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Separation axioms via ${}^{\star}\delta$-set in topological vector spaces

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Abstract
In this paper we introduce a new sort of spaces as ${}^{\star}\delta$-Homogenous space, ${}^{\star}\delta$-Hausdorff space and ${}^{\star}\delta$-Compact space. It provides a new connection between ${}^{\star}\delta$-Vector spaces and ${}^{\star}\delta$-homogenous spaces. Also we investigated the relationship between the translation and scalar multiplication mappings and ${}^{\star}\delta$-homeomorphism on ${}^{\star}\delta$-Topological vector spaces. Finally we derive ${}^{\star}\delta$-Topological vector space is ${}^{\star}\delta$-Hausdorff and ${}^{\star}\delta$-Compact spaces.

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${}^{\star}\delta$-topological vector spaces, ${}^{\star}\delta$-homeomorphism, ${}^{\star}\delta$-continuous, ${}^{\star}\delta$-Hausdorff, ${}^{\star}\delta$-compact.

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Contents

1. Introduction

A topological space is a vector space with a topological structure such that the algebraic operations addition and scalar multiplication are continuous. The concept of vector spaces was introduced by Kolmogroff [1]. In 2015, Khan et al [2] introduced the s-topological vector spaces which are generalization of topological vector spaces. In 2016 Khan and Iqbal [3] introduced the irresolute independent of topological vector spaces. In 2019, β-topological vector spaces have been introduced by Sharma and M.Ram [8]. In 2019, S.Sharma et al. [9] investigated almost β-topological vector spaces. Maki et al [4] introduced the notions of generalized homeomorphism in topological spaces. In this paper we introduce a new sort of spaces as ${}^{\star}\delta$-Homogenous space, ${}^{\star}\delta$-Hausdorff space and ${}^{\star}\delta$-Compact space. It provides a new connection between ${}^{\star}\delta$-Vector spaces and ${}^{\star}\delta$-homogenous spaces. Also we investigated the relationship between the translation and scalar multiplication mappings and ${}^{\star}\delta$-homeomorphism on ${}^{\star}\delta$-Topological vector spaces. Finally we derive ${}^{\star}\delta$-Topological vector space is ${}^{\star}\delta$-Hausdorff and ${}^{\star}\delta$-Compact spaces. Throughout the present paper (X,τ) (Simply X) always mean topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of a space $X, cl(A)$ and $int(A)$ denote the closure and the interior of A respectively. Now we recall some of the basic definitions and results in topology.

2. Preliminaries

In this section, we recall some definitions and basic results of fractional calculus which will be used throughout the paper.

Definition 2.1. *[7]* A subset A of a topological space (X,τ) is

(i) Regular*-open if $A = int(cl^{*}(A))$

(ii) Regular*-closed if $A = cl(int^{*}(A))$

Definition 2.2. *[5]* The ${}^{\star}\delta$-interior of a subset A of X is called ${}^{\star}\delta$-open if $A = int_{\star\delta}(A)$ ie, a set is if it is the union of *Regular**-open sets. The complement of a ${}^{\star}\delta$-open set is called ${}^{\star}\delta$-closed set in X.

Definition 2.3. *[6]* A $^\star\delta$-topological vector space is a vector space X over the field F (real or complex) with a topology τ with the following conditions.

(i) Vector addition mapping $m : X \to Y$ defined by $m((x,y)) = x+y$, for each x,y in X is $^\star\delta$-continuous

(ii) Scalar multiplication mapping $M : F \times X \to X$ which define by $M((\lambda,x)) = \lambda x$ for each λ in F and x,y in X is $^\star\delta$-continuous.

The pair $(X_{(F)},\tau)$ is said to be Topological vector space. In short, it is denoted by X, a $^\star\delta$-TVS.

Definition 2.4. *[2]* IfX is a Vector space then e denotes its identity element, and for a fixed $x \in X$, ${}_xT : X \to X; y \to x+y$ and $T_x : X \to X; y \to y+x$ denote the left and right translation by x respectively.

Definition 2.5. *[6]* Let Y be a Linear Subspace of (X,τ) which means $Y+Y \subseteq Y$ and for all $\alpha \in F, \alpha Y \subseteq Y$.

Result 2.6. *[6]* Let $(X_{(F)},\tau)$ be a $^\star\delta$-TVS. If A is open in $(X_{(F)},\tau)$, then the following are true.
(i) $x+A$ is a $^\star\delta$-open for each $x \in X$
(ii) αA is a $^\star\delta$-open for all non-zero scalar α in X.

Result 2.7. *[6]* In a $^\star\delta$-TVS $(X_{(F)},\tau)$, for any $^\star\delta$-open set U containing 0, there exists a symmetric $^\star\delta$-open set V containing 0 such that $V+V \subseteq U$.

Result 2.8. *[6]* Let X be $^\star\delta$-TVS. If A is open subset of X then $A+B$ is a $^\star\delta$-open in X for any subset B of X.

3. Translation Mappings

In this section we prove that translation mappings are $^\star\delta$-continuous in a $^\star\delta$-Topological Vector Spaces. Also it's basic properties have been derived.

Theorem 3.1. In a $^\star\delta$-TVS $(X_{(F)},\tau)$, for any $x \in X$, the translation mapping $T_x : X \to X$ defined by $T_x(y) = y+x$ for all $y \in X$ is $^\star\delta$-continuous function.

Proof. Suppose that $(X_{(F)},\tau)$ is a $^\star\delta$-topological vector space. Let $x \in X$ be arbitrary. Let K be any open set in the codomain X containing $T_x(y) = y+x$. By hypothesis, there exists a $^\star\delta$-open set U containing y and V containing x such that $U+V \subseteq K$. Then $T_x(U) = U+x \subseteq U+V \subseteq K$. It is proved that for every open set K containing $T_x(y), \exists$ a $^\star\delta$-open set U containing y such that $T_x(U) \subseteq K$. Therefore the translation mapping T_x is $^\star\delta$-continuous function. □

Theorem 3.2. In a $^\star\delta$-TVS $(X_{(F)},\tau)$, for any $\alpha \in F$, the multiplication mapping $M_\alpha : X \to X$ defined by $M_\alpha(x) = \alpha.x$ is $^\star\delta$-continuous mapping.

Proof. Suppose that $(X_{(F)},\tau)$ is a $^\star\delta$-topological vector space. Let K be any open set in the X containing $M_\alpha(x) = \alpha.x$. By hypothesis, there exists a $^\star\delta$-open set U in F containing α and V in X containing x such that $UV \subseteq K$. Then $M_\alpha(V) = \alpha V \subseteq UV \subseteq K$. It is proved that for every open set K containing $M_\alpha(x), \exists$ a $^\star\delta$-open set V in X containing x such that $M_\alpha(V) \subseteq K$. Hence M_α is $^\star\delta$-continuous mapping. □

Theorem 3.3. Let $(X_{(F)},\tau)$ be a $^\star\delta$-TVS. If U is open in X, then $U+x$ is a $^\star\delta$-open subset of X $\forall\, x \in X$.

Proof. Let $u+x \in U+x$ be arbitrary. Now U is an open set in X containing $u = u+x-x = T_{-x}(U+x)$. Since the translation map T_{-x} is $^\star\delta$-continuous, $\exists$ a $^\star\delta$-open set V containing $u+x$ such that $T_{-x}(V) \subseteq U$. That is, $V+(-x) \subseteq U$ and hence $V \subseteq U+x$. It is proved that for any point $u+x \in U+x$, there exists $^\star\delta$-open set V containing $u+x$ such that $u+x \in V \subseteq U+x$. therefore $u+x$ is $^\star\delta$-open subset of X, $\forall\, x \in X$. □

Theorem 3.4. Let $(X_{(F)},\tau)$ be a $^\star\delta$-TVS. If U is open in X, then $\alpha.U$ is a $^\star\delta$-open in X for any nonzero element $\alpha \in F$.

Proof. Let $x \in \alpha.U$ be arbitrary. Then $x = \alpha u$ for some $u \in U$. Now U is open set in the codomain X containing $u = \frac{1}{\alpha}(\alpha u) = M_{\frac{1}{\alpha}}(\alpha U) = M_{\frac{1}{\alpha}}(x)$. Since the multiplication mapping $M_{\frac{1}{\alpha}} : X \to X$ is $^\star\delta$-continuous, there exists a $^\star\delta$-open set V containing $\alpha u = x$ such that $M_{\frac{1}{\alpha}}(V) \subseteq U$. That is, $\frac{1}{\alpha}(V) \subseteq U$. Hence $V \subseteq \alpha U$. Therefore αu is $^\star\delta$-open subset of X for any non-zero element $\alpha \in F$. □

4. $^\star\delta$-Closure in a $^\star\delta$-TVS

Definition 4.1. The $^\star\delta$-interior of a subset A of X is the union of all regular*-open sets of X contained in A and is denoted by $int_{^\star\delta}(A)$.

Definition 4.2. A subset A of a topological space (X,τ) is called $^\star\delta$-open if $A = int_{^\star\delta}(A)$. i.e., a set is $^\star\delta$-open if it is the union of regular*-open sets. The complement of a $^\star\delta$-open is called $^\star\delta$-closed set in X.

Note 4.3. The $^\star\delta$-closure of a subset A of (X,τ) is denoted by $cl_{^\star\delta}(A)$.

Theorem 4.4. In a $^\star\delta$-TVS $(X_{(F)},\tau)$, a scalar multiple of a $^\star\delta$-closed set is $^\star\delta$-closed for any $\alpha \in F$.

Proof. Let U be any $^\star\delta$-closed subset of X and $\alpha \in F$ be arbitrary. $(\alpha U)^c = X \setminus \alpha U = \alpha(X \setminus U) = \alpha U^c$. Since U is $^\star\delta$-closed subset of X, U^c is $^\star\delta$-open subset of X. Since every $^\star\delta$-open set is open, U^c is an open subset of X. By Result 2.6, αU^c is a $^\star\delta$-open subset of X. Then $(\alpha U)^c$ is a $^\star\delta$-open. So αU is a $^\star\delta$-closed subset of X. □

Theorem 4.5. Let A be any closed subset of a $^\star\delta$-topological vector space $(X_{(F)},\tau)$. Then the following are true.
(i) $x+A$ is $^\star\delta$-closed for each $x \in X$
(ii) αA is a $^\star\delta$-closed for each non-zero scalar α in F.

Proof. (i) Let $y \in cl_{\star_\delta}(x+A)$. Now consider $z = -x+y$ and let K be any open set in X containing z. Then by definition of $^\star\delta$-topological vector space, there exists $^\star\delta$-open sets U and V in X such that $-x \in U, y \in V$ and $U+V \subseteq K$. Since $y \in cl_{\star_\delta}(x+A)$, $(x+A)\cap V \neq \phi$. Then there is $a \in (x+A)\cap V$. Now $-x+a \in A\cap(U+V) \subseteq A\cap K \Rightarrow A\cap K \neq \phi$ which implies $z \in cl(A) = A \Rightarrow y \in x+A$. Hence $cl_{\star_\delta}(x+A) \subseteq x+A$. Always $x+A \subseteq cl_{\star_\delta}(x+A)$. Thus $x+A = cl_{\star_\delta}(x+A)$. Hence $x+A$ is $^\star\delta$-closed in X.
(ii) Let $x \in cl_{\star_\delta}(\alpha A)$ and let K be any open neighborhood of $y = \frac{1}{\alpha}x$ in X. Since $(X_{(F)},\tau)$ is $^\star\delta$-TVS, $\exists^\star\delta$-open sets U in F containing $\frac{1}{\alpha}$ and V in X containing x such that $U.V \subseteq K$. By hypothesis, $(\alpha A)\cap V \neq \phi$. Therefore there is $a \in (\alpha A)\cap V$. Now $\frac{1}{\alpha}a \in A\cap(U.V) \subseteq A\cap K \Rightarrow A\cap K \neq \phi \Rightarrow y \in cl(A) = A \Rightarrow x \in \alpha A$. Then $cl_{\star_\delta}(\alpha A) \subseteq \alpha A$. Always $\alpha A \subseteq cl_{\star_\delta}(\alpha A)$. Hence $\alpha A = cl_{\star_\delta}(\alpha A)$. Thus αA is $^\star\delta$-closed. □

Theorem 4.6. Let $(X_{(F)},\tau)$ be a $^\star\delta$-TVS. If U is $^\star\delta$-open set in X, then there exists a $^\star\delta$-open set V in X containing 0 such that $u+V \subseteq U$ for all $u \in U$.

Proof. Let U be any $^\star\delta$-open set in $(X_{(F)},\tau)$. Since every $^\star\delta$-open set is open, U is an open subset of X. By Result 2.6, $U+x$ is $^\star\delta$-open set in X for all $x \in X$. In particular $U-u$ is a $^\star\delta$-open set in X containing 0 for all $u \in U$. By taking $V = U-u$, we get a $^\star\delta$-open set V containing 0 such that $u+V \subseteq U$. □

Theorem 4.7. Let S and T be any subsets of a $^\star\delta$-TVS $(X_{(F)},\tau)$, then $cl_{\star_\delta}(S)+cl_{\star_\delta}(T) \subseteq cl_{\star_\delta}(S+T)$.

Proof. Let $z \in cl_{\star_\delta}(S)+cl_{\star_\delta}(T)$ be arbitrary. Then $z = x+y$ where $x \in cl_{\star_\delta}(S)$ and $y \in cl_{\star_\delta}(T)$. Let K be any $^\star\delta$-open set in X containing $z = x+y$. Since every $^\star\delta$-open set is open, K is open in X containing z. Since X is a $^\star\delta$-TVS, the condition of $^\star\delta$-Topological vector space, there exists $^\star\delta$-open sets U in X containing x, V in X containing y such that $U+V \subseteq K$. Since $x \in cl_{\star_\delta}(S), y \in cl_{\star_\delta}(T)$, there are $a \in S\cap U$ and $b \in T\cap V$. Then $a+b \in (S+T)\cap(U+V) \subseteq (S+T)\cap K$. So, $K\cap(S+T) \neq \phi$. Therefore $z = x+y \in cl_{\star_\delta}(S+T)$. □

Theorem 4.8. Let $(X_{(F)},\tau)$ be a $^\star\delta$-TVS and let S, T be subsets of $(X_{(F)},\tau)$. If T is $^\star\delta$-open, then $S+T = cl_{\star_\delta}(S)+T$.

Proof. Let S and T be any two subsets of a $^\star\delta$-TVS X. Always, $S \subseteq cl_{\star_\delta}(S)$. So $S+T \subseteq cl_{\star_\delta}(S)+T$. Now let $y \in cl_{\star_\delta}(S)+T$ be arbitrary. Then $y = x+b$ where $x \in cl_{\star_\delta}(S)$ and $b \in T$. Since T is $^\star\delta$-open, by Theorem 4.6, $\exists$ a $^\star\delta$-open set V containing 0 such that $V+b \subseteq T$. Since V is $^\star\delta$-open in X containing zero and $-V$ is also a $^\star\delta$-open in X containing zero. Now $x+(-V)$ is $^\star\delta$-open set containing x. Since $x \in cl_{\star_\delta}(S)$, $S\cap(x-V) \neq \phi$. Choose $a \in S\cap(x-V)$. Then $a \in S$ and $a \in x-V$ or $x-a \in V$. Now $y = x+b = x+b-a+a = a+(x-a)+b \in a+V+b \subseteq S+T$. Hence $cl_{\star_\delta}(S)+T \subseteq S+T$. Therefore $S+T = cl_{\star_\delta}(S)+T$. □

5. $^\star\delta$-Homeomorphism in $^\star\delta$-TVS

In this section, it has been shown that translation and scalar multiplication mappings are $^\star\delta$-homeomorphism on a $^\star\delta$-TVS.

Definition 5.1. A bijective function f from a $^\star\delta$-TVS X to itself is called $^\star\delta$-homeomorphism if $f$and f^{-1} are $^\star\delta$-continuous on a $^\star\delta$-TVS X.

Definition 5.2. A TVS $(X_{(F)},\tau)$ is called as $^\star\delta$-homogeneous space, if for all $x,y \in X$, there is $^\star\delta$-homeomorphism f of the space X onto itself such that $f(x) = y$.

Theorem 5.3. Translation mapping on a $^\star\delta$-topological vector space is $^\star\delta$-homeomorphism.

Proof. Let $(X_{(F)},\tau)$ be a $^\star\delta$-TVS, $\forall\, x \in X$, translation mapping $T_x : X \to X$ is defined by $T_x(z) = z+x$ for all $z \in X$. Clearly, T_x is a bijective mapping for all $x \in X$. By Theorem 3.1, T_x is $^\star\delta$-continuous. Let U be any open set containing the point z, where z in X. By Theorem 3.3,$U+x = T_x(U)$ is $^\star\delta$-open in X. Therefore T_x is a $^\star\delta$-homeomorphism. □

Theorem 5.4. Multiplication mapping on a $^\star\delta$-TVS is $^\star\delta$-homeomorphism.

Proof. Let $(X_{(F)},\tau)$ be a $^\star\delta$-TVS and let the arbitrary scalar $\alpha \in F$. Multiplication mapping $M_\alpha : X \to X$ is $M_\alpha(x) = \alpha.x$ Obviously, it is a bijective mapping. By Theorem 3.2, M_α is $^\star\delta$-continuous for any $\alpha \in F$. Then $M_\alpha(U) = \alpha.U$ where U is any pen set in X. By Theorem 3.4, $\alpha.U$ is $^\star\delta$-open in X. Hence M_α is $^\star\delta$-homeomorphism. □

Theorem 5.5. $^\star\delta$-closure of a linear subspace of a $^\star\delta$-TVS is a $^\star\delta$-TVS.

Proof. Let $(X_{(F)},\tau)$ be a $^\star\delta$-TVS and H be any linear subspace of X. Then $H+H \subseteq H$ and $\alpha H \subseteq H$ for all $\alpha \in F$. So $cl_{\star_\delta}(H+H) \subseteq cl_{\star_\delta}(H)$ and $cl_{\star_\delta}(\alpha H) \subseteq cl_{\star_\delta}(H)$ for all $\alpha \in F$. By Theorem 4.7, $cl_{\star_\delta}(H)+cl_{\star_\delta}(H) \subseteq cl_{\star_\delta}(H+H) \subseteq cl_{\star_\delta}(H)$. Also since scalar multiplication is a $^\star\delta$-homeomorphism, by Theorem 4.4, it maps $^\star\delta$-closure of a set into $^\star\delta$-closure of its image. So $\alpha(cl_{\star_\delta}(H)) = cl_{\star_\delta}(\alpha H)$. □

Theorem 5.6. Every $^\star\delta$-TVS is $^\star\delta$-homogeneous space.

Proof. Let $(X_{(F)},\tau)$ be a $^\star\delta$-TVS. Take $x,y \in X$ and take $z = (-x)+y$. Define a translation map $T_z : X \to X$ by $T_z(x) = x+z\ \forall\ x \in X$. Then $T_z(x) = y$ for all $x \in X$. By Theorem 5.3,$T_z : X \to X$ is $^\star\delta$-homeomorphism. Hence $(X_{(F)},\tau)$ is an $^\star\delta$-homogeneous space. □

Theorem 5.7. $g : (X_{(F)},\tau_1) \to (X_{(F)},\tau_2)$ be a homeomorphism of $^\star\delta$-TVS. If g is $^\star\delta$-continuous at 0 in X then g is $^\star\delta$-continuous on X.

Proof. Let $x \in X$ be arbitrary. Suppose that K is $^\star\delta$-open set in Y containing $y = g(x)$. By Theorem 3.1, $T_y : Y \to Y$, defined by $T_y(x) = x+y$ for all $x \in Y$ is $^\star\delta$-continuous. Therefore there is a $^\star\delta$-open set V of 0 such that $T_y(V) = V + y \subseteq K$. Since g is $^\star\delta$-continuous at 0 in X, $\exists$ $^\star\delta$-open set U in X containing 0 such that $f(U) \subseteq V$. Since $T_x : X \to X$ is $^\star\delta$-homeomorphism, $U + x$ is $^\star\delta$-open set containing x. Then $f(U+x) = f(U) + f(x) = f(U) + y \subseteq V + y \subseteq K$. Therefore g is $^\star\delta$-continuous at x of X and hence on X. □

Theorem 5.8. In a $^\star\delta$-TVS $(X_{(F)}, \tau)$, every $^\star\delta$-open subspace of X is $^\star\delta$-closed.

Proof. Let Y be a $^\star\delta$-open subspace of X. By Theorem 3.1, $T_x : X \to X$ defined by $T_x(y) = x + y$ for all $y \in X$ is $^\star\delta$-homeomorphism. Therefore $Y + x$ is $^\star\delta$-open subset of X for all $x \in X$. Since arbitrary union of $^\star\delta$-open subsets is a $^\star\delta$-open, $Y = \bigcup_{x \in Y^c}(Y + x) = U(\text{say})$ is $^\star\delta$-open sunset of X. Now $Y = X \setminus U$ is $^\star\delta$-closed subsets of X. Hence every $^\star\delta$-open subspace of X is $^\star\delta$-closed in X. □

6. $^\star\delta$-Hausdorff and $^\star\delta$-Compact in $^\star\delta$-TVS

In this section, we defined $^\star\delta$-Hausdorff and $^\star\delta$-Compact spaces. Also we derive $^\star\delta$- topological vector space is a $^\star\delta$-Hausdorff and $^\star\delta$-Compact spaces.

Definition 6.1. *A* Topological space X is said to be $^\star\delta$-Hausdorff if for every $x \neq y \in X$, there exists a $^\star\delta$-open sets U_x, V_y such that $x \in U_x, y \in V_y$ and $U_x \cap V_y = \phi$.

Definition 6.2. *A* Topological space X is called $^\star\delta$-Compact if every cover of X by $^\star\delta$-open sets has finite subcover. A subset A of X is said to be $^\star\delta$-compact if every cover of A by $^\star\delta$-open sets of X has a finite subcover.

Theorem 6.3. Every $^\star\delta$- TVS $(X_{(F)}, \tau)$ is $^\star\delta$-Hausdorff

Proof. Let $a \in X, a \neq 0$. Since every singleton set in a $^\star\delta$-TVS is $^\star\delta$-closed, $\{a\}$ is $^\star\delta$-closed in X. Then $\{a\}^c = X \setminus \{a\} = U(\text{say})$ is $^\star\delta$-open set containing 0. By Result 2.7, $\exists$ a symmetric $^\star\delta$-open set V containing 0 such that $V + V \subseteq U$. Then by Result 2.8, $a + V = a - V$ is $^\star\delta$-open set. If $V \cap (a - V) \neq \phi$, then take $y \in V \cap (a - V)$. $y \in a - V \Rightarrow y = a - x$ for some $x \in V \Rightarrow x + y = a \Rightarrow a \in V + V$ as $x, y \in V \Rightarrow a \in U$ which is a contradiction. Therefore $V \cap (a - V) = \phi$. Hence the points 0 and $a \neq 0$ are separated by $^\star\delta$-open sets in X. Thus $(X_{(F)}, \tau)$ is $^\star\delta$-Hausdorff space. □

Theorem 6.4. Let A be $^\star\delta$- compact set in a $^\star\delta$-TVS $(X_{(F)}, \tau)$. Then $x + A$ is compact $\forall\, x \in X$.

Proof. Let A be $^\star\delta$-compact subset of $^\star\delta$-TVS X. Let $\{U_\alpha : \alpha \in I\}$ be a $^\star\delta$-open cover for $x + A$. Then $x + A \subseteq \bigcup_{\alpha \in I} U_\alpha$ which implies that $A \subseteq (-x) + \bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I}(-x + U_\alpha)$. Since U_α is $^\star\delta$-open subset of $^\star\delta$-topological vector space, $(-x + U_\alpha)$ is also $^\star\delta$-open subset of X for each $x \in X$. Since A is $^\star\delta$-compact, there exists a finite subset I_0 of I such that $A \subseteq \bigcup_{\alpha \in I_0}(-x + U_\alpha)$. This implies that $x + A \subseteq \bigcup_{\alpha \in I_0} U_\alpha$. Thus $x + A$ is compact. □

Theorem 6.5. Let $(X_{(F)}, \tau)$ be an $^\star\delta$-TVS . The scalar multiple of $^\star\delta$-compact set is $^\star\delta$-compact.

Proof. If $\lambda = 0$, we are nothing to prove. Assume that λ is non-zero. Let A be a $^\star\delta$-compact subset of X and let $\{U_\alpha : \alpha \in I\}$ be a $^\star\delta$-open cover of λA for some non-zero $\lambda \in F$, then $\lambda A \subseteq \bigcup_{\alpha \in I} U_\alpha$. Then $A \subseteq (\frac{1}{\lambda}) \bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I_0}((\frac{1}{\lambda})U_\alpha)$. Since U_α is $^\star\delta$-open subset of $^\star\delta$-topological vector space $(X_{(F)}, \tau)$, $(\frac{1}{\lambda})U_\alpha$ is $^\star\delta$-open subset of X for each $\alpha \in I$. Since A is $^\star\delta$-compact, there exists a finite subset I_0 of I such that $A \subseteq \bigcup_{\alpha \in I_0}((\frac{1}{\lambda})U_\alpha)$. this implies that $\lambda A \subseteq \bigcup_{\alpha \in I_0}(U_\alpha)$. Thus λA is $^\star\delta$-compact in $(X_{(F)}, \tau)$. □

Theorem 6.6. Let $(X_{(F)}, \tau)$ be an $^\star\delta$-TVS . If K is a $^\star\delta$-compact set of X and G is $^\star\delta$-closed subset of X such that $K \cap G = \phi$, then $\exists$ a $^\star\delta$-open set U containing 0 such that $(K + U) \cap (G + U) = \phi$.

Proof. If $K = \phi$, then the proof is trivial. Otherwise, let $0 = x \in K$, where K is $^\star\delta$-compact. Given that G is $^\star\delta$-closed set. So G^c is an $^\star\delta$-open subset of X containing $0 = x$. Since the addition mapping is $^\star\delta$-continuous and $0 = 0 + 0 + 0$, therefore there is an $^\star\delta$-open set U containing 0 satisfy $3U = U + U + U \subset G^c$. Define $U_x = U \cap (-U)$ which is $^\star\delta$-open set, symmetric and $3U_x = U_x + U_x + U_x \subset G^c$. Hence $\{x + x + x, x \in U_x\} \cap G = \phi$. Since U_x is symmetric, $(x + U_x + U_x) \cap (G + U_x) = \phi$. By hypothesis, for each $x \in K$ and K is $^\star\delta$-compact, then by the above argument, we have a symmetric $^\star\delta$-open set V_x such that $(x + 2V_x) \cap (G + V_x) = \phi$. The sets $\{V_x : x \in K\}$ are a $^\star\delta$-open that covers K and since K is $^\star\delta$-compact, for finitely number of points $x_i \in K$ where $i = 1, 2, ...n$, we have $K \subset \bigcup_{\text{i=1,2,... n}}(x_i + V_{x_i})$. Define the $^\star\delta$-open set containing 0 by $V = \bigcap_{\text{i=1,2,... n}} V_{x_i}$. Therefore $(K + V) \cap (G + V) \subset \bigcup_{\text{i=1,2,... n}}(x_i + V_{xi} + V) \cap (G + V) \subset \bigcup_{\text{i=1,2,... n}}(x_i + 2V_{xi}) \cap (G + V_{xi}) = \phi$. Hence $(K + U) \cap (G + U) = \phi$. □

Lemma 6.7. Let $(X_{(F)}, \tau)$ be a $^\star\delta$-TVS, let U be $^\star\delta$-open subset of X. If A is any subset of X such that $U \cap A = \phi$ then $U \cap cl_{^\star\delta}(A) = \phi$

Proof. Suppose $U \cap cl_{^\star\delta}(A) \neq \phi$. Let $x \in U \cap cl_{^\star\delta}(A) = \phi$. Then $x \in cl_{^\star\delta}(A)$ and $x \in U$. Since U is $^\star\delta$-open subset of X, $X - U$ is $^\star\delta$-closed subset that contain A. Therefore $cl_{^\star\delta}(A) \subseteq X - U$, so $x \notin cl_{^\star\delta}(A)$ which implies a contradiction. Hence $U \cap cl_{^\star\delta}(A) = \phi$. □

Corollary 6.8. Let $(X_{(F)}, \tau)$ be $^\star\delta$-TVS. If $^\star\delta$-closed set G and $^\star\delta$-compact set K are disjoint then there is $^\star\delta$-open set U containing 0 such that $cl_{^\star\delta}(K + U) \cap (G + U) = \phi$.

Proof. Given that G is $^\star\delta$-closed and K is $^\star\delta$-compact and $G \cap K = \phi$. By Theorem 6.6, there exists $^\star\delta$-open set U containing 0 satisfy $(K + U) \cap (G + U) = \phi$. The set $G + U = \{y + U : y \in G\}$ is an $^\star\delta$-open set then by Lemma 6.7, $cl_{^\star\delta}(K + U) \cap (G + U) = \phi$. □

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