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A study on Ss-Semilocal modules in view of singularity

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Abstract. In this paper, we define weakly δ_{ss}- supplemented modules and give a characterization for them named with δ_{ss}-semilocal modules. In particular, we determine the suitable conditions for a δ_{ss}- semilocal module to be δ-semilocal and ss-semilocal, respectively. In addition to these we supply contrast examples pointing the relations are proper between these classes of modules.

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Contents

1. Introduction and Background

Firstly, note that throughout this study the symbols R and M will denote an associative ring with identity and a unitary left R-module, respectively. The notations $A \leq M$ and $A \leq_{\oplus} M$ will indicate that A is a submodule of M and A is a direct summand of M. A submodule A of M is called *essential* (denoted by $A \trianglelefteq M$) if $A \cap K \neq \{0\}$ for any proper submodule K of M except for $\{0\}$. The intersection of all essential submodules of a module M is denoted by $Soc(M)$ which is the largest semisimple submodule of M. A submodule $B^{'} \leq M$ is called a *complement* of A in M if it is maximal in the set of submodules $B \leq M$ with $A \cap B = \{0\}$. A submodule A of M is called *small* (denoted by $A \ll M$) if $A + K \neq M$ for any proper submodule K of M. The sum of all small submodules of a module M is denoted by $Rad(M)$. A *(weak) supplement submodule* T of A in M is a submodule such that $A + T = M$ and $A \cap T \ll T$ ($A \cap T \ll M$). A module M is called *(weakly) supplemented* if every submodule of M has a (weak) supplement in M [14].

In [15] and [6], the authors updated the small and supplemented modules via singularity as follows. A submodule $A \leq M$ is δ-small if and only if for all submodules $X \leq M$: if $A + X = M$, then $M = Y \oplus X$ for a projective semisimple submodule Y of A. Also the submodule A is called δ-*small* in M if $A + K \neq M$ for every proper submodule K of M with $\frac{M}{K}$ is singular (denoted by $A \ll_{\delta} M$) and the sum of all δ-small submodules of M denoted by $\delta(M)$. Clearly $Rad(M) \leq \delta(M)$.

A δ-supplement submodule T of A in M is a submodule such that $A + T = M$ and $A \cap T \ll_{\delta} T$. A *(generalized) weak δ-supplement submodule* T of A in M is a submodule such that $A + T = M$ and $(A \cap T \leq$

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$\delta(M)$) $A \cap T \ll_\delta M$ [11]. The module M is called *(weakly) δ-supplemented,* if every submodule of M has a (weak) δ-supplement in M. Clearly every (weakly) supplemented module is (weakly) δ-supplemented. In [5], the authors introduced ss-supplemented modules which are stronger than supplemented modules. A module M is called *ss-supplemented* if for every submodule A of M there exists a submodule T of M such that $A+T = M$ and $A \cap T \leq Soc_s(T)$ where $Soc_s(T) = Soc(T) \cap Rad(T)$. In [9], the authors generalized ss-supplemented modules to *weakly ss-supplemented modules* by taking $Soc_s(M)$ instead of $Soc_s(T)$ and gave a characterization for these modules named with ss-semilocal. In [13] the authors generalized (amply) ss-supplemented modules in view of singularity and introduced *(amply) δ_{ss}-supplemented modules and δ_{ss}-supplemented rings*.

In this article, in the light of the given studies we define weakly δ_{ss}-supplemented modules and obtain a new characterization for them named with *δ_{ss}-semilocal modules. A* module M is called δ_{ss}-semilocal whenever $\frac{M}{Soc_\delta(M)}$ is semisimple where $Soc_\delta(M) = Soc(M) \cap \delta(M)$. A module M is called δ-semilocal if $\frac{M}{\delta(M)}$ is semisimple. As $Soc_\delta(M) \leq \delta(M) \leq M$, every δ_{ss}-semilocal module is δ-semilocal and every ss-semilocal module is δ_{ss}-semilocal. We give examples on the converse implications might not be true. Also, we investigate suitable conditions when δ_{ss}semilocal modules are δ-semilocal and ss-semilocal. In particular, we obtain new characterizations for δ_{ss}-semilocal rings.

For undefined algebraic structures used here, such as δ-(semi)perfect and δ_{ss}-perfect rings, we refer to [15] and [13], respectively.

2. Weakly δ_{ss}-supplemented modules

A module M is called weakly δ-supplemented if for any submodule A of M there exists a submodule T of M such that $A + T = M$ and $A \cap T \ll_\delta M$ [11]. By means of this concept and the usuful lemma given in the following we will define weakly δ_{ss}-supplemented modules as a strongly version of weakly δ-supplemented modules.

Lemma 2.1. *Let $f : A \longrightarrow B$ be a module homomorphism. Then $f(Soc_\delta(A)) \leq Soc_\delta(B)$. In particular, we have $Soc_\delta(A) \leq Soc_\delta(B)$ whenever $A \leq B$.*

Proof. As f is a homomorphism we have $f(Soc(A)) \leq Soc(B)$ and $f(\delta(A)) \leq \delta(B)$. Therefore we get $f(Soc_\delta(A)) = f(Soc(A) \cap \delta(A)) \leq f(Soc(A)) \cap f(\delta(A)) \leq Soc(B) \cap \delta(B) = Soc_\delta(B)$. In particular, if the inclusion map from A to B is taken instead of f, then $Soc_\delta(A) \leq Soc_\delta(B)$ is obtained clearly. ■

Definition 2.2. *A module M is called weakly δ_{ss}-supplemented if for any submodule A of M there exists a submodule T of M such that $A + T = M$ and $A \cap T \leq Soc_\delta(M)$.*

It is a clear fact that every weakly δ_{ss}-supplemented module is weakly δ-supplemented but not vice versa. To verify this with an example we need the following lemma.

Lemma 2.3. *Let M be a weakly δ-supplemented module with $Soc(M) = 0$. Then $M = 0$.*

Proof. Let $A \leq M$. By hypothesis there exists a submodule T of M such that $A + T = M$ and $A \cap T \leq Soc_\delta(M)$. Since $Soc_\delta(M) = Soc(M) \cap \delta(M) = 0 \cap \delta(M) = 0$ then we have $A \cap T = \{0\}$. Therefore, M is a semisimple module as each submodule is a direct summand. Hence $M = Soc(M) = 0$. ■

Example 2.4. *It is a known fact that $\mathbb{Z}$-module $\mathbb{Q}$ is weakly δ-supplemented as it is weakly supplemented [3, 17.15 Example, 213p.]. On the orher hand, it is not weakly δ_{ss}-supplemented by Lemma 2.3.*

Now we give a characterization lemma for weak δ_{ss}-supplement submodules of a module.

Lemma 2.5. *Let M be a module and $A, T \leq M$. Then the following implications are equivalent:*

1. $M = A + T$ and $A \cap T \leq Soc_\delta(M)$.

2. T is a weak δ-supplement of A in M and $A \cap T$ is semisimple.
3. T is a generalized weak δ-supplement of A in M and $A \cap T$ is semisimple.

Proof. $(1) \Rightarrow (2)$: By hypothesis we have $M = A + T$, $A \cap T \leq \delta(M)$ and $A \cap T \leq Soc(M)$ as $Soc_\delta(M)$ is a submodule of both $\delta(M)$ and $Soc(M)$. Therefore, $A \cap T$ is semisimple and it is also δ-small in M by [13, Lemma 2.2].

$(2) \Rightarrow (3)$: It is clear.

$(3) \Rightarrow (1)$: By hypothesis we have $M = A + T$, $A \cap T \leq \delta(M)$ and $A \cap T$ is semisimple. Thus, $A \cap T \leq Soc(M)$. Hence, $A \cap T \leq Soc(M) \cap \delta(M) = Soc_\delta(M)$. ■

We say that a module M is called δ-semilocal if $\frac{M}{\delta(M)}$ is semisimple. And it is proven in [10, Theorem 3.7] that a module M with $\delta(M) \ll_\delta M$ and $\frac{M}{\delta(M)}$ is singular is δ-semilocal if and only if M is a generalized weakly δ-supplemented module. Motivated by this we give a similar characterization for our modules in the following theorem.

Theorem 2.6. *The following implications are equivalent for a module M:*

1. $\frac{M}{Soc_\delta(M)}$ is semisimple.
2. M is weakly δ_{ss}-supplemented.
3. M is a direct sum of two submodules M_1 and M_2 such that M_1 and $\frac{M_2}{Soc_\delta(M)}$ are semisimple, also $Soc_\delta(M) \trianglelefteq M_2$.

Proof. $(3) \Rightarrow (1)$: Let $M = M_1 \oplus M_2$. Then $\frac{M}{Soc_\delta(M)} = \frac{M_1 + Soc_\delta(M)}{Soc_\delta(M)} \oplus \frac{M_2}{Soc_\delta(M)}$ is semisimple as a direct sum of two semisimple modules.

$(1) \Rightarrow (2)$: For any $A \leq M$, $\frac{A + Soc_\delta(M)}{Soc_\delta(M)} \oplus \frac{T}{Soc_\delta(M)} = \frac{M}{Soc_\delta(M)}$ can be written by hypothesis. Then, $M = A + T$ and by modularity $(A + Soc_\delta(M)) \cap T = (A \cap T) + Soc_\delta(M) = Soc_\delta(M)$ are obtained. Thus, $A \cap T \leq Soc_\delta(M)$ is got.

$(1) \Rightarrow (3)$: Let M_1 be a complement of $Soc_\delta(M)$. Then, $M_1 \cong \frac{M_1 + Soc_\delta(M)}{Soc_\delta(M)} \leq_\oplus \frac{M}{Soc_\delta(M)}$ and so M_1 is semisimple as it is isomorphic to a submodule of a semisimple module. Additionally, there exists a semisimple direct summand $\frac{M_2}{Soc_\delta(M)}$ satisfying $\frac{M_1 + Soc_\delta(M)}{Soc_\delta(M)} \oplus \frac{M_2}{Soc_\delta(M)} = \frac{M}{Soc_\delta(M)}$. Clearly, $M = M_1 + M_2$. Furthermore, since $Soc_\delta(M) = (M_1 + Soc_\delta(M)) \cap M_2 = Soc_\delta(M) \oplus (M_1 \cap M_2)$ by modularity. Then we get $M_1 \cap M_2 \leq Soc_\delta(M)$ and $M_1 \cap M_2 \leq M_1$ which means $M_1 \cap M_2 \leq M_1 \cap Soc_\delta(M) = 0$ by the property of a complement. Thus $M = M_1 \oplus M_2$. For the remaining part of the proof let us show that $Soc_\delta(M) \trianglelefteq M_2$. As M_1 is the complement of $Soc_\delta(M)$ we have $M_1 \oplus Soc_\delta(M) \trianglelefteq M = M_1 \oplus M_2$ [3, 1.11(1)]. For the second injection map $i_2 : M_2 \longrightarrow M_1 \oplus M_2$, $i_2^{-1}(M_1 \oplus Soc_\delta(M)) \trianglelefteq M_2$ by [1, Theorem 9.1(3)].

$2 \Rightarrow 1$: For any $\frac{A}{Soc_\delta(M)} \leq \frac{M}{Soc_\delta(M)}$ we have $A + T = M$ and $A \cap T \leq Soc_\delta(M)$ for a submodule $T \leq M$ by hypothesis. Thus $\frac{A}{Soc_\delta(M)} \oplus \frac{T+M}{Soc_\delta(M)} = \frac{M}{Soc_\delta(M)}$, that is, $\frac{M}{Soc_\delta(M)}$ is semisimple. ■

From now on, we will call a module M is δ_{ss}-semilocal whenever M satisfies one of the equvalent conditions of the theorem given above.

3. δ_{ss}-Semilocal modules

In this part we will present the fundamental properties of our modules firstly. Before of all we need a useful lemma.

Lemma 3.1. *For a given family of R-modules $\{M_i\}_{i \in I}$, $Soc_\delta(\oplus_{i \in I} M_i) = \oplus_{i \in I} Soc_\delta(M_i)$.*

Proof. It is clear by Lemma 2.1 and [3, 6.2(3)]. ■

Theorem 3.2. *Let $\{M_i\}_{i\in I}$ be a family of δ_{ss}-semilocal modules. Then $M = \oplus_{i\in I} M_i$ is δ_{ss}-semilocal.*

Proof. As each $\frac{M_i}{Soc_\delta(M_i)}$ is semisimple, $\frac{M}{Soc_\delta(M)} = \frac{\oplus_{i\in I} M_i}{Soc_\delta(\oplus_{i\in I} M_i)} = \frac{\oplus_{i\in I} M_i}{\oplus_{i\in I} Soc_\delta(M_i)} \cong \oplus_{i\in I} \frac{M_i}{Soc_\delta(M_i)}$ is also semisimple by [4, Cor. 8.1.5] and Lemma 3.1. Hence, M is is δ_{ss}-semilocal. ■

Corollary 3.3. *The sum of δ_{ss}-semilocal modules is also δ_{ss}-semilocal.*

Theorem 3.4. *If M is a δ_{ss}-semilocal module, then so is any homomorphic image.*

Proof. Let us consider the module epimorphism $h : M \longrightarrow K$ where M is δ_{ss}-semilocal. Then the homomorphism $\overline{h} : \frac{M}{Soc_\delta(M)} \longrightarrow \frac{K}{Soc_\delta(K)}$ defined by $\overline{h}(x + Soc_\delta(M)) = h(x) + Soc_\delta(K)$ for every $x + Soc_\delta(M) \in \frac{M}{Soc_\delta(M)}$ is epic. As $\frac{M}{Soc_\delta(M)}$is semisimple, then the homomorphic image $\frac{K}{Soc_\delta(K)}$ is also semisimple by [4, Cor. 8.1.5], that is $h(M) = K$ is δ_{ss}-semilocal. ■

Proposition 3.5. *Let M be a δ_{ss}-semilocal module and A be a submodule of M satisfying $\delta(A) = A \cap \delta(M)$. Then A is δ_{ss}-semilocal.*

Proof. Let $B \leq A$. Then there exists a submodule T of M such that $B + T = M$ and $B \cap T \leq Soc_\delta(M)$. Following this $A = (B + T) \cap A = B + (T \cap A)$ is obtained by using modularity. Now we will verify that $T \cap A$ is a weak δ_{ss}-supplement of B in A. As $B \cap (T \cap A) = B \cap T \leq Soc_\delta(M) \leq \delta(M)$ we have $B \cap (T \cap A) \leq \delta(M) \cap A = \delta(A)$. Thus, $B \cap T = B \cap (T \cap A) \leq Soc(A) \cap \delta(A) = Soc_\delta(A)$. Hence, A is δ_{ss}-semilocal. ■

Corollary 3.6. *Every δ_{ss}-supplement (and so δ-supplement) submodule of a δ_{ss}-semilocal module is δ_{ss}-semilocal.*

Recall that a module K is said to be M-generated, if there exists an epimorphism from $M^{(I)}$ to K where I is an index set.

Lemma 3.7. *Let M be a module. M is δ_{ss}-semilocal if and only if every M-generated module is δ_{ss}-semilocal.*

Proof. $(\Longrightarrow)$: It is clear by Corollary 3.3 and Theorem 3.4.
$(\Longleftarrow)$: It is clear. ■

In general, every amply δ_{ss}-supplemented module is δ_{ss}-supplemented [13]. now it is possible to think whether the analogous idea is valid for our modules. In the following proposition we show that δ_{ss}-semilocal modules already contain this property by themselves.

Proposition 3.8. *Let M be a δ_{ss}-semilocal module and $A, T \leq M$ with $A + T = M$. Then A has a weak δ_{ss}-supplement in M contained by T.*

Proof. As $A \cap T \leq M$, there is a submodule $B \leq M$ such that $(A \cap T) + B = M$ and $(A \cap T) \cap B \leq Soc_\delta(M)$ by hypothesis. By modularity, we have $T = T \cap M = T \cap [(A \cap T) + B] = (A \cap T) + (B \cap T)$. Thus, $M = A + T = A + (A \cap T) + (B \cap T) = A + (B \cap T)$ and $A \cap (B \cap T) = (A \cap B) \cap T \leq Soc_\delta(M)$. Hence, $B \cap T$ is a weak δ_{ss}-supplement of A in M contained by T. ■

As we pointed before every δ-semilocal module is δ_{ss}-semilocal. Under suitable conditions the converse might be provided as follows.

Proposition 3.9. *Let M be a δ-semilocal module with $\delta(M) \leq Soc(M)$. Then M is δ_{ss}-semilocal.*

Proof. Clearly, $Soc_\delta(M) = \delta(M)$ as $\delta(M) \leq Soc(M)$. Therefore, $\frac{M}{\delta(M)} = \frac{M}{Soc_\delta(M)}$ is semisimple. Hence, M is δ_{ss}-semilocal by Theorem 2.6. ■

Due to the consequences of the proposition given in the following we will obtain the ring characterization of δ_{ss}-semilocal modules in the next.

Proposition 3.10. *Let M be a δ_{ss}-semilocal module and $A \ll_\delta M$. Then $A \leq Soc_\delta(M)$.*

Proof. By hypothesis there exists a submodule T of M such that $A + T = M$ and $A \cap T \leq Soc_\delta(M)$. As $A \ll_\delta M$ we have $Y \oplus T = M$ for a projective semisimple submodule Y of A. From modularity we get $Y \oplus (T \cap A) = A$ and so A is semisimple as a direct sum of two semisimple modules. Hence $A \leq Soc(M) \cap \delta(M) = Soc_\delta(M)$. ■

Corollary 3.11. *Let M be a δ_{ss}-semilocal module and $\delta(M) \ll_\delta M$. Then $\delta(M) \leq Soc(M)$.*

As finitely generated modules have δ-small δ-radical we have the following corollary.

Corollary 3.12. *Let M be a finitely generated module. Then M is δ_{ss}-semilocal if and only if M is δ-semilocal and $\delta(M)$ is semisimple.*

Proof. $(\Longrightarrow)$: By hypothesis M is weakly δ_{ss}-supplemented and so it is weakly δ-supplemented. hence it can be shown that M is δ-semilocal by the similar way from [7, Prop. 2.1]. Also $\delta(M) \leq Soc(M)$ by Corollary 3.11 as $\delta(M) \ll_\delta M$.

$(\Longleftarrow)$: Let M be δ-semilocal with a semisimple δ-radical. Then $\delta(M) \leq Soc(M)$. Hence M is δ_{ss}-semilocal from Proposition 3.9. ■

Definition 3.13. *A module M is called weakly δ-radical δ-supplemented if every submodule of M containing $\delta(M)$ has a weak δ-supplement in M.*

Theorem 3.14. *Let M be a module with $\delta(M) \ll_\delta M$. Then the statements given in the following are equivalent:*

1. M is δ_{ss}-semilocal
2. M is δ-semilocal and $\delta(M)$ has a weak δ_{ss}-supplement in M.
3. M is δ-semilocal and $\delta(M) \leq Soc(M)$.
4. M is weakly δ-supplemented and $\delta(M) \leq Soc(M)$.
5. M is weakly δ-radical supplemented and $\delta(M) \leq Soc(M)$.

Proof. $(1) \Rightarrow (2)$: It is clear.

$(2) \Rightarrow (3)$: Let T be a weak δ-supplement of $\delta(M)$ in M. Then $\delta(M) + T = M$ and $\delta(M) \cap T \leq Soc_\delta(M) \leq Soc(M)$ and so $\delta(M) \cap T$ is semisimple. As $\delta(M) \ll_\delta (M)$ and $\delta(M) + T = M$ we have $M = Y \oplus T$ for a projective semisimple submodule Y of $\delta(M)$. By modularity we get $\delta(M) = Y \oplus (\delta(M) \cap T)$ and so $\delta(M)$ is semisimple by [4, Cor. 8.1.5]. Thus, $\delta(M) \leq Soc(M)$.

$(3) \Rightarrow (4)$: By hypothesis, for any $A \leq M$ there is a submodule $T \leq M$ such that $A + T = M$, $A \cap T \leq \delta(M)$ and $A \cap T$ is semisimple. Hence M is weakly δ-radical supplemented as $\delta(M) \ll_\delta M$.

$(4) \Rightarrow (5)$: It is clear.

$(5) \Rightarrow (1)$: For any $A \leq M$, $A \leq A + \delta(M)$ and so, there exists $T \leq M$ such that $[A + \delta(M)] + T = M$, $[A + \delta(M)] \cap T \ll_\delta M$. Following that $[A + \delta(M)] \cap T \leq \delta(M) \leq Soc(M)$. As $\delta(M) \ll_\delta M$, we have $P \oplus [A + T] = M$ for a projective semisimple submodule P of $\delta(M)$. Therefore, we get $A + (P \oplus T) = M$ and $A \cap (P \oplus T) \leq [P \cap (A + T)] + [T \cap (A + P)]$ where $P \cap (A + T)$ is δ-small and semisimple in M as P is projective semisimple and, $T \cap (A + P)$ is δ-small and semisimple in M as a submodule of $T \cap (A + \delta(P))$. ■

It is a clear fact that every δ_{ss}-semilocal module is weakly δ-supplemented but not vice versa. In the following example this is verified via Theorem 3.14.

Example 3.15. *Let us consider the $\mathbb{Z}$-module $\mathbb{Z}_8$. As it is local, it is supplemented and so δ-supplemented. Therefore, $\mathbb{Z}$-module $\mathbb{Z}_8$ is weakly δ-supplemented. On the other hand, since $\delta(\mathbb{Z}_8) = Rad(\mathbb{Z}_8) = 2\mathbb{Z}_8 \ll_\delta \mathbb{Z}_8$ and $Soc(\mathbb{Z}_8) = 4\mathbb{Z}_8$, $\mathbb{Z}_8$ is not a δ_{ss}-semilocal $\mathbb{Z}$-module by Theorem 3.14.*

Now we give a ring characterization theorem for δ_{ss}-perfect rings to be δ_{ss}-semilocal.

Corollary 3.16. *The following statements are equivalent for a ring R.*

1. ${}_RR$ is δ_{ss}-semilocal.
2. ${}_RR$ is δ-semilocal and $\delta(R) \leq Soc(R)$.
3. ${}_RR$ is δ_{ss}-perfect (δ_{ss}-supplemented).

Proof. $(1) \Longleftrightarrow (2)$: It is clear by Corollary 3.12

$(2) \Longrightarrow (3)$: As a ring R with unit is locally projective [8], $Soc(R) \ll_\delta R$ is got from [13, Prop. 5.2]. Thus, $\delta(R) = Soc(R)$ is obtained. Since $\frac{R}{\delta(R)} = \frac{R}{Soc(R)}$ and $Soc(R)$ is semisimple Artinian by hypothesis, then R is also Artinian and so it is δ-supplemented. Therefore R is δ-semiperfect by [6, Theorem 3.3]. Hence, R is δ_{ss}-perfect by [13, Theorem 5.3].

$(3) \Longrightarrow (1)$: Let R be a δ_{ss}-perfect ring. Then by [13, Theorem 5.3 (2)] R is δ-semiperfect and $\delta(R) = Soc(R)$. Therefore $Soc_\delta(R) = \delta(R)$ and so $\frac{R}{Soc_\delta(R)} = \frac{R}{Soc(R)}$ is semisimple by [15, Theorem 3.6]. Hence ${}_RR$ is δ_{ss}-semilocal. ■

Owing to the following we will construct rings whose modules are δ_{ss}-semilocal. In addition to this a proper class of δ-perfect rings is obtained. It will be verified via Example 3.18.

Theorem 3.17. *The following statements are equivalent for a ring R:*

1. ${}_RR$ is δ_{ss}-semilocal.
2. Every R-module is δ_{ss}-semilocal.
3. R is δ-semilocal and $\delta(R) \leq Soc(R)$.

Proof. $(1) \Longrightarrow (2)$: Let M be an R-module. Since each R-module is R-generated, then there exists an epimorphism $h : R^{(I)} \longrightarrow M$. By hypothesis M is δ_{ss}-semilocal by Lemma 3.7.

$(2) \Longrightarrow (3)$: By hypothesis ${}_RR$ is δ_{ss}-semilocal. Then the proof is clear from Corollary 3.16.

$(3) \Longrightarrow (1)$:It is clear by Corollary 3.16. ■

Example 3.18. *Let $\mathcal{F}$ be a field, $I = \begin{pmatrix} \mathcal{F} & \mathcal{F} \\ 0 & \mathcal{F} \end{pmatrix}$ and $R = \{(x_1, x_2, ..., x_n, x, x...) : n \in \mathbb{N}, x_i \in M_2(\mathcal{F}), x \in I\}$ be a ring with component-wise operations. Then, $Soc(R) = \{(x_1, x_2, ..., x_n, 0, 0...) : n \in \mathbb{N}, x_i \in M_2(\mathcal{F})\}$ and $\delta(R) = \{(x_1, x_2, ..., x_n, x, x...) : n \in \mathbb{N}, x_i \in M_2(\mathcal{F}), x \in J = \begin{pmatrix} 0 & \mathcal{F} \\ 0 & 0 \end{pmatrix}$. From [15, Example 4.3] it can be seen that R is a δ-perfect ring. But as $\delta(R) \neq Soc(R)$, R is not a δ_{ss}-semilocal ring by [13, Proposition 5.2].*

Every ss-semilocal module is δ_{ss}-semilocal. Now we investigate the suitable conditions satisfying the vice versa inspired by [2, Prop. 4.2].

Proposition 3.19. *Let M be a projective, semilocal and δ_{ss}-semilocal module with $Rad(M) \ll M$. Then M is ss-semilocal.*

Proof. As $Soc(M)$ is semisimple, the submodule $Soc_\delta(M)$ is a direct summand of $Soc(M)$. Then for a submodule X of M it can be written that $Soc(M) = Soc_s(M) \oplus X$. Besides there exists a submodule Y of M such that $M = X + Y$ and $X \cap Y \ll M$ since M is semilocal. Clearly, $X \cap Y \leq Rad(M)$. Following this we have $X \cap Y \leq X \cap Rad(M) = [X \cap Soc(M)] \cap Rad(M) = X \cap [Soc(M) \cap Rad(M)] = X \cap Soc_s(M) = 0$. Also we get $Rad(M) = Rad(X) \oplus Rad(Y) = Rad(Y)$ as X is semisimple. Here Y is projective as a direct summand of the projective module M. Now let us show that $\delta(Y) = Rad(Y)$. For thiswe have to verify that Y has no simple projective direct summand [12, Prop. 2.4]. Assume that S is a simple projective direct summand of Y. Then $Y = S \oplus K$ for $K \leq Y$. Therefore, $S \ll_\delta S \leq Y$ and so $S \leq Soc_\delta(Y) \leq Soc(Y)$ because S is semisimple projective. By modularity, $Soc(Y) = Soc(M) \cap Y = [Soc_s(M) \oplus X] \cap Y = [(Soc(M) \cap Rad(M)) \oplus X] \cap Y = [(Soc(M) \cap Rad(Y)) \oplus X] \cap Y =$

$[Soc(M) \cap Rad(Y)) \oplus (X \cap Y) = Soc(M) \cap Rad(Y) \leq Rad(Y)$ is got and using this $S \leq Soc(Y) \leq Rad(Y) = Rad(M) \ll M$ is obtained. As $Y \leq_\oplus M$, S is also small in Y and so this creates the contradiction $K = Y$. According to this it must be true that $\delta(Y) = Rad(Y)$. However, Y is also δ_{ss}-semilocal by Theorem 3.4 as M is δ_{ss}-semilocal. Then for any $U \leq Y$ there is a submodule V of Y such that $U + V = Y$ and $U \cap V \leq Soc_\delta(Y)$. From this fact $U \cap V \leq \delta(Y) = Rad(Y)$ and so $U \cap V \ll Y$. Thus, $U \cap V \leq Soc_s(Y)$. Hence, Y is an ss-semilocal module. By taking into account that X is an ss-semilocal by [9, Corollary 2.13]. ■

Corollary 3.20. *The following statements are equivalent for a ring R:*

1. ${}_RR$ is δ_{ss}-semilocal.
2. R is left δ_{ss}-perfect and semilocal.
3. R is left δ_{ss}-perfect and $\frac{Soc({}_RR)}{Soc_s({}_RR)}$ is finitely generated.

Proof. $(1) \Leftrightarrow (2)$: It is clear by Proposition 3.19 and Corollary 3.16.

$(2) \Leftrightarrow (3)$: It is clear by [13, Corollary 5.10] ■

Example 3.21. *Let $\mathcal{F}_i = \mathcal{Z}_2$ and $Q = \prod_{i=1}^{\infty} \mathcal{F}_i$. Then Q is a regular $(Rad(R) = 0)$ commutative ring with unity via component-wise operations. Let R be the subring of Q generated by $\bigoplus_{i=1}^{\infty} \mathcal{F}_i$ and 1_Q. Then it can be seen that $\delta(R) = Soc(R) = \bigoplus_{i=1}^{\infty} \mathcal{F}_i$. Since $\frac{R}{Soc_\delta(R)} \cong \mathcal{F}_i$ is simple then ${}_RR$ is δ_{ss}-semilocal. On the other hand, R is not a semilocal ring as $\frac{R}{Rad(R)} \cong R$ is not semisimple. Hence, R is not an ss-semilocal ring.*

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