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Exponential stability to a laminated beam in thermoelasticity of type III with delay

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Abstract. In this paper, we study the well-posedness and asymptotic behaviour of solutions to a laminated beam in thermoelasticity of type III with delay term in the fourth equation. We first give the well-posedness of the system by using semigroup method and Lumer-Philips theorem. Then, by using the perturbed energy method and construct some Lyapunov functionals, we obtain the exponential decay result for the case of equal wave speeds.
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Keywords: Laminated beam, thermoelasticity of type III, delay, well-posedness, exponential stability.

Contents

1. Introduction

In this work, we consider a coupled system of a laminated beam with thermoelasticity of type III and delay term in the fourth equation, which has the form

$$\begin{cases} \rho_1\varphi_{tt} + G\left(\psi - \varphi_x\right)_x = 0, & (x,t) \in (0,1)\times(0,+\infty), \\ \rho_2\left(3\omega - \psi\right)_{tt} - G\left(\psi - \varphi_x\right) - D\left(3\omega - \psi\right)_{xx} + \alpha\theta_x = 0, & (x,t) \in (0,1)\times(0,+\infty), \\ \rho_2\omega_{tt} + G\left(\psi - \varphi_x\right) + \frac{4}{3}\gamma\omega + \frac{4}{3}\beta\omega_t - D\omega_{xx} = 0, & (x,t) \in (0,1)\times(0,+\infty), \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \sigma\left(3\omega - \psi\right)_{ttx} - \mu_1\theta_{txx}\left(x,t\right) - \mu_2\theta_{txx}\left(x,t-\tau\right) = 0, & (x,t) \in (0,1)\times(0,+\infty), \end{cases} \tag{1.1}$$

with the following initial and boundary conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), & x \in [0,1], \\ \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x), & x \in [0,1], \\ \omega(x,0) = \omega_0(x), \omega_t(x,0) = \omega_1(x), & x \in [0,1], \\ \theta(x,0) = \theta_0(x),\ \theta_t(x,0) = \theta_1(x), & x \in [0,1], \\ \theta_{tx}\left(x,t-\tau\right) = f_0\left(x,t-\tau\right), & (x,t) \in (0,1)\times(0,\tau), \\ \varphi_x(0,t) = \varphi_x(1,t) = \psi(0,t) = \psi(1,t) = 0, & t \in [0,+\infty), \\ \omega(0,t) = \omega\left(1,t\right) = \theta_x\left(0,t\right) = \theta_x\left(1,t\right) = 0, & t \in [0,+\infty), \end{cases} \tag{1.2}$$

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where $\varphi(x,t)$ denotes the transverse displacement, $\psi(x,t)$ represents the rotation angle. $\omega(x,t)$ is proportional to the amount of slip along the interface at time t and longitudinal spatial variable x. $\theta(x,t)$ is the differential temperature, and $\rho_1,\rho_2,\rho_3,G,D,\alpha,\beta,\gamma,\delta,\sigma,\mu_1$ are positive constants, μ_2 is a real number and $\tau>0$ represents the time delay. Moreover, $\sqrt{\frac{G}{\rho_1}}$ and $\sqrt{\frac{D}{\rho_2}}$ are two wave speeds.

Laminated beam, which is a relevant research subject due to the high applicability of such materials in the industry, was firstly introduced by Hansen and Spies, see, for instance [15, 16]. Hansen [15] proposed a model of laminated beam based on the Timoshenko system which is one of particular interest. In [16], Hansen and Spies derived three mathematical models for two-layered beams with structural damping due to the interfacial slip. The system is given by the following equations

$$\begin{cases} \rho_1\varphi_{tt}+G\left(\psi-\varphi_x\right)_x=0, & (x,t)\in(0,1)\times(0,+\infty),\\ \rho_2\left(3\omega-\psi\right)_{tt}-D\left(3\omega-\psi\right)_{xx}-G\left(\psi-\varphi_x\right)=0, & (x,t)\in(0,1)\times(0,+\infty),\\ 3\rho_2\omega_{tt}+3G\left(\psi-\varphi_x\right)+4\gamma\omega+4\beta\omega_t-3D\omega_{xx}=0, & (x,t)\in(0,1)\times(0,+\infty), \end{cases}$$

the coefficients ρ_1, G, ρ_2, D, γ and β are positive constants and represent density, shear stiffness, mass moment of inertia, flexural rigidity, adhesive stiffness, and adhesive damping parameter, respectively. The third equation describes the dynamics of the slip. For asymptotic behavior results to laminated beams, we refer the reader to [1, 19, 21, 22, 31] and the references therein. In [26], Rivera and Racke established several exponential decay results for linear Timoshenko systems in classical thermoelasticity where the heat flux is given by Fourier's law. Since this theory predicts an infinite speed of heat propagation, many theories have emerged, to overcome this physical paradox. Green and Naghdi [11–13], suggest a replacing Fourier's law by the so- called thermoelasticity of type III. This is for heat conduction modeling thermal disturbances as wave-like pulses traveling at finite speed. For more details, see [2]. A large number of interesting decay results depending on the stability number have been established, (see [9, 24, 25, 27] and references therein). W. Liu et al. [23] considered a coupled system of a laminated beam with thermoelasticity of type III, which has the form

$$\begin{cases} \rho_1\varphi_{tt}+G\left(\psi-\varphi_x\right)_x=0, & (x,t)\in(0,1)\times(0,+\infty),\\ I_{\rho_1}\left(3\omega-\psi\right)_{tt}-D\left(3\omega-\psi\right)_{xx}-G\left(\psi-\varphi_x\right)+\alpha\theta_x=0, & (x,t)\in(0,1)\times(0,+\infty),\\ I_{\rho_1}\omega_{tt}-D\omega_{xx}+G\left(\psi-\varphi_x\right)+\frac{4}{3}\beta_1\omega+\frac{4}{3}\beta_2\omega_t=0, & (x,t)\in(0,1)\times(0,+\infty),\\ \rho_2\theta_{tt}-\delta\theta_{xx}+\gamma\left(3\omega-\psi\right)_{ttx}-k\theta_{txx}=0, & (x,t)\in(0,1)\times(0,+\infty), \end{cases}$$

they used the energy method to prove an exponential decay result for the case of equal wave speeds.

Time delay appears in many physical, biological and economic problems, because, in most instances, the present state system does not depend only on the current state but also on some past occurrences. In recent years, the control of PDEs with time delay effects has become an active area of research.The presence of delay may be a source of instability. It may turn a well-behaved system into a wild one. For example, it was shown in [4, 5, 14, 28, 32] that an arbitrarily small delay may destabilize a system that is uniformly asymptotically stable in the absence of delay unless additional control terms have been used. The stability issue of systems with delay is, therefore, of theoretical and practical great importance. In [29], Nicaise, Pignotti and Valein replaced the constant delay term in the boundary condition of [28] by a time-varying delay term and obtained an exponential decay result under an appropriate assumption on the weights of the damping and delay. Moreover, Kafini et al. [18] studied the following Timoshenko system of thermoelasticity of type III with delay of the form

$$\begin{cases} \rho_1\phi_{tt}-K\left(\phi_x+\psi\right)_x+\mu_1\phi_t\left(x,t\right)+\mu_2\phi_t\left(x,t-\tau\right)=0, & (x,t)\in(0,1)\times(0,+\infty),\\ \rho_2\psi_{tt}-b\psi_{xx}+K\left(\phi_x+\psi\right)+\beta\theta_{tx}=0, & (x,t)\in(0,1)\times(0,+\infty),\\ \rho_3\theta_{tt}-\delta\theta_{xx}+\gamma\psi_{tx}-k\theta_{txx}=0, & (x,t)\in(0,1)\times(0,+\infty), \end{cases}$$

under the initial and boundary conditions

$$\begin{cases} \theta(.,0)=\theta_0,\ \theta_t(.,0)=\theta_1,\psi(.,0)=\psi_0, & x\in[0,1],\\ \psi_t(.,0)=\psi_1,\phi(.,0)=\phi_0,\phi_t(.,0)=\phi_1, & x\in[0,1],\\ \phi_t\left(x,t-\tau\right)=f_0\left(x,t-\tau\right), & t\in(0,\tau),\\ \phi(0,t)=\phi(1,t)=\psi(0,t)=\psi(1,t)=\theta_x\left(0,t\right)=\theta_x\left(1,t\right)=0, & t\in[0,+\infty), \end{cases}$$

the energy of system decays exponentially in the case of equal wave speeds. For other related results, we refer the reader to [3, 6–8, 17, 20]. Motivated by the above results, in the present work, we study the well-posedness and asymptotic behaviour of solutions to the laminated beam (1.1)-(1.2) in thermoelasticity of type III with delay term. The plan of the paper is as follows. In Section 2, we introduce some preliminaries. In Section 3, by using semigroup method and Lumer-Philips theorem, we state and prove the well posedness of the system. In Section 4, by using the perturbed energy method and construct some Lyapunov functionals, we then establish the exponential result if and only if $\frac{G}{\rho_1} = \frac{D}{\rho_2}$.

2. Preliminaries

In this section, we present some material that we shall use in order to present our results, to exhibit the dissipative nature of the system (1.1), we introduce some new variables

$$\Phi = \varphi_t \,, \Psi = \psi_t \,, W = \omega_t,$$

and we introduce as in [28] the new variable

$$z\left(x,\rho,t\right) = \theta_{tx}\left(x,t-\tau\rho\right), (x,\rho,t) \in (0,1)\times(0,1)\times(0,\infty).$$

Then we have

$$\tau z_t\left(x,\rho,t\right) + z_\rho\left(x,\rho,t\right) = 0, (x,\rho,t) \in (0,1)\times(0,1)\times(0,\infty).$$

Therefore, system (1.1) takes the form

$$\begin{cases} \rho_1\Phi_{tt} + G\left(\Psi-\Phi_x\right)_x = 0, \\ \rho_2\left(3W-\Psi\right)_{tt} - G\left(\Psi-\Phi_x\right) - D\left(3W-\Psi\right)_{xx} + \alpha\theta_{tx} = 0, \\ \rho_2 W_{tt} + G\left(\Psi-\Phi_x\right) + \frac{4}{3}\gamma W + \frac{4}{3}\beta W_t - DW_{xx} = 0, \\ \rho_3\theta_{tt} - \delta\theta_{xx} - \mu_1\theta_{txx} - \mu_2 z_x\left(x,1,t\right) + \sigma\left(3W-\Psi\right)_{tx} = 0, \\ \tau z_t\left(x,\rho,t\right) + z_\rho\left(x,\rho,t\right) = 0, \end{cases} \tag{2.1}$$

where $(x,\rho,t) \in (0,1)\times(0,1)\times(0,\infty)$, with the initial data and boundary conditions

$$\begin{cases} \Phi(x,0) = \Phi_0(x), \Phi_t(x,0) = \Phi_1(x), & x\in[0,1], \\ \Psi(x,0) = \Psi_0(x), \Psi_t(x,0) = \Psi_1(x), & x\in[0,1], \\ W(x,0) = W_0(x), W_t(x,0) = W_1(x), & x\in[0,1], \\ \theta(x,0) = \theta_0(x),\ \theta_t(x,0) = \theta_1(x), & x\in[0,1], \\ z\left(x,\rho,0\right) = f_0\left(x,-\tau\rho\right), & (x,\rho)\in(0,1)\times(0,1), \\ z\left(x,0,t\right) = \theta_{tx}\left(x,t\right), & (x,t)\in(0,1)\times(0,\infty), \\ \Phi_x(0,t) = \Phi_x(1,t) = \Psi(0,t) = \Psi(1,t) = 0, & t\in[0,+\infty), \\ W(0,t) = W\left(1,t\right) = \theta_x\left(0,t\right) = \theta_x\left(1,t\right) = 0, & t\in[0,+\infty), \end{cases} \tag{2.2}$$

where

$$\begin{aligned} \Phi_0(x) &= \varphi_1,\ \ \Phi_1(x) = -\frac{G}{\rho_1}\left(\psi_0-\varphi_{0x}\right)_x,\ \Psi_0\left(x\right) = \psi_1, \\ \Psi_1(x) &= -\frac{4G}{\rho_2}\left(\psi_0-\varphi_{0x}\right) - \frac{D}{\rho_2}\left(3\omega_0-\psi_0\right)_{xx} + \frac{\alpha}{\rho_2}\theta_{1x} - \frac{4\gamma}{\rho_2}\omega_0 - \frac{4\beta}{\rho_2}\omega_1 + \frac{3D}{\rho_2}\omega_{0xx}, \\ W_0(x) &= \omega_1, W_1(x) = -\frac{G}{\rho_2}\left(\psi_0-\varphi_{0x}\right) - \frac{4\gamma}{3\rho_2}\omega_0 - \frac{4\beta}{3\rho_2}\omega_1 + \frac{D}{\rho_2}\omega_{0xx}, \end{aligned}$$

where $x\in[0,1]$. From equations $(2.1)_4$ and (2.2), we easily verify that

$$\frac{d^2}{dt^2}\int_0^1 \theta\left(x,t\right)dx = 0.$$

So, if we set

$$\overline{\theta}(x,t) := \theta(x,t) - \int_0^1 \theta_0(x)\,dx - t\int_0^1 \theta_1(x)\,dx,$$

then simple substitution shows that $\left(\Phi, \Psi, W, \overline{\theta}, z\right)$ satisfies (2.1), the boundary conditions in (2.2) and more importantly

$$\int_0^1 \overline{\theta}(x,t)\,dx = 0, \ \ \forall t > 0.$$

In this case, Poincaré's inequality is applicable for $\overline{\theta}$. In the sequel, we work with $\overline{\theta}$ but for convenience, we write θ instead. We will assume that

$$\mu_1 > |\mu_2|, \tag{2.3}$$

and show the well-posedness of the problem and that this condition is sufficient to prove the uniform decay of the solution energy.

3. Well-posedness of the problem

In this Section, we prove the existence and uniqueness of solutions for (2.1)-(2.2). Introducing the vector function

$$U = (\Phi, 3W - \Psi, W, \theta, \Phi_t, 3W_t - \Psi_t, W_t, \theta_t, z)^T,$$

system (2.1)-(2.2) can be written as

$$\begin{cases} \dfrac{dU(t)}{dt} = \mathcal{A}U(t), t > 0, \\ U(0) = U_0 = (\Phi_0, 3W_0 - \Psi_0, W_0, \theta_0, \Phi_1, 3W_1 - \Psi_1, W_1, \theta_1, f_0)^T, \end{cases} \tag{3.1}$$

where $\mathcal{A}$ is a linear operator defined by

$$\mathcal{A}\begin{pmatrix} \Phi \\ 3W - \Psi \\ W \\ \theta \\ \Phi_t \\ 3W_t - \Psi_t \\ W_t \\ \theta_t \\ z \end{pmatrix} = \begin{pmatrix} \Phi_t \\ 3W_t - \Psi_t \\ W_t \\ \theta_t \\ -\dfrac{G}{\rho_1}(\psi - \Phi_x)_x \\ \dfrac{G}{\rho_2}(\psi - \Phi_x) + \dfrac{D}{\rho_2}(3W - \Psi)_{xx} - \dfrac{\alpha}{\rho_2}\theta_{tx} \\ -\dfrac{G}{\rho_2}(\psi - \Phi_x) - \dfrac{4\gamma}{3\rho_2}W - \dfrac{4\beta}{3\rho_2}W_t + \dfrac{D}{\rho_2}W_{xx} \\ \dfrac{\delta}{\rho_3}\theta_{xx} - \dfrac{\sigma}{\rho_3}(3W - \Psi)_{tx} + \dfrac{\mu_1}{\rho_3}\theta_{txx} + \dfrac{\mu_2}{\rho_3}z_x(x,1,t) \\ -\tau^{-1}z_\rho \end{pmatrix}.$$

We consider the following spaces

$$L^2_*(0,1) = \left\{w \in L^2(0,1) : \int_0^1 w(s)\,ds = 0\right\}, \ H^1_*(0,1) = H^1(0,1) \cap L^2_*(0,1),$$
$$H^2_*(0,1) = \left\{w \in H^2(0,1) : w_x(0) = w_x(1) = 0\right\}.$$

Let

$$\mathcal{H} = H^1_*(0,1) \times H^1_0(0,1) \times H^1_0(0,1) \times H^1_*(0,1) \times L^2_*(0,1) \times L^2(0,1) \times L^2(0,1) \times L^2_*(0,1) \times L^2\left((0,1), L^2(0,1)\right),$$

be the Hilbert space equipped with the inner product

$$\left\langle U,\widetilde{U}\right\rangle_{\mathcal{H}} = \sigma\rho_1\int_0^1 \Phi_t\overline{\Phi}_t dx + \sigma G\int_0^1 (\Psi-\Phi_x)\left(\overline{\Psi}-\overline{\Phi}_x\right)dx + 4\sigma\gamma\int_0^1 W\overline{W}dx + 3\sigma\int_0^1 \rho_2 W_t\overline{W}_t dx$$
$$+\sigma\rho_2\int_0^1 (3W-\Psi)_t\left(3\overline{W}-\overline{\Psi}\right)_t dx + \sigma\int_0^1 D(3W-\Psi)_x\left(3\overline{W}-\overline{\Psi}\right)_x dx$$
$$+3\sigma D\int_0^1 W_x\overline{W}_x dx + \alpha\rho_3\int_0^1 \theta_t\overline{\theta}_t dx + \alpha\delta\int_0^1 \theta_x\overline{\theta}_x dx + \lambda\int_0^1\int_0^1 z\overline{z}d\rho dx,$$

where λ is the positive constant satisfying

$$\begin{cases} \tau\alpha|\mu_2| < \lambda < \tau\alpha(2\mu_1-|\mu_2|), & \text{if} \quad |\mu_2| < \mu_1, \\ \lambda = \tau\alpha\mu_1, & \text{if} \quad |\mu_2| = \mu_1. \end{cases} \tag{3.2}$$

Then, the domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ \begin{array}{c} U\in\mathcal{H} \mid \Phi,\theta\in H_*^2(0,1)\cap H_*^1(0,1), \Psi, W\in H^2(0,1)\cap H_0^1(0,1), \\ \Psi_t, W_t\in H_0^1(0,1), \Phi_t,\theta_t\in H_*^1(0,1), (\delta+e^{-\tau}\mu_2)\theta+\mu_1\theta_t\in H_*^2(0,1), \\ z, z_\rho\in L^2\left((0,1), L^2(0,1)\right), z(x,0)=\theta_{tx}(x) \end{array} \right\}. \tag{3.3}$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

We have the following existence and uniqueness result.

Theorem 3.1. *Assume that $U_0\in\mathcal{H}$ and (2.3) holds. Then there exists a unique solution $U\in C(\mathbb{R}^+;\mathcal{H})$ of problem (3.1). Moreover, if $U_0\in D(\mathcal{A})$, then*

$$U\in C\left(\mathbb{R}^+; D(\mathcal{A})\cap C^1\left(\mathbb{R}^+;\mathcal{H}\right)\right).$$

Proof. The result follows from Lumer-Phillips theorem provided we prove that $\mathcal{A}$ is a maximal monotone operator. For this purpose, we need the following two steps: $\mathcal{A}$ is dissipative and $Id-\mathcal{A}$ surjective.

Step 1. $\mathcal{A}$ is dissipative.

For any $U\in D(\mathcal{A})$, and using the inner product, we obtain

$$\langle\mathcal{A}U,U\rangle_{\mathcal{H}} = -4\sigma\beta\int_0^1 W_t^2dx - \alpha\mu_1\int_0^1\theta_{tx}^2 + \alpha\mu_2\int_0^1 z_x(x,1,t)\theta_t dx - \frac{\lambda}{\tau}\int_0^1\int_0^1 zz_\rho(x,\rho,t)d\rho dx. \tag{3.4}$$

By using integration by parts and the fact that $z(x,0)=\theta_{tx}(x)$, the last term in the right-hand side of (3.4) gives

$$-\int_0^1\int_0^1 zz_\rho(x,\rho,t)d\rho dx = \frac{1}{2}\int_0^1\theta_{tx}^2dx - \frac{1}{2}\int_0^1 z^2(x,1,t)dx. \tag{3.5}$$

Substituting (3.5) in (3.4) yields

$$\langle\mathcal{A}U,U\rangle_{\mathcal{H}} = -4\sigma\beta\int_0^1 W_t^2dx - \alpha\mu_1\int_0^1\theta_{tx}^2 + \alpha\mu_2\int_0^1 z_x(x,1,t)\theta_t dx + \frac{\lambda}{2\tau}\int_0^1\theta_{tx}^2dx$$
$$-\frac{\lambda}{2\tau}\int_0^1 z^2(x,1,t)dx. \tag{3.6}$$

Also, using integration by parts and Young's inequality we obtain, from (3.6)

$$\langle\mathcal{A}U,U\rangle_{\mathcal{H}} \le -\left(\alpha\mu_1 - \frac{\alpha|\mu_2|}{2} - \frac{\lambda}{2\tau}\right)\int_0^1\theta_{tx}^2dx - \left(\frac{\lambda}{2\tau} - \frac{\alpha|\mu_2|}{2}\right)\int_0^1 z^2(x,1,t)dx - 4\sigma\beta\int_0^1 W_t^2dx.$$

Keeping in mind condition (3.2), we observe that

$$\alpha\mu_1 - \frac{\alpha\left|\mu_2\right|}{2} - \frac{\lambda}{2\tau} \geq 0, \qquad \frac{\lambda}{2\tau} - \frac{\alpha\left|\mu_2\right|}{2} \geq 0.$$

Consequently, the operator $\mathcal{A}$ is dissipative.

Step 2. $Id - \mathcal{A}$ is surjective.

To prove that the operator $Id - \mathcal{A}$ is surjective, that is, for any $F = (f_1, ..., f_9) \in \mathcal{H}$, there exists $U = (\Phi, 3W - \Psi, W, \theta, \Phi_t, 3W_t - \Psi_t, W_t, \theta_t, z) \in D(\mathcal{A})$ satisfying

$$\left(Id - \mathcal{A}\right) U = F, \tag{3.7}$$

which is equivalent to

$$\begin{cases} \Phi - \Phi_t = f_1, \\ (3W - \Psi) - (3W - \Psi)_t = f_2, \\ W - W_t = f_3, \\ \theta - \theta_t = f_4, \\ \rho_1\Phi_t - G\Phi_{xx} - G\left(3W - \Psi\right)_x + 3GW_x = \rho_1 f_5, \\ \rho_2\left(3W - \Psi\right)_t + G\Phi_x + G\left(3W - \Psi\right) - 3GW - D\left(3W - \Psi\right)_{xx} + \alpha\theta_{tx} \\ = \rho_2 f_6, \\ \rho_2 W_t - G\left(3W - \Psi\right) + 3GW - G\Phi_x + \frac{4\gamma}{3}W + \frac{4\beta}{3}W_t - DW_{xx} = \rho_2 f_7, \\ \rho_3\theta_t - \delta\theta_{xx} + \sigma\left(3W - \Psi\right)_{tx} - \mu_1\theta_{txx} - \mu_2 z_x\left(x, 1, t\right) = \rho_3 f_8, \\ \tau z + z_\rho = \tau f_9. \end{cases} \tag{3.8}$$

From $(3.8)_1$–$(3.8)_4$, we have

$$\begin{cases} \Phi_t = \Phi - f_1, \\ (3W - \Psi)_t = (3W - \Psi) - f_2, \\ W_t = W - f_3, \\ \theta_t = \theta - f_4. \end{cases} \tag{3.9}$$

By combining (3.9) and (3.8), it can be $\Phi, 3W - \Psi, W, \theta$ shown that satisfy

$$\begin{cases} \rho_1\Phi - G\Phi_{xx} - G\left(3W - \Psi\right)_x + 3GW_x = \rho_1\left(f_1 + f_5\right), \\ \rho_2\left(3W - \Psi\right) + G\Phi_x + G\left(3W - \Psi\right) - 3GW - D\left(3W - \Psi\right)_{xx} + \alpha\theta_x \\ = \rho_2\left(f_2 + f_6\right) + \alpha\partial_x f_4, \\ \rho_2 W - G\left(3W - \Psi\right) + 3GW - G\Phi_x + \frac{4\gamma}{3}W + \frac{4\beta}{3}W - DW_{xx} \\ = \rho_2\left(f_3 + f_7\right) + \frac{4\beta}{3}f_3, \\ \rho_3\theta - \delta\theta_{xx} + \sigma\left(3W - \Psi\right)_x - \mu_1\theta_{xx} - \mu_2 z_x\left(x, 1, t\right) \\ = \rho_3\left(f_4 + f_8\right) + \sigma\partial_x f_2 + \mu_1\partial_{xx} f_4, \\ \tau z + z_\rho = \tau f_9. \end{cases} \tag{3.10}$$

Using the last equation in (3.10) we can find z with

$$z\left(x, 0\right) = \theta_{tx}\left(x\right), \ x \in (0, 1).$$

Following the same approach as in [28], we obtain, by using $(3.10)_5$,

$$z\left(x, \rho, \tau\right) = \theta_{tx}\left(x\right) e^{-\tau\rho} + \tau e^{-\tau\rho}\int_0^\rho e^{\tau s} f_9\left(x, s\right) ds.$$

From $(3.9)_4$, we obtain

$$z\left(x, \rho, \tau\right) = \theta_x e^{-\tau\rho} - \partial_x f_4\left(x\right) e^{-\tau\rho} + \tau e^{-\tau\rho}\int_0^\rho e^{\tau s} f_9\left(x, s\right) ds, \tag{3.11}$$

and in particular,

$$z\left(x,1,\tau\right)=\theta_{x}e^{-\tau}+z_{0}\left(x,\tau\right),$$

where

$$z_{0}\left(x,\tau\right)=-\partial_{x}f_{4}\left(x\right)e^{-\tau}+\tau e^{-\tau}\int_{0}^{1}e^{\tau s}f_{9}\left(x,s\right)ds.$$

In order to solve (3.8), we consider the following variational formulation

$$B\left(\left(\Phi,3W-\Psi,W,\theta\right)^{T},\left(\tilde{\Phi},3\tilde{W}-\tilde{\Psi},\tilde{W},\tilde{\theta}\right)^{T}\right)=G\left(\left(\tilde{\Phi},3\tilde{W}-\tilde{\Psi},\tilde{W},\tilde{\theta}\right)^{T}\right), \tag{3.12}$$

where $B:\left[H_{*}^{1}\left(0,1\right)\times H_{0}^{1}\left(0,1\right)\times H_{0}^{1}\left(0,1\right)\times H_{*}^{1}\left(0,1\right)\right]^{2}\longrightarrow\mathbb{R}$ is the bilinear form

$$\begin{aligned}
&B\left(\left(\Phi,3W-\Psi,W,\theta\right)^{T},\left(\tilde{\Phi},3\tilde{W}-\tilde{\Psi},\tilde{W},\tilde{\theta}\right)^{T}\right)\\
&=\sigma\int_{0}^{1}G(\Psi-\Phi_{x})(\tilde{\Psi}-\tilde{\Phi}_{x})dx+\sigma\int_{0}^{1}\rho_{1}\Phi\tilde{\Phi}dx+\sigma\int_{0}^{1}\rho_{2}\left(3W-\Psi\right)\left(3\tilde{W}-\tilde{\Psi}\right)dx+\alpha\int_{0}^{1}\rho_{3}\theta\tilde{\theta}dx\\
&\quad+\left(3\sigma\rho_{2}+4\sigma\gamma+4\sigma\beta\right)\int_{0}^{1}W\tilde{W}dx+\sigma\int_{0}^{1}D\left(3W-\Psi\right)_{x}\left(3\tilde{W}-\tilde{\Psi}\right)_{x}dx+3\sigma\int_{0}^{1}DW_{x}\tilde{W}_{x}dx\\
&\quad+\alpha\left(\delta+\mu_{1}+e^{-\tau}\mu_{2}\right)\int_{0}^{1}\theta_{x}\tilde{\theta}_{x}dx+\sigma\alpha\int_{0}^{1}\left(3W-\Psi\right)_{x}\tilde{\theta}dx+\sigma\alpha\int_{0}^{1}\theta_{x}\left(3\tilde{W}-\tilde{\Psi}\right)dx,
\end{aligned}$$

and $G:H_{*}^{1}\left(0,1\right)\times H_{0}^{1}\left(0,1\right)\times H_{0}^{1}\left(0,1\right)\times H_{*}^{1}\left(0,1\right)\longrightarrow\mathbb{R}$ is the linear form

$$\begin{aligned}
&F\left(\left(\tilde{\Phi},3\tilde{W}-\tilde{\Psi},\tilde{W},\tilde{\theta}\right)^{T}\right)\\
&=\sigma\int_{0}^{1}\rho_{1}\left(f_{1}+f_{5}\right)\tilde{\Phi}dx+\sigma\int_{0}^{1}\rho_{2}\left(f_{2}+f_{6}\right)\left(3\tilde{W}-\tilde{\Psi}\right)dx+3\sigma\int_{0}^{1}\rho_{2}\left(f_{3}+f_{7}\right)\tilde{W}dx\\
&\quad+4\sigma\int_{0}^{1}\beta f_{3}\tilde{W}dx+\alpha\int_{0}^{1}\rho_{3}\left(f_{4}+f_{8}\right)\tilde{\theta}dx+\alpha\sigma\int_{0}^{1}\partial_{x}f_{2}\tilde{\theta}dx+\alpha\mu_{1}\int_{0}^{1}\partial_{x}f_{4}\partial_{x}\tilde{\theta}dx\\
&\quad+\sigma\alpha\int_{0}^{1}\partial_{x}f_{4}\left(3\tilde{W}-\tilde{\Psi}\right)dx-\alpha\mu_{2}\int_{0}^{1}\partial_{x}z_{0}\tilde{\theta}dx.
\end{aligned}$$

Now, for

$$V=H_{*}^{1}\left(0,1\right)\times H_{0}^{1}\left(0,1\right)\times H_{0}^{1}\left(0,1\right)\times H_{*}^{1}\left(0,1\right),$$

equipped with the norm

$$\left\|\left(\Phi,3W-\Psi,W,\theta\right)\right\|_{V}^{2}=\left\|\Psi-\Phi_{x}\right\|_{2}^{2}+\left\|\Phi\right\|_{2}^{2}+\left\|\left(3W-\Psi\right)_{x}\right\|_{2}^{2}+\left\|W_{x}\right\|_{2}^{2}+\left\|\theta\right\|_{2}^{2}+\left\|\theta_{x}\right\|_{2}^{2},$$

one can easily see that $B(.,.)$ and $G(.)$ are bounded. Furthermore, using integration by parts, we obtain

$$B\left(\left(\Phi,3W-\Psi,W,\theta\right)^{T},\left(\Phi,3W-\Psi,W,\theta\right)^{T}\right)\geq c\left\|\left(\Phi,3W-\Psi,W,\theta\right)\right\|_{V}^{2},$$

for some $c>0$. Thus, $B(.,.)$ is coercive.

Consequently, by Lax-Milgram lemma, we obtain that (3.12) has a unique solution

$$\Phi\in H_{*}^{1}\left(0,1\right),\quad\left(3W-\Psi\right)\in H_{0}^{1}\left(0,1\right),\quad W\in H_{0}^{1}\left(0,1\right),\quad\theta\in H_{*}^{1}\left(0,1\right).$$

The substitution of $\Phi,3W-\Psi,W$ and θ into (3.9) yields

$$\Phi_{t}\in H_{*}^{1}\left(0,1\right),\quad\left(3W-\Psi\right)_{t}\in H_{0}^{1}\left(0,1\right),\quad W_{t}\in H_{0}^{1}\left(0,1\right),\quad\theta_{t}\in H_{*}^{1}\left(0,1\right).$$

Next, it remains to show that

$$\Phi \in \left(H_*^2(0,1) \cap H_*^1(0,1)\right), \quad (3W-\Psi) \in \left(H^2(0,1) \cap H_0^1(0,1)\right),$$
$$W \in \left(H^2(0,1) \cap H_0^1(0,1)\right), \quad \theta \in \left(H_*^2(0,1) \cap H_*^1(0,1)\right).$$

Taking $\left(3\tilde{W}-\tilde{\Psi}, \tilde{W}, \tilde{\theta}\right) = (0,0,0) \in H_0^1(0,1) \times H_0^1(0,1) \times H_*^1(0,1)$ in (3.12), we get

$$\begin{aligned} & B\left((\Phi, 3W-\Psi, W, \theta)^T, \left(\tilde{\Phi}, 0,0,0\right)^T\right) \\ & = \sigma \int_0^1 \rho_1 \Phi \tilde{\Phi} dx + \sigma \int_0^1 G(-\Phi_{xx}\tilde{\Phi} - (3W-\Psi)_x \tilde{\Phi} + 3W_x \tilde{\Phi}) dx \\ & = \sigma \int_0^1 \rho_1 (f_1+f_5) \tilde{\Phi} dx, \quad \forall \tilde{\Phi} \in H_*^1(0,1), \end{aligned} \tag{3.13}$$

which implies

$$G\Phi_{xx} = \rho_1 \Phi - G(3W-\Psi)_x + 3GW_x - \rho_1 (f_1+f_5) \in L_*^2(0,1). \tag{3.14}$$

Consequently, by the regularity theory for the linear elliptic equations, it follows that

$$\Phi \in H^2(0,1) \cap H_*^1(0,1).$$

Moreover, (3.13) is also true for any $\phi \in C^1[0,1] \subset H_*^1(0,1)$. Hence, we have

$$\int_0^1 G\Phi_x \phi_x dx + \int_0^1 (\rho_1 \Phi - G(3W-\Psi)_x + 3GW_x - \rho_1(f_1+f_5)) \phi dx = 0$$

for all $\phi \in C^1[0,1]$. Thus, using integration by parts and bearing in mind (3.14), we obtain

$$\Phi_x(1)\phi(1) - \Phi_x(0)\phi(0) = 0, \forall \phi \in C^1[0,1].$$

Therefore, $\Phi_x(0) = \Phi_x(1) = 0$. Consequently, we obtain

$$\Phi \in H_*^2(0,1) \cap H_*^1(0,1).$$

In the same way, taking $\left(\tilde{\Phi}, \tilde{W}, \tilde{\theta}\right) = (0,0,0) \in H_*^1(0,1) \times H_0^1(0,1) \times H_*^1(0,1)$ in (3.12), we get

$$\begin{aligned} & B\left((\Phi, 3W-\Psi, W, \theta)^T, \left(0, 3\tilde{W}-\tilde{\Psi}, 0,0\right)^T\right) \\ & = \sigma \int_0^1 G\left(\Phi_x\left(3\tilde{W}-\tilde{\Psi}\right) + (3W-\Psi)\left(3\tilde{W}-\tilde{\Psi}\right) - 3W\left(3\tilde{W}-\tilde{\Psi}\right)\right) dx \\ & \quad + \sigma \int_0^1 \rho_2 (3W-\Psi)\left(3\tilde{W}-\tilde{\Psi}\right) dx + \sigma \int_0^1 D(3W-\Psi)_x \left(3\tilde{W}-\tilde{\Psi}\right)_x dx + \sigma\alpha \int_0^1 \theta_x \left(3\tilde{W}-\tilde{\Psi}\right) dx \\ & = \sigma \int_0^1 \rho_2 (f_2+f_6)\left(3\tilde{W}-\tilde{\Psi}\right) dx + \sigma\alpha \int_0^1 \partial_x f_4 \left(3\tilde{W}-\tilde{\Psi}\right) dx. \end{aligned}$$

Recalling $(3.8)_2$ and $(3.8)_4$, we arrive at

$$\begin{aligned} & \int_0^1 D(3W-\Psi)_x \left(3\tilde{W}-\tilde{\Psi}\right)_x dx \\ & = \int_0^1 \left[\rho_2 f_6 - G(\Phi_x + (3W-\Psi) - 3W) - \alpha\theta_{tx} - \rho_2 (3W-\Psi)_t\right] \left(3\tilde{W}-\tilde{\Psi}\right) dx \end{aligned} \tag{3.15}$$

for all $\left(3\tilde{W}-\tilde{\Psi}\right)\in H^{1}\left(0,1\right)$, which implies

$$\rho_2 f_6 - G\left(\Phi_x + (3W-\Psi) - 3W\right) - \alpha\theta_{tx} - \rho_2 \left(3W-\Psi\right)_t \in L^2\left(0,1\right).$$

Consequently, (3.15) takes the form

$$\int_0^1 \left[-D\left(3W-\Psi\right)_{xx} + G\Phi_x + G\left(3W-\Psi\right) - 3GW + \alpha\theta_{tx} + \rho_2\left(3W-\Psi\right)_t - \rho_2 f_6\right]\left(3\tilde{W}-\tilde{\Psi}\right)dx = 0.$$

We obtain

$$-D\left(3W-\Psi\right)_{xx} + G\left(\Phi_x + G\left(3W-\Psi\right) - 3W\right) + \alpha\theta_{tx} + \rho_2\left(3W-\Psi\right)_t = \rho_2 f_6,$$

and

$$\left(3W-\Psi\right)\in H^2\left(0,1\right)\cap H_0^1\left(0,1\right),$$

which gives $(3.8)_6$. Similarly, we can show that

$$W\in H^2\left(0,1\right)\cap H_0^1\left(0,1\right),$$

and $(3.8)_7$ are satisfied. Also, if we take $\left(\tilde{\Phi},3\tilde{W}-\tilde{\Psi},\tilde{W}\right) = (0,0,0)\in H_*^1\left(0,1\right)\times H_0^1\left(0,1\right)\times H_0^1\left(0,1\right)$ in (3.12), then using $(3.8)_2$ and $(3.8)_4$, we get

$$\left(\delta + e^{-\tau}\mu_2\right)\theta_{xx} + \mu_1\theta_{txx} = \rho_3\theta_t - \rho_3 f_8 + \sigma\left(3W-\Psi\right)_{tx} + \mu_2\partial_x z_0,$$

and we conclude that

$$\left(\delta + e^{-\tau}\mu_2\right)\theta + \mu_1\theta_t \in H^2\left(0,1\right).$$

Furthermore, it is obvious from

$$\left(\delta + e^{-\tau}\mu_2\right)\theta_x + \mu_1\theta_{tx} = \rho_3\int_0^x \theta_t dx - \rho_3\int_0^x f_8 dx + \sigma\left(3W-\Psi\right)_t + \mu_2 z_0,$$

that

$$\left(\left(\delta + e^{-\tau}\mu_2\right)\theta_x + \mu_1\theta_{tx}\right)(0) = \left(\left(\delta + e^{-\tau}\mu_2\right)\theta_x + \mu_1\theta_{tx}\right)(1) = 0,$$

then, we get

$$\left(\delta + e^{-\tau}\mu_2\right)\theta + \mu_1\theta_t \in H_*^2\left(0,1\right).$$

Finally, it follows, from (3.11), that

$$z\left(x,0\right) = \theta_{tx}\left(x\right) \text{ and } z, z_\rho \in L^2\left(\left(0,1\right),L^2\left(0,1\right)\right).$$

Hence, there exists a unique $U\in D\left(\mathcal{A}\right)$ such that (3.7) is satisfied, the operator $Id-\mathcal{A}$ is surjective. Moreover, it is easy to see that $D\left(\mathcal{A}\right)$ is dense in $\mathcal{H}$.

At last, by Lumer-Philips theorem (see [10, 30]) we have the well-posedness result stated in Theorem 3.1. ■

4. Exponential stability

In this section, we state and prove our stability result for the solution of problem (2.1)-(2.2), by using the multiplier technique. We first introduce the following energy functional

$$\begin{aligned} E\left(t\right) := \frac{1}{2}\int_0^1 \Big[&\sigma\rho_1\Phi_t^2 + \sigma G\left(\Psi-\Phi_x\right)^2 + \sigma\rho_2\left(3W-\Psi\right)_t^2 + \sigma D\left(3W-\Psi\right)_x^2 \\ &+3\sigma\rho_2 W_t^2 + 4\sigma\gamma W^2 + 3\sigma D W_x^2 + \alpha\rho_3\theta_t^2 + \alpha\delta\theta_x^2 + \lambda\int_0^1 z^2\left(x,\rho,t\right)d\rho\Big]dx. \end{aligned} \tag{4.1}$$

To achieve our goal, we need the following lemmas.

Lemma 4.1. *Let* (Φ,Ψ,W,θ,z) *be the solution of problem (2.1)-(2.2). Then the energy functional* $E(t)$ *defined by (4.1) satisfies*

$$\frac{d}{dt}E\left(t\right)=-4\beta\sigma\int_0^1 W_t^2dx-C_1\int_0^1\theta_{tx}^2dx-C_2\int_0^1 z^2\left(x,1,t\right)dx\le 0, \tag{4.2}$$

where

$$C_1=\mu_1\alpha-\frac{|\mu_2|\,\alpha}{2}-\frac{\lambda}{2\tau}\ge 0 \quad , \quad C_2=\frac{\lambda}{2\tau}-\frac{|\mu_2|\,\alpha}{2}\ge 0.$$

Proof. Multiplying the first four equations in (2.1) by $\sigma\Phi_t$, $\sigma\left(3W-\Psi\right)_t$, $3\sigma W_t$, $\alpha\theta_t$ respectively, then, integrating over $(0,1)$, and multiplying $(2.1)_5$ by $\frac{\lambda}{\tau}z$ and integrating over $(0,1)\times(0,1)$ with respect to ρ and x, summing them up, we obtain

$$\begin{aligned}&\frac{d}{dt}\frac{\sigma}{2}\int_0^1\left[\rho_1\Phi_t^2+G\left(\Psi-\Phi_x\right)^2+\rho_2\left(3W_t-\Psi_t\right)^2+D\left(3W_x-\Psi_x\right)^2+3\rho_2W_t^2+4\gamma W^2+3DW_x^2\right]dx\\&+\frac{d}{dt}\frac{\alpha}{2}\int_0^1\left(\rho_3\theta_t^2+\delta\theta_x^2\right)dx+\frac{d}{dt}\frac{\lambda}{2}\int_0^1\int_0^1 z^2\left(x,\rho,t\right)d\rho dx\\=&-4\beta\sigma\int_0^1 W_t^2dx-\mu_1\alpha\int_0^1\theta_{tx}^2dx+\mu_2\alpha\int_0^1\theta_tz_x\left(x,1,t\right)dx-\frac{\lambda}{\tau}\int_0^1\int_0^1 zz_\rho\left(x,\rho,t\right)d\rho dx.\end{aligned} \tag{4.3}$$

The last two terms of the right side of (4.3) can be estimated as follows.

$$-\frac{\lambda}{\tau}\int_0^1\int_0^1 zz_\rho\left(x,\rho,t\right)d\rho dx=\frac{\lambda}{2\tau}\int_0^1\theta_{tx}^2dx-\frac{\lambda}{2\tau}\int_0^1 z^2\left(x,1,t\right)dx,$$

$$\mu_2\alpha\int_0^1\theta_tz_x\left(x,1,t\right)dx\le\frac{|\mu_2|\,\alpha}{2}\int_0^1\theta_{tx}^2dx+\frac{|\mu_2|\,\alpha}{2}\int_0^1 z^2\left(x,1,t\right)dx.$$

Hence,

$$\frac{d}{dt}E\left(t\right)\le-4\beta\sigma\int_0^1 W_t^2dx-\left(\mu_1\alpha-\frac{|\mu_2|\,\alpha}{2}-\frac{\lambda}{2\tau}\right)\int_0^1\theta_{tx}^2dx-\left(\frac{\lambda}{2\tau}-\frac{|\mu_2|\,\alpha}{2}\right)\int_0^1 z^2\left(x,1,t\right)dx.$$

Using (3.2), we obtain the result. ■

Lemma 4.2. *Let* (Φ,Ψ,W,θ,z) *be the solution of problem (2.1)-(2.2). The functional*

$$F_1\left(t\right):=-\rho_1\int_0^1\Phi\Phi_tdx+\rho_2\int_0^1 WW_tdx \tag{4.4}$$

satisfies the estimate

$$\begin{aligned}F_1'\left(t\right)\le&-\rho_1\int_0^1\Phi_t^2dx-\frac{2\gamma}{3}\int_0^1 W^2dx-\frac{D}{2}\int_0^1 W_x^2dx+C_3\int_0^1 W_t^2dx+C_4\int_0^1\left(\Psi-\Phi_x\right)^2dx\\&+\frac{D}{18}\int_0^1\left(3W_x-\Psi_x\right)^2dx,\end{aligned} \tag{4.5}$$

where

$$C_3=\rho_2+\frac{4\beta^2}{3\gamma} \quad , \quad C_4=G+\frac{9G^2}{2D}+\frac{3G^2}{4\gamma}.$$

Proof. By differentiating F_1 with respect to t, using $(2.1)_1$, $(2.1)_3$ and integrating by parts, we obtain

$$F_1'(t) = -\rho_1 \int_0^1 \Phi_t^2 dx - G \int_0^1 \Phi_x (\Psi - \Phi_x) dx + \rho_2 \int_0^1 W_t^2 dx - D \int_0^1 W_x^2 dx - G \int_0^1 W (\Psi - \Phi_x) dx$$
$$-\frac{4\gamma}{3} \int_0^1 W^2 dx - \frac{4\beta}{3} \int_0^1 W W_t dx.$$

Note that

$$-G \int_0^1 \Phi_x (\Psi - \Phi_x) dx = G \int_0^1 (\Psi - \Phi_x)^2 dx - G \int_0^1 \Psi (\Psi - \Phi_x) dx.$$

Then, we deduce that

$$F_1'(t) = -\rho_1 \int_0^1 \Phi_t^2 dx + G \int_0^1 (\Psi - \Phi_x)^2 dx - G \int_0^1 \Psi (\Psi - \Phi_x) dx + \rho_2 \int_0^1 W_t^2 dx - D \int_0^1 W_x^2 dx$$
$$-G \int_0^1 W (\Psi - \Phi_x) dx - \frac{4\gamma}{3} \int_0^1 W^2 dx - \frac{4\beta}{3} \int_0^1 W W_t dx.$$

Making use of Young's and Poincaré inequalities, we obtain

$$F_1'(t) \le -\rho_1 \int_0^1 \Phi_t^2 dx - \frac{2\gamma}{3} \int_0^1 W^2 dx - D \int_0^1 W_x^2 dx + \frac{D}{36} \int_0^1 \Psi_x^2 dx + \left(\rho_2 + \frac{4\beta^2}{3\gamma}\right) \int_0^1 W_t^2 dx$$
$$+\left(G + \frac{9G^2}{2D} + \frac{3G^2}{4\gamma}\right) \int_0^1 (\Psi - \Phi_x)^2 dx.$$

Note that

$$\int_0^1 \Psi_x^2 dx = \int_0^1 (\Psi_x - 3W_x + 3W_x)^2 dx \le 2 \int_0^1 (3W_x - \Psi_x)^2 + 18 \int_0^1 W_x^2 dx.$$

Then the estimate (4.5) is established. ■

Lemma 4.3. *Let $(\Phi, \Psi, W, \theta, z)$ be the solution of problem (2.1)-(2.2). The functional*

$$F_2(t) := \rho_2 \int_0^1 (3W - \Psi)(3W - \Psi)_t dx \tag{4.6}$$

satisfies the estimate

$$F_2'(t) \le -\frac{D}{2} \int_0^1 (3W_x - \Psi_x)^2 dx + \rho_2 \int_0^1 (3W_t - \Psi_t)^2 dx + \frac{G^2}{2D} \int_0^1 (\Psi - \Phi_x)^2 dx + \frac{\alpha^2}{D} \int_0^1 \theta_t^2 dx, \tag{4.7}$$

Proof. By differentiating F_2 with respect to t, using $(2.1)_2$ and integrating by parts, we get

$$F_2'(t) = G \int_0^1 (3W - \Psi)(\Psi - \Phi_x) dx - D \int_0^1 (3W_x - \Psi_x)^2 dx + \alpha \int_0^1 (3W_x - \Psi_x) \theta_t dx$$
$$+\rho_2 \int_0^1 (3W_t - \Psi_t)^2 dx.$$

Using Young's and Poincaré inequalities, we obtain the result. ■

Lemma 4.4. *Let $(\Phi, \Psi, W, \theta, z)$ be the solution of problem (2.1)-(2.2). The functional*

$$F_3(t) := \rho_2\rho_3 \int_0^1 (3W - \Psi)_t \int_0^x \theta_t(y,t) dy dx - \delta\rho_2 \int_0^1 \theta_x (3W - \Psi) dx \tag{4.8}$$

satisfies the estimate

$$F_3'(t) \leq -\frac{\rho_2\sigma}{2}\int_0^1 (3W-\Psi)_t^2\,dx + \varepsilon_1\int_0^1 (\Psi-\Phi_x)^2\,dx + C_5(\varepsilon_1)\int_0^1 \theta_{xt}^2 dx + \varepsilon_1\int_0^1 (3W_x-\Psi_x)^2\,dx$$
$$+\frac{\rho_2\mu_2^2}{\sigma}\int_0^1 z^2(x,1,t)\,dx, \tag{4.9}$$

for any $\varepsilon_1 > 0$, *where*

$$C_5(\varepsilon_1) = \frac{\alpha\rho_3}{2} + \frac{\rho_2\mu_2^2}{\sigma} + \frac{D^2\rho_3^2}{8\varepsilon_1} + \frac{\delta^2\rho_2^2}{4\varepsilon_1} + \frac{G^2\rho_3^2}{16\varepsilon_1}.$$

Proof. By differentiating F_3 with respect to t, using $(2.1)_2$, $(2.1)_4$ and integrating by parts, we obtain

$$F_3'(t) = \rho_3\int_0^1 G(\Psi-\Phi_x)\int_0^x \theta_t(y,t)\,dydx - \delta\rho_2\int_0^1 \theta_{xt}(3W-\Psi)\,dx$$
$$+\left[\rho_3\left(-G\Phi + D(3W-\Psi)_x - \alpha\theta_t\right)\int_0^x \theta_t(y,t)\,dy\right]_{x=0}^{x=1} + \alpha\rho_3\int_0^1 \theta_t^2 dx - \rho_2\sigma\int_0^1 (3W-\Psi)_t^2\,dx$$
$$+\rho_2\mu_1\int_0^1 (3W-\Psi)_t\,\theta_{tx}dx - D\rho_3\int_0^1 \theta_t(3W-\Psi)_x\,dx + \rho_2\mu_2\int_0^1 (3W-\Psi)_t\,z(x,1,t)\,dx.$$

Note that

$$\int_0^1 \theta_t(y,t)\,dy = \frac{d}{dt}\int_0^1 \theta(y,t)\,dy = 0,$$

then, by Young's and Poincaré inequalities, with $\varepsilon_1 > 0$ to obtain (4.9). ∎

Lemma 4.5. *Let* (Φ,Ψ,W,θ,z) *be the solution of problem (2.1)-(2.2). The functional*

$$F_4(t) := \int_0^1 \left[\rho_3\theta_t\theta + \frac{\mu_1}{2}\theta_x^2 + \sigma(3W-\Psi)_x\,\theta\right]dx \tag{4.10}$$

satisfies the estimate

$$F_4'(t) \leq -\frac{\delta}{2}\int_0^1 \theta_x^2 dx + \left(\rho_3 + \frac{\sigma^2}{4\varepsilon_2}\right)\int_0^1 \theta_t^2 dx + \varepsilon_2\int_0^1 (3W-\Psi)_x^2\,dx + \frac{\mu_2^2}{2\delta}\int_0^1 z^2(x,1,t)\,dx, \tag{4.11}$$

for any $\varepsilon_2 > 0$.

Proof. By differentiating F_4 with respect to t, using $(2.1)_4$ and integrating by parts, we obtain

$$F_4'(t) = \int_0^1 \delta\theta_{xx}\theta dx + \int_0^1 \rho_3\theta_t^2 dx + \int_0^1 \mu_2 z_x(x,1,t)\,\theta dx + \int_0^1 \sigma(3W-\Psi)_x\,\theta_t dx.$$

Using Young's inequality with $\varepsilon_2 > 0$, we establish (4.11). ∎

Lemma 4.6. *Let* (Φ,Ψ,W,θ,z)*be the solution of (2.1)-(2.2). Then the functional*

$$F_5(t) := \rho_2\int_0^1 (3W-\Psi)_t(\Phi_x-\Psi)\,dx + \frac{D\rho_1}{G}\int_0^1 (3W-\Psi)_x\,\Phi_t dx \tag{4.12}$$

satisfies the estimate

$$F_5'(t) \leq -\frac{G}{2}\int_0^1 (\Psi-\Phi_x)^2\,dx + \frac{\alpha^2}{2G}\int_0^1 \theta_{tx}^2 dx + (\rho_2+\varepsilon_3)\int_0^1 (3W-\Psi)_t^2\,dx + \frac{9\rho_2^2}{4\varepsilon_3}\int_0^1 W_t^2 dx$$
$$+\left(\frac{D\rho_1}{G} - \rho_2\right)\int_0^1 (3W-\Psi)_{xt}\,\Phi_t dx, \tag{4.13}$$

for any $\varepsilon_3 > 0$.

Proof. By differentiating F_5 with respect to t, using $(2.1)_1$, $(2.1)_2$ and integrating by parts, we obtain

$$\begin{aligned} F_5'(t) &= -\int_0^1 G(\Psi-\Phi_x)^2\,dx + \int_0^1 \alpha\theta_{tx}(\Psi-\Phi_x)\,dx - \rho_2\int_0^1 (3W-\Psi)_t\,\Psi_t dx \\ &+ \left(\frac{D\rho_1}{G}-\rho_2\right)\int_0^1 (3W-\Psi)_{xt}\,\Phi_t dx. \end{aligned}$$

Using Young's inequality with $\varepsilon_3 > 0$, we establish (4.13). ■

Lemma 4.7. *Let* (Φ,Ψ,W,θ,z)*be the solution of (2.1)-(2.2). Then the functional*

$$F_6(t) := \int_0^1\int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\,d\rho dx \tag{4.14}$$

satisfies, for some $m, c > 0$*, the following estimate*

$$F_6'(t) \le -m\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho dx - \frac{c}{\tau}\int_0^1 z^2(x,1,t)\,dx + \frac{1}{\tau}\int_0^1 \theta_{tx}^2 dx, \tag{4.15}$$

Proof. By differentiating F_6 with respect to t, using $(2.1)_5$ and integrating by parts, we obtain

$$\begin{aligned} F_6'(t) &= -\frac{2}{\tau}\int_0^1\int_0^1 e^{-2\tau\rho} z(x,\rho,t)\,z_\rho(x,\rho,t)\,d\rho dx \\ &= -2\int_0^1\int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\,d\rho dx - \frac{1}{\tau}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}\left(e^{-2\tau\rho} z^2(x,\rho,t)\right) d\rho dx \\ &\le -m\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho dx - \frac{c}{\tau}\int_0^1 z^2(x,1,t)\,dx + \frac{1}{\tau}\int_0^1 \theta_{tx}^2 dx. \end{aligned}$$

This gives (4.15). ■

The stability result reads as follows.

Theorem 4.8. *Assume that* $\frac{G}{\rho_1} = \frac{D}{\rho_2}$ *and (2.3) holds. Let* $U_0 \in \mathcal{H}$*, then there exist two positive constants* c_0 *and* c_1*, such that the energy* $E(t)$ *associated with problem (2.1)-(2.2) satisfies*

$$E(t) \le c_0 E(0)\, e^{-c_1 t},\ t \ge 0.$$

Proof. To establish the decay result, we assume $\frac{G}{\rho_1} = \frac{D}{\rho_2}$ and define a Lyapunov functional $\mathcal{L}$ as follows

$$\mathcal{L}(t) := \delta_1 E(t) + F_1(t) + \delta_2 F_2(t) + \delta_3 F_3(t) + F_4(t) + \delta_4 F_5(t) + F_6(t),$$

where $\delta_1, \delta_2, \delta_3, \delta_4$ are positive constants to be chosen properly later.

Using Cauchy-Schwarz inequality and the Poincaré's inequality, one can easily see that all $F_i(t), i = 1, ..., 6$ are bounded by an expression with the existing terms in the energy $E(t)$. This leads to the equivalence of $\mathcal{L}(t)$ and $E(t)$.

Gathering the estimates in the previous lemmas and using

$$\int_0^1 \theta_t^2 dx \le \int_0^1 \theta_{tx}^2 dx,$$

we arrive at

$$\begin{aligned}\mathcal{L}'(t) \leq & -\left[4\beta\sigma\delta_1 - C_3 - \frac{9\rho_2^2}{4\varepsilon_3}\delta_4\right]\int_0^1 W_t^2 dx - \frac{D}{2}\int_0^1 W_x^2 dx - \frac{\delta}{2}\int_0^1 \theta_x^2 dx \\ & -\left[\delta_1 C_1 - \frac{\alpha^2}{D}\delta_2 - C_5(\varepsilon_1)\delta_3 - \left(\rho_3 + \frac{\sigma^2}{4\varepsilon_2}\right) - \frac{\alpha^2}{2G}\delta_4 - \frac{1}{\tau}\right]\int_0^1 \theta_{tx}^2 dx \\ & -\left[\frac{G}{2}\delta_4 - C_4 - \frac{G^2}{2D}\delta_2 - \varepsilon_1\delta_3\right]\int_0^1 (\Psi - \Phi_x)^2 dx - \rho_1\int_0^1 \Phi_t^2 dx - \frac{2\gamma}{3}\int_0^1 W^2 dx \\ & -\left[\frac{D}{2}\delta_2 - \frac{D}{18} - \varepsilon_1\delta_3 - \varepsilon_2\right]\int_0^1 (3W_x - \Psi_x)^2 dx - \left[\frac{\rho_2\sigma}{2}\delta_3 - \rho_2\delta_2 - (\rho_2 + \varepsilon_3)\delta_4\right]\int_0^1 (3W_t - \Psi_t)^2 dx \\ & -\left[\delta_1 C_2 + \frac{c}{\tau} - \frac{\rho_2\mu_2^2}{\sigma}\delta_3 - \frac{\mu_2^2}{2\delta}\right]\int_0^1 z^2(x,1,t)\,dx - m\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho dx. \end{aligned} \tag{4.16}$$

At this point we will choose all the constants, carefully. First, we take δ_2 large enough and ε_2 small, such that

$$\frac{D}{2}\delta_2 - \frac{D}{18} - \varepsilon_2 > 0.$$

Then we can take δ_4 sufficiently large such that

$$\frac{G}{2}\delta_4 - C_4 - \frac{G^2}{2D}\delta_2 > 0.$$

Next, we pick ε_3 small and choose δ_3 large enough such that

$$\frac{\rho_2\sigma}{2}\delta_3 - \rho_2\delta_2 - (\rho_2 + \varepsilon_3)\delta_4 > 0.$$

After that, we then select ε_1 so small that

$$\frac{D}{2}\delta_2 - \frac{D}{18} - \varepsilon_2 - \varepsilon_1\delta_3 > 0\,, \quad \frac{G}{2}\delta_4 - C_4 - \frac{G^2}{2D}\delta_2 - \varepsilon_1\delta_3 > 0.$$

Finally, we choose δ_1 so large such that

$$\begin{aligned} & 4\beta\sigma\delta_1 - C_3 - \frac{9\rho_2^2}{4\varepsilon_3}\delta_4 > 0\,, \quad \delta_1 C_2 + \frac{c}{\tau} - \frac{\rho_2\mu_2^2}{\sigma}\delta_3 - \frac{\mu_2^2}{2\delta} > 0, \\ & \delta_1 C_1 - \frac{\alpha^2}{D}\delta_2 - C_5(\varepsilon_1)\delta_3 - \left(\rho_3 + \frac{\sigma^2}{4\varepsilon_2}\right) - \frac{\alpha^2}{2G}\delta_4 - \frac{1}{\tau} > 0. \end{aligned}$$

On the hand, from the above, we deduce that for some positive constants α_1, α_2 one has

$$\alpha_1 E(t) \leq \mathcal{L}(t) \leq \alpha_2 E(t)\,.$$

Therefore, (4.16) becomes

$$\mathcal{L}'(t) \leq -cE(t)\,.$$

For $c_1 = \dfrac{c}{\alpha_2}$, we get

$$\mathcal{L}'(t) \leq -c_1\mathcal{L}(t)\,, \forall t \geq 0. \tag{4.17}$$

Integrating (4.17) over $(0,t)$, yields

$$\mathcal{L}(t) \leq \mathcal{L}(0)\,e^{-c_1 t}, \forall t \geq 0. \tag{4.18}$$

At last, estimate (4.18) gives the desired result Theorem 4.8 when combined with the equivalence of $\mathcal{L}(t)$ and $E(t)$. ■

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