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Coupled Caputo-Fabrizio fractional differential systems in generalized Banach spaces

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Abstract
This paper deals with existence and uniqueness of solutions for some coupled systems of Caputo-Fabrizio fractional differential equations. Some applications are made of generalizations of classical fixed point theorems on generalized Banach spaces. An illustrative example is presented in the last section.

Keywords
Fractional differential equation, Caputo–Fabrizio integral of fractional order, Caputo–Fabrizio fractional derivative, coupled system, generalized Banach space, fixed point.

AMS Subject Classification
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Contents

1. Introduction

There has been a significant development in the area of the theory of fractional calculus and fractional differential equations [27]. For some fundamental results in this subject, we refer the reader to the monographs [4, 7, 8, 20, 25, 29], and the papers [6, 14]. These fractional differential equations involve Riemann-Liouville, Caputo, Hadamard and Hilfer fractional differential operators. In recent times, a new fractional differential operator having a kernel with exponential decay has been introduced by Caputo and Fabrizio [15]. The approach of with a fractional derivative is known as the Caputo-Fabrizio operator which has attracted many research scholars due to the fact that it has a non-singular kernel. Several mathematicians are involved in development of Caputo-Fabrizio fractional differential equations, see; [13, 16, 17, 21, 28], and the references therein.

Coupled fractional differential equations have received much attention and its research has developed very rapidly. They are amongst the strongest tools of modern mathematics as they play a key role in developing differential models for highly complex systems. Some of the latest studies on initial and boundary value problems for coupled fractional differential equations are presented in [5, 9–11, 18, 19, 24].

Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations in generalized Banach spaces [1–3, 23, 26]. In this paper we discuss the existence and uniqueness of solutions for the coupled system of Caputo-Fabrizio fractional differential equations,

$$\begin{cases} ({}^{CF}D_0^{r_1}u)(t) = f_1(t,u(t),v(t)) \\ ({}^{CF}D_0^{r_2}v)(t) = f_2(t,u(t),v(t)) \end{cases} ; \ t \in I := [0,T], \quad (1.1)$$

with the initial conditions

$$\begin{cases} u(0) = u_0, \\ v(0) = v_0, \end{cases} \quad (1.2)$$

where $T > 0,\ u_0, v_0 \in \mathbb{R}^m,\ f_i : I \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m,\ i = 1,2,$ are given continuous functions, $\mathbb{R}^m,\ m \in \mathbb{N}^*$ is the Euclidian

Banach space with a suitable norm $\|\cdot\|$, and ${}^{CF}D_{t_k}^{r_i}$ is the Caputo–Fabrizio fractional derivative of order $r_i \in (0,1)$.

As far as we know, this is the first paper considering the existence of solutions for a coupled system of Caputo-Fabrizio fractional differential equations on generalized Banach spaces.

2. Preliminaries

Let C be the Banach space of all continuous functions from I into $\mathbb{R}^m$ with the supremum (uniform) norm $\|\cdot\|_\infty$, and $\mathscr{C} := C \times C$ be the product Banach space with the norm

$$\|(u,v)\|_C = \|u\|_\infty + \|v\|_\infty.$$

By $L^1(I)$, we denote the space of Lebesgue-integrable functions $v : I \to \mathbb{R}^m$ with the norm

$$\|v\|_1 = \int_0^T \|v(t)\| dt.$$

By $AC(I)$ we denote the space of absolutely continuous functions.

Definition 2.1. *[15, 21] The Caputo-Fabrizio fractional integral of order $0 < r < 1$ for a function $h \in L^1(I)$ is defined by*

$${}^{CF}I^r h(\tau) = \frac{2(1-r)}{M(r)(2-r)} h(\tau) + \frac{2r}{M(r)(2-r)} \int_0^\tau h(x)dx, \ \ \tau \geq 0$$

where $M(r)$ is normalization constant depending on r.

Definition 2.2. *[15, 21] The Caputo-Fabrizio fractional derivative for a function $h \in AC(I)$ of order $0 < r < 1$, is defined by, for $\tau \in I$,*

$${}^{CF}D^r h(\tau) = \frac{(2-r)M(r)}{2(1-r)} \int_0^\tau \exp\left(-\frac{r}{1-r}(\tau - x)\right) h'(x)dx.$$

Note that $({}^{CF}D^r)(h) = 0$ if and only if h is a constant function.

Definition 2.3. *By a solution of the problem (1.1)-(1.2) we mean a coupled ordered pair of continuous functions $(u,v) \in \mathscr{C}$ that satisfy (1.1) and (1.2).*

Lemma 2.4. *Let $h \in L^1(I)$. Then the linear problem*

$$\begin{cases} ({}^{CF}D_0^r u)(t) = h(t); \ \ t \in I := [0,T] \\ u(0) = u_0, \end{cases} \tag{2.1}$$

has a unique solution given by

$$u(t) = C + a_r h(t) + b_r \int_0^t h(s)ds, \tag{2.2}$$

where

$$a_r = \frac{2(1-r)}{(2-r)M(r)}, \ b_r = \frac{2r}{(2-r)M(r)}, \ C = u_0 - a_r h(0).$$

Proof. Suppose that u satisfies (2.1). From [21, Proposition 1], the equation

$$({}^{CF}D_0^r u)(t) = h(t)$$

implies that

$$u(t) - u(0) = a_r(h(t) - h(0)) + b_r \int_0^t h(s)ds.$$

Thus from the initial condition $u(0) = u_0$, we obtain

$$u(t) = u_0 - a_r h(0) + a_r h(t) + b_r \int_0^t h(s)ds.$$

Hence we get (2.2).

Coversely, if u satisfies (2.2), then $u(0) = u_0$, and for each $t \in I := [0,T]$, we have

$$({}^{CF}D_0^r u)(t) = h(t).$$

Hence, u satisfies (2.1). □

From the above Lemma, we can conclude the following Lemma.

Lemma 2.5. *A coupled pair of functions (u,v) is a solution of the system (1.1)-(1.2), if and only if (u,v) satisfies the following integral equations*

$$\begin{cases} u(t) = c_1 + a_{r_1} f_1(t,u(t),v(t)) + b_{r_1} \int_0^t f_1(s,u(s),v(s))ds, \\ v(t) = c_2 + a_{r_2} f_2(t,u(t),v(t)) + b_{r_2} \int_0^t f_2(s,u(s),v(s))ds, \end{cases} \tag{2.3}$$

where $c_1 = u_0 - a_{r_1} f_1(0,u_0,v_0)$, and $c_2 = v_0 - a_{r_2} f_2(0,u_0,v_0)$.

Let $x,,y \in \mathbb{R}^m$ with $x = (x_1,x_2,\ldots,x_m)$, $y = (y_1,y_2,\ldots,y_m)$. By $x \leq y$ we mean $x_i \leq y_i$; $i = 1,\ldots,m$. Also $|x| = (|x_1|,|x_2|,\ldots,|x_m|)$, $\max(x,y) = (\max(x_1,y_1),\max(x_2,y_2),\ldots,\max(x_m,y_m))$, and $\mathbb{R}_+^m = \{x \in \mathbb{R}^m : x_i \in \mathbb{R}_+, \ i = 1,\ldots,m\}$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c, \ i = 1,\ldots,m$.

Definition 2.6. *Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^m$ with the following properties:*

(i) $d(x,y) \geq 0$ *for all* $x,y \in X$, *and if* $d(x,y) = 0$, *then* $x = y$;

(ii) $d(x,y) = d(y,x)$ *for all* $x,y \in X$;

(iii) $d(x,z) \leq d(x,y) + d(y,z)$ *for all* $x,y,z \in X$.

We call the pair (X,d) a generalized metric space with

$$d(x,y) := \begin{pmatrix} d_1(x,y) \\ d_2(x,y) \\ \cdot \\ \cdot \\ \cdot \\ d_m(x,y) \end{pmatrix}.$$

Notice that d is a generalized metric space on X if and only if $d_i, \ i = 1,\ldots,m$, are metrics on X.

Definition 2.7. *[12] A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc i.e. $|\lambda| < 1$; for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$; where I denotes the unit matrix of $M_{m\times m}(\mathbb{R})$.*

Example 2.8. *The matrix $A \in M_{2\times 2}(\mathbb{R})$ defined by*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

converges to zero in the following cases:

(1) $b = c = 0,\ a,d > 0$ *and* $\max\{a,d\} < 1$.

(2) $c = 0,\ a,d > 0,\ a+d < 1$ *and* $-1 < b < 0$.

(3) $a+b = c+d = 0,\ a > 1,\ c > 0$ *and* $|a-c| < 1$.

In the sequel we will make use of the following fixed point theorems in Generalized Banach spaces.

Theorem 2.9. *[23] Let (X,d) be a complete generalized metric space and $N : X \to X$ a contractive operator with Lipschitz matrix M. Then N has a unique fixed point x_0 and for each $x \in X$ we have*

$$d(N^k(x),x_0) \le M^k(M)^{-1}d(x,N(x)),\ \text{for all } k \in \mathbb{N}.$$

For $n = 1$, we recover the classical Banach's contraction fixed point result.

Theorem 2.10. *[22] Let X be a generalized Banach space and $N : X \to X$ be a continuous and compact mapping. Then either,*

(a) The set

$$\mathscr{A} := \{x \in X : x = \lambda N(x)\ for\ some\ \lambda \in (0,1)\}$$

in unbounded, or

(b) The operator N has a fixed point.

3. Existence and Uniqueness Results

In this section, we are concerned with the existence and uniqueness results of the system (1.1)-(1.2). The following hypotheses will be used in the sequel.

(H_1) There exist continuous functions $p_i, q_i : I \to (0,\infty)$; $i = 1,2$ such that

$$\|f_i(t,u_1,v_1) - f_i(t,u_2,v_2)\| \le p_i(t)\|u_1 - u_2\| + q_i(t)\|v_1 - v_2\|,$$

$for\ a.e.\ t \in I,\ and\ each\ u_i, v_i \in \mathbb{R}^m,\ i = 1,2.$

(H_2) There exist continuous functions $a_i, b_i : I \to (0,\infty)$; $i = 1,2$ such that for a.e. $t \in I$ and each $u,v \in \mathbb{R}^m$,

$$\|f_i(t,u,v)\| \le a_i(t)\|u\| + b_i(t)\|v\|,$$

$for\ a.e.\ t \in I,\ and\ each\ u,v \in \mathbb{R}^m,$

(H_3) For any bounded set $B \subset \mathscr{C}$, the sets

$$\{t \mapsto f_i(t,u(t),v(t)) : (u,v) \in B\};\ i = 1,2,$$

are equicontinuous in $\mathscr{C}$.

First, we prove an existence and uniqueness result for the coupled system (1.1)- (1.2) by using a Banach's fixed point theorem type in generalized Banach spaces. Set

$$p_i^* := \sup_{t\in I} p_i(t),\ q_i^* := \sup_{t\in I} q_i(t);\ i = 1,2.$$

Theorem 3.1. *Assume that the hypothesis (H_1) holds. If the matrix*

$$M := \begin{pmatrix} (a_{r_1} + Tb_{r_1})p_1^* & (a_{r_1} + Tb_{r_1})q_1^* \\ (a_{r_2} + Tb_{r_2})p_2^* & (a_{r_2} + Tb_{r_2})q_2^* \end{pmatrix}$$

converges to 0, then the coupled system (1.1)-(1.2) has a unique solution.

Proof. Define the operators $N_i : \mathscr{C} \to C$; $i = 1,2$ by

$$(N_1(u,v))(t) = c_1 + a_{r_1} f_1(t,u(t),v(t)) + b_{r_1}\int_0^t f_1(s,u(s),v(s))ds, \tag{3.1}$$

and

$$(N_2(u,v))(t) = c_2 + a_{r_2} f_2(t,u(t),v(t)) + b_{r_2}\int_0^t f_2(s,u(s),v(s))ds. \tag{3.2}$$

Consider the operator $N : \mathscr{C} \to \mathscr{C}$ defined by

$$(N(u,v))(t) = ((N_1(u,v))(t),(N_2(u,v))(t)). \tag{3.3}$$

Clearly, the fixed points of the operator N are solutions of the system (1.1)-(1.2).
For any $i \in \{1,2\}$ and each $(u_1,v_1),(u_2,v_2) \in \mathscr{C}$ and $t \in I$, we have

$$\begin{aligned}
& \|(N_1(u_1,v_1))(t) - (N_1(u_2,v_2))(t)\| \\
& \le a_{r_1}\|f_1(t,u_1(t),v_1(t)) - f_1(t,u_2(t),v_2(t))\| \\
+ \quad & b_{r_1}\int_0^t \|f_1(s,u_1(s),v_1(s)) - f_1(s,u_2(s),v_2(s))\|ds \\
& \le a_{r_1}(p_1(t)\|u_1(t) - u_2(t)\| + q_1(t)\|v_1(t) - v_2(t)\|) \\
+ \quad & b_{r_1}\int_0^t (p_1(s)\|u_1(s) - u_2(s)\| + q_1(s)\|v_1(s) - v_2(s)\|)ds \\
& \le a_{r_1}(p_1^*\|u_1 - u_2\|_\infty + q_1^*\|v_1 - v_2\|_\infty) \\
+ \quad & Tb_{r_1}(p_1^*\|u_1 - u_2\|_\infty + q_1^*\|v_1 - v_2\|_\infty) \\
& \le (a_{r_1} + Tb_{r_1})(p_1^*\|u_1 - u_2\|_\infty + q_1^*\|v_1 - v_2\|_\infty).
\end{aligned}$$

Thus, we get,

$$\|N_1(u_1,v_1)-N_1(u_2,v_2)\|_\infty \le (a_{r_1}+Tb_{r_1})(p_1^*\|u_1-u_2\|_\infty + q_1^*\|v_1-v_2\|_\infty).$$

Also, for each $(u_1,v_1),(u_2,v_2)\in\mathscr{C}$ and $t\in I$, we get

$$\|N_2(u_1,v_1)-N_2(u_2,v_2)\|_\infty \le (a_{r_2}+Tb_{r_2})(p_2^*\|u_1-u_2\|_\infty + q_2^*\|v_1-v_2\|_\infty).$$

Hence,

$$d(N(u_1,v_1),N(u_2,v_2)) \le Md((u_1,v_1),(u_2,v_2)),$$

where

$$d((u_1,v_1),(u_2,v_2)) = \begin{pmatrix} \|u_1-u_2\|_\infty \\ \|v_1-v_2\|_\infty \end{pmatrix}.$$

Since the matrix M converges to zero, then Theorem 2.9 implies that the system (1.1)- (1.2) has a unique solution. □

Now, we prove an existence result for the coupled system (1.1)-(1.2) by using the nonlinear alternative of Leray–Schauder type in generalized Banach space. Set

$$a_i^* := \sup_{t\in I} a(t),\ b_i^* := \sup_{t\in I} b(t):\ i=1,2,$$

$$A = \max\{a_{r_1}a_1^*+a_{r_2}a_2^*, a_{r_1}b_1^*+a_{r_2}b_2^*\},$$

and

$$B = \max\{b_{r_1}a_1^*+b_{r_2}a_2^*, b_{r_1}b_1^*+b_{r_2}b_2^*\}.$$

Theorem 3.2. *Assume that the hypotheses* (H_2) *and* (H_3) *hold. If* $A<1$, *then the coupled system (1.1)-(1.2) has at least one solution.*

Proof. . We show that the operator $N:\mathscr{C}\to\mathscr{C}$ defined in (3.3) satisfies all conditions of Theorem 2.10. The proof will be given in four steps.

Step 1. *N is continuous.*
Let $(u_n,v_n)_n$ be a sequence such that $(u_n,v_n)\to(u,v)\in\mathscr{C}$ as $n\to\infty$. For any $i\in\{1,2\}$ and each $t\in I$, we have

$$\begin{aligned} & \|(N_i(u_n,v_n))(t)-(N_i(u,v))(t)\| \\ & \le a_{r_i}\|f_i(t,u_n(t),v_n(t))-f_i(t,u(t),v(t))\| \\ + \quad & b_{r_i}\int_0^t \|f_i(s,u_n(s),v_n(s))-f_i(s,u(s),v(s))\|ds \\ \le \quad & (a_{r_i}+Tb_{r_i})\|f_i(\cdot,u_n(\cdot),v_n(\cdot))-f_i(\cdot,u(\cdot),v(\cdot))\|_\infty. \end{aligned}$$

Since f_i is continuous, then by the Lebesgue dominated convergence theorem, we get

$$\|N_i(u_n,v_n)-N_i(u,v)\|_\infty \to 0 \text{ as } n\to\infty.$$

Hence N is continuous.

Step 2. *N maps bounded sets into bounded sets in $\mathscr{C}$.*
Let $R>0$ and set

$$B_R := \{(\mu,\nu)\in\mathscr{C} : \|\mu\|_\infty \le R, \|\nu\|_\infty \le R\}.$$

For each $(u,v)\in B_R$ and $t\in I$, we have

$$\begin{aligned} \|(N_1(u,v))(t)\| & \le \|c_1\| + a_{r_1}\|f_1(t,u(t),v(t))\| \\ & \quad + b_{r_1}\int_0^t \|f_1(s,u(s),v(s))\|ds \\ & \le \|c_1\| + a_{r_1}(a_1(t)\|u(t)\| + b_1(t)\|v(t)\|) \\ & \quad + b_{r_1}\int_0^t (a_1(s)\|u(s\| + b_1(s)\|v(s)\|)ds \\ & \le \|c_1\| + (a_{r_1}+Tb_{r_1})(a_1^*+b_1^*)R \\ & := \ell_1. \end{aligned}$$

Thus,

$$\|N_1(u,v)\|_\infty \le \ell_1.$$

Also, for each $(u,v)\in B_R$ and $t\in I$, we get

$$\begin{aligned} \|N_2(u,v)\|_\infty & \le \|c_2\| + (a_{r_2}+Tb_{r_2})(a_2^*+b_2^*)R \\ & := \ell_2. \end{aligned}$$

Hence,

$$\|N(u,v)\|_{\mathscr{C}} \le (\ell_1,\ell_2) := \ell.$$

Step 3. *N maps bounded sets into equicontinuous sets in $\mathscr{C}$.*
Let B_R be the ball defined in Step 2. For each $t_1,t_2\in I$ with $t_1\le t_2$ and $(u,v)\in B_R$, we have

$$\begin{aligned} & \|(N_1(u,v))(t_1)-(N_1(u,v))(t_2)\| \\ \le\ & a_{r_1}\|f_1(t_2,u(t_2),v(t_2))-f_1(t_1,u(t_1),v(t_1))\| \\ & + b_{r_1}\int_{t_1}^{t_2}\|f_1(s,u(s),v(s))\|ds \\ \le\ & a_{r_1}\|f_1(t_2,u(t_2),v(t_2))-f_1(t_1,u(t_1),v(t_1))\| \\ & + Rb_{r_1}(a_1^*+b_1^*)(t_2-t_1) \\ \to\ & 0 \text{ as } t_1\to t_2. \end{aligned}$$

Also, from (H_3), we get

$$\begin{aligned} & \|(N_2(u,v))(t_1)-(N_2(u,v))(t_2)\| \\ \le\ & a_{r_2}\|f_2(t_2,u(t_2),v(t_2))-f_2(t_1,u(t_1),v(t_1))\| \\ +\ & Rb_{r_2}(a_2^*+b_2^*)(t_2-t_1) \\ \to\ & 0 \text{ as } t_1\to t_2. \end{aligned}$$

Hence, the set $N(B_R)$ is equicontinuous in $\mathscr{C}$.

As a consequence of Steps 1 to 3, with the Arzela–Ascoli theorem, we conclude that N maps B_R into a precompact set in $\mathscr{C}$.

Step 4. *The set E consisting of $(u,v)\in\mathscr{C}$ such that $(u,v)=\lambda N(u,v)$ for some $\lambda\in(0,1)$ is bounded in $\mathscr{C}$.*
Let $(u,v)\in\mathscr{C}$ such that $(u,v)=\lambda N(u,v)$. Then $u=\lambda N_1(u,v)$

and $v=\lambda N_2(u,v)$. Thus, for each $t\in I$, we have

$$\begin{aligned}\|u(t)\| &\le \|c_1\|+a_{r_1}\|f_1(t,u(t),v(t))\|\\ &\quad +b_{r_1}\int_0^t\|f_1(s,u(s),v(s))\|ds\\ &\le \|c_1\|+a_{r_1}(a_1^*\|u(t\|+b_1^*\|v(t)\|)\\ &\quad +b_{r_1}\int_0^t(a_1^*\|u(s\|+b_1^*\|v(s)\|)ds.\end{aligned}$$

Also, we get

$$\begin{aligned}\|v(t)\| &\le \|c_2\|+a_{r_2}(a_2^*\|u(t)\|+b_2^*\|v(t)\|)\\ &\quad +b_{r_2}\int_0^t(a_2^*\|u(s)\|+b_2^*\|v(s)\|)ds.\end{aligned}$$

Thus, we get

$$\begin{aligned}\|u(t)\|+\|v(t)\| &\le \|c_1\|+\|c_2\|+(a_{r_1}a_1^*+a_{r_2}a_2^*)\|u(t)\|\\ &+ (a_{r_1}b_1^*+a_{r_2}b_2^*)\|v(t)\|\\ &+ \int_0^t[(b_{r_1}a_1^*+b_{r_2}a_2^*)\|u(s)\|\\ &\quad +(b_{r_1}b_1^*+b_{r_2}b_2^*)\|v(s\|]ds\\ &\le \|c_1\|+\|c_2\|+A(\|u(t\|+\|v(t\|)\\ &+ B\int_0^t(\|u(s)\|+\|v(s)\|)ds.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\|u(t)\|+\|v(t)\| &\le \frac{\|c_1\|+\|c_2\|}{1-A}\\ &+ \frac{B}{1-A}\int_0^t(\|u(s)\|+\|v(s)\|)ds.\end{aligned}$$

By applying a classical Gronwall's lemma, we get

$$\begin{aligned}\|u(t)\|+\|v(t)\| &\le \frac{\|c_1\|+\|c_2\|}{1-A}\exp\left(\frac{B}{1-A}\int_0^t ds\right)\\ &\le \frac{\|c_1\|+\|c_2\|}{1-A}\exp\left(\frac{BT}{1-A}\right)\\ &= L.\end{aligned}$$

This gives

$$\|u\|_\infty+\|v\|_\infty\le L.$$

Hence

$$\|(u,v)\|_{\mathscr{C}}\le L.$$

This shows that the set E is bounded.

As a consequence of Steps 1 to 4, together with Theorem 2.10, we can conclude that N has at least one fixed point in B_R which is a solution of the system (1.1)-(1.2). □

4. An Example

Consider the following coupled system of Caputo-Fabrizio fractional differential equations,

$$\begin{cases}({}^{CF}D_0^{\frac{1}{2}}u)(t)=f(t,u(t),v(t));\\ ({}^{CF}D_0^{\frac{1}{4}}v)(t)=g(t,u(t),v(t));\\ u_n(0)=1,\\ v_n(0)=0,\end{cases} \quad : \; t\in[0,1], \qquad (4.1)$$

where

$$f(t,u,v)=\frac{t^{\frac{-1}{4}}(u(t)+v(t))\sin t}{64(1+\sqrt{t})(1+|u|+|v|)};\; t\in[0,1],$$

$$g(t,u,v)=\frac{(u(t)+v(t))\cos t}{64(1+|u|+|v|)};\; t\in[0,1].$$

Set $r_1=\frac{1}{2}$ and $r_1=\frac{1}{4}$. The hypothesis (H_1) is satisfied with

$$p_1^*=p_2^*=q_1^*=q_2^*=\frac{1}{64}.$$

Also the matrix

$$\begin{aligned}M &:= \begin{pmatrix}(a_{r_1}+Tb_{r_1})p_1^* & (a_{r_1}+Tb_{r_1})q_1^*\\ (a_{r_2}+Tb_{r_2})p_2^* & (a_{r_2}+Tb_{r_2})q_2^*\end{pmatrix}\\ &= \frac{1}{64}\begin{pmatrix}a_{r_1}+b_{r_1} & a_{r_1}+b_{r_1}\\ a_{r_2}+b_{r_2} & a_{r_2}+b_{r_2}\end{pmatrix}\end{aligned}$$

converges to 0. Hence, Theorem 3.1 implies that the system (4.1) has a unique solution defined on $[0,1]$.

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