0$, then

$$d(x_n,u)=0\precsim_{i_2} d(x_n,Sx_n). \tag{3.4}$$

Considering all the inequalities (3.1), (3.2), (3.3), (3.4), we conclude that for some positive real quantity k and for each $n\geq 0$,

$$d(x_n,u)\precsim_{i_2} kd(x_n,Sx_n)$$

$$i.e., \|d(x_n,u)\|\leq k\,\|d(x_n,Sx_n)\|$$

Taking limit as $n\to\infty$ with in view of Lemma 2.1 and Lemma 2.2 we have $\|d(x_n,u)\|\to 0$ that is $\lim_{n\to\infty} x_n=u$. Hence the common fixed point problem of S and T is well-posed.

If we consider $\lim_{n\to 0} d(Tx_n,x_n)=0$, we can get the same result. This completes the proof. □

Theorem 3.7. *Let $(X,\precsim)$ be a partially ordered set such that there exists a complete bicomplex valued metric d on X with degenerated $d(x,y)+d(u,v)$ for all $x,y,u,v\in X$. Let T be a self-map on X as in Corollary 3.3. If for each $x\in X$, $O(x,\infty)$ is totally ordered, then T has the property P.*

Proof. From Corollary 3.3, T has a unique fixed point u (say)$\in X$. Then u is obviously also a fixed point of T^n, for each $n\in\mathbb{N}$. Therefore $F(T)\subseteq F(T^n)$, for each $n\in\mathbb{N}$. Let $u\in F(T^n)$ be fixed point of T^n i.e., $T^nu=u$ for each $1<n\in\mathbb{N}$. We have to show that u is also fixed point of T. Since for each $x\in X$, $O(x,\infty)$ is totally ordered, by the condition (i) of the Corollary 3.3, we have

$$d(u,Tu)=d(TT^{n-1}u,TT^nu).$$

$$\begin{aligned}\text{i.e.}, d(u,Tu)\precsim_{i_2}\ &\alpha\frac{\begin{bmatrix}\{d(T^{n-1}u,TT^nu)\}^2\\+\{d(T^nu,TT^{n-1}u)\}^2\end{bmatrix}}{d(T^{n-1}u,TT^nu)+d(T^nu,TT^{n-1}u)}\\&+\beta[d(T^{n-1}u,TT^{n-1}u)+d(T^nu,TT^nu)]\\&+\gamma[d(T^{n-1}u,TT^nu)+d(T^nu,TT^{n-1}u)]\\&+\delta d(T^{n-1}u,T^nu).\end{aligned}$$

$$\begin{aligned}\text{i.e.}, d(u,Tu)\precsim_{i_2}\ &\alpha\frac{\begin{bmatrix}\{d(T^{n-1}u,T^{n+1}u)\}^2\\+\{d(T^nu,T^nu)\}^2\end{bmatrix}}{d(T^{n-1}u,T^{n+1}u)+d(T^nu,T^nu)}\\&+\beta[d(T^{n-1}u,T^nu)+d(T^nu,T^{n+1}u)]\\&+\gamma[d(T^{n-1}u,T^{n+1}u)+d(T^nu,T^nu)]\\&+\delta d(T^{n-1}u,T^nu).\end{aligned}$$

$$\begin{aligned}\text{i.e.}, d(u,Tu)\precsim_{i_2}\ &\alpha d(T^{n-1}u,T^{n+1}u)+\beta[d(T^{n-1}u,T^nu)\\&+d(T^nu,T^{n+1}u)]\\&+\gamma d(T^{n-1}u,T^{n+1}u)+\delta d(T^{n-1}u,T^nu).\end{aligned}$$

$$\begin{aligned}\text{i.e.}, d(u,Tu)\precsim_{i_2}\ &\alpha d(T^{n-1}u,Tu)+\beta[d(T^{n-1}u,u)+d(u,Tu)]\\&+\gamma d(T^{n-1}u,Tu)+\delta d(T^{n-1}u,u).\end{aligned}$$

$$\begin{aligned}\text{i.e.}, d(u,Tu)\precsim_{i_2}\ &(\alpha+\gamma)d(T^{n-1}u,Tu)+(\beta+\delta)d(T^{n-1}u,u)\\&+\beta d(u,Tu).\end{aligned}$$

$$\begin{aligned}\text{i.e.}, d(u,Tu)\precsim_{i_2}\ &(\alpha+\gamma)[d(T^{n-1}u,u)+d(u,Tu)]\\&+(\beta+\delta)d(T^{n-1}u,u)+\beta d(u,Tu).\end{aligned}$$

$$\begin{aligned}\text{i.e.}, d(u,Tu)\precsim_{i_2}\ &\frac{(\alpha+\beta+\gamma+\delta)}{(1-\alpha-\beta-\gamma)}d(T^{n-1}u,u)\\&=hd(T^{n-1}u,T^nu),\end{aligned}$$

where $0\le h=\frac{(\alpha+\beta+\gamma+\delta)}{(1-\alpha-\beta-\gamma)}<1$, as $2\alpha+2\beta+2\gamma+\delta<1$. Repeating the above calculation, we have for each $1<n\in\mathbb{N}$,

$$\begin{aligned}d(u,Tu)&\precsim_{i_2} hd(T^{n-1}u,T^nu)\\&\precsim_{i_2} h^2d(T^{n-2}u,T^{n-1}u)\\&\quad\cdots\\&\precsim_{i_2} h^nd(u,Tu).\\\text{i.e.}, (1-h^n)&d(u,Tu)\precsim_{i_2} 0.\end{aligned}$$

Since $0\le h<1$, this implies that $d(u,Tu)=0$ i.e., $u=Tu$. Therefore u is also a fixed point of T. Thus $F(T^n)\subseteq F(T)$. So $F(T)=F(T^n)$. Hence T has the property P. $\square$

The supporting example to the Theorem 3.7 is given as follows:

Example 3.8. *Let $X=[0,1]$ together with the the partial order on X as: $x\precsim y$ if and only if $y\le x$, where $\le$ is the usual order on $\mathbb{R}$. Let us define $d:X\times X\to\mathbb{C}_2$ as*

$$d(x,y)=i_1i_2\,|x-y|\,,\ \text{for all } x,y\in X.$$

Then one can check that (X,d) is a complete bicomplex valued metric sapce with degenerated $d(x,y)+d(u,v)$ for all $x,y,u,v\in X$. Now we define the self map T on X by

$$Tx=\frac{2x}{7},\ \text{for all } x\in X.$$

Clearly T is continuous on X. Now

$$Tx=\frac{2x}{7}\precsim\frac{4x}{49}=T^2x.$$

Therefore T is weakly increasing. Set $k=i_1i_2$. For any comparable $x,y\in X$,

$$\begin{aligned}d(Tx,Ty)&=k\,|Tx-Ty|\\&=k\left|\frac{2x}{7}-\frac{2y}{7}\right|\\&\precsim_{i_2} k\left(\frac{2x}{7}+\frac{2y}{7}\right)\\&=\frac{2k}{5}\left(\frac{5x}{7}+\frac{5y}{7}\right).\end{aligned}$$

$$\begin{aligned}\text{i.e.}, d(Tx,Ty)&\precsim_{i_2}\frac{2}{5}k\left(\frac{5x}{7}+\frac{5y}{7}\right).\\&\precsim_{i_2}\frac{1}{30}k\frac{\left|x-\frac{2y}{7}\right|^2+\left|y-\frac{2x}{7}\right|^2}{\left|x-\frac{2y}{7}\right|+\left|y-\frac{2x}{7}\right|}\\&\quad+\frac{2}{5}k\left(\left|x-\frac{2x}{7}\right|+\left|y-\frac{2y}{7}\right|\right)\\&\quad+\frac{1}{30}k\left(\left|x-\frac{2y}{7}\right|+\left|y-\frac{2x}{7}\right|\right)\\&\quad+\frac{1}{30}k\,|x-y|\,.\end{aligned}$$

$$i.e., d(Tx,Ty) \precsim_{i_2} \alpha \frac{[\{d(x,Ty)\}^2+\{d(y,Tx)\}^2]}{d(x,Ty)+d(y,Tx)} +\beta[d(x,Tx)+d(y,Ty)] +\gamma[d(x,Ty)+d(y,Tx)] +\delta d(x,y),$$

where $\alpha=\frac{1}{30}, \beta=\frac{2}{5}, \gamma=\frac{1}{30}, \delta=\frac{1}{30}$ *with* $2\alpha+2\beta+2\gamma+\delta=\frac{29}{30}<1$. *Since* $x \lesssim \frac{2x}{7} \lesssim \frac{4x}{49} \lesssim \frac{8x}{343}...$, *then* $O(x,\infty)=\{x,Tx,T^2x,....\}$ *is totally ordered. Thus all the conditions of Theorem 3.7 are satisfied and for each* $n\in\mathbb{N}$, T^n *have the same unique fixed point* $0\in X$. *Hence* T *has the property P.*

Theorem 3.9. *: Let* $(X,\lesssim)$ *be a partially ordered set such that there exists a complete bicomplex valued metric d on X with degenerated* $d(x,y)+d(u,v)$, *for all* $x,y,u,v\in X$ *. Let* T *be a self-map on X as in Corollary 3.5. If for each* $x\in X$, $O(x,\infty)$ *is totally ordered, then* T *has the property P.*

Proof. From Corollary (3.5), T has a unique fixed point u (say)$\in X$. Then u is obviously also a fixed point of T^n, for each $n\in\mathbb{N}$. Therefore for each $n\in\mathbb{N}$, $F(T)\subseteq F(T^n)$. Let $u \in F(T^n)$ be fixed point of T^n i.e., $T^n u=u$ for each $1<n\in\mathbb{N}$. Since for each $x\in X$, $O(x,\infty)$ is totally ordered, by the condition (i) of Corollary 3.5,

$$d(u,Tu)=d(TT^{n-1}u,TT^nu).$$

$$i.e., d(u,Tu) \precsim_{i_2} \alpha \frac{\begin{bmatrix} d(T^{n-1}u,TT^{n-1}u)d(T^{n-1}u,TT^nu) \\ +d(T^nu,TT^nu)d(T^nu,TT^{n-1}u)\end{bmatrix}}{d(T^{n-1}u,TT^nu)+d(T^nu,TT^{n-1}u)} + \frac{d(T^{n-1}u,TT^nu)d(T^nu,TT^{n-1}u)}{d(T^{n-1}u,TT^{n-1}u)+d(T^nu,TT^nu)}$$

$$= \alpha \frac{\begin{bmatrix} d(T^{n-1}u,T^nu)d(T^{n-1}u,T^{n+1}u) \\ +d(T^nu,T^{n+1}u)d(T^nu,T^nu)\end{bmatrix}}{d(T^{n-1}u,T^{n+1}u)+d(T^nu,T^nu)} + \frac{d(T^{n-1}u,T^{n+1}u)d(T^nu,T^nu)}{d(T^{n-1}u,T^nu)+d(T^nu,T^{n+1}u)}.$$

$$i.e., d(u,Tu) \precsim_{i_2} \alpha d(T^{n-1}u,T^nu). \tag{3.5}$$

Repeating the calculation by which (3.5) has been established, we have for each $1<n\in\mathbb{N}$,

$$\begin{aligned} d(u,Tu) &\precsim_{i_2} \alpha d(T^{n-1}u,T^nu). \\ &\precsim_{i_2} \alpha^2 d(T^{n-2}u,T^{n-1}u). \\ &\precsim_{i_2} \alpha^3 d(T^{n-3}u,T^{n-2}u). \\ &\precsim_{i_2} \cdots \precsim_{i_2} \alpha^n d(u,Tu). \end{aligned}$$

$$i.e., (1-\alpha^n)d(u,Tu) \precsim_{i_2} 0.$$

Since $0\le\alpha<1$, this implies that $d(u,Tu)=0$ i.e., $u=Tu$. Therefore u is also a fixed point of T. Thus $F(T^n)\subseteq F(T)$. So $F(T)=F(T^n)$. Hence T has the property P. $\square$

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