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Oscillatory properties of third-order delay difference neutral equations

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Abstract
The aim of article is to investigate oscillatory manner for remediation of thirdorder linear delay difference neutral equation term

$$\Delta(c_2(t)\Delta(c_1(t)\Delta y(t))) + p(t)x(t-\sigma) = 0, \quad t \geq t_0 > 0$$

here $y(t) = x(t) + q(t)x(t-\xi)$. By using comparability concepts with related 1^{st} and 2^{nd} order difference delay inequality. Examples are given to major outcomes.

Keywords
Linear difference equation, delay, third-order.

AMS Subject Classification
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1. Introduction

In research, considered with oscillation for the third order linear delay difference neutral equation term

$$\Delta(c_2(t)\Delta(c_1(t)\Delta y(t))) + p(t)x(t-\sigma) = 0, \quad t \geq t_0 > 0 \tag{1.1}$$

here $y(t) = x(t) + q(t)x(t-\xi)$. Create following presumption:

(LH_1) : $c_1(t)$ and $c_2(t)$ sequences for non-negative integers;

(LH_2) : $p(t)$ and $q(t)$ are the positive real sequences such that $q(t) \geq q_0 > 1$ and $p(t) \neq 0$;

(LH_3) : σ, ξ are positive integers, such that $\sigma > \xi$

(LH_4) : $t + \xi - \sigma \leq t$ and $(t + \xi - \sigma) \geq (t - \sigma)$

Specify operators

$$E_0 y = y, \quad E_1 y = c_1 \Delta y,$$
$$E_2 y = c_2 \Delta(c_1(\Delta y)), \quad E_3 y = \Delta(c_2 \Delta(c_1(\Delta y)))$$

and assuming that $E_3 y$ for non canonical , (ie)

$$\sum_{s=t_0}^{\infty} \frac{1}{c_1(s)} < \infty \text{ and } \sum_{s=t_0}^{\infty} \frac{1}{c_2(s)} < \infty \tag{1.2}$$

By remediation for (1.1), real sequence $\{x(t)\}$ explained for $t \geq t_0$ and satisfy this (1.1). We taken single remediation $\{x(t)\}$ for (1.1) satisfy this $\sup\{|x(t)| : t \geq T\} > 0$ for absolutely $t \geq T$ and assuming (1.1) possession suchlike solutions. A remediation for equation (1.1) call on oscillatory whether it's not either positive eventually nor yet negative eventually; or else, it call non oscillatory.

Convey the (1.1) have characteristic V_2 whether any remediation $x(t)$ for (1.1) not either is oscillatory of satisfy this $\lim_{t\to\infty} x(t) = 0$.

Oscillation concepts for difference third-order equations uses have continuous attention of previous years, example, $[2-10, 12-15]$ and the sources of references placed there are in.

In [13] author consider the following equation

$$\Delta\left(a_n \Delta\left(b_n\left(\Delta x_n\right)^\alpha\right)\right)+p_n\left(\Delta x_{n+1}\right)^\alpha+q_n f\left(x_{\sigma(n)}\right)=0,\ n \geq n_0 \tag{1.3}$$

and established some oscillation for certain difference third-order equations uses comparability concepts couple for difference first order equations.

Above observation motivated us to study oscillatory for third order difference neutral delay with non-canonical operators. Section 2, we present oscillatory for all remediation of (1.1) and Section 3, issue few examples illustrative major result.

2. Main Outcomes

Following notations uses in research article.

$$\mu_1(\mathrm{t})=\sum_{\mathrm{s}=\mathrm{t}_1}^{\mathrm{t}-1} \frac{1}{\mathrm{c}_1(\mathrm{s})}, \quad \mu_2(\mathrm{t})=\sum_{\mathrm{s}=\mathrm{t}_1}^{\mathrm{t}-1} \frac{1}{\mathrm{c}_2(\mathrm{s})}, \quad \mu(\mathrm{t})=\sum_{\mathrm{s}=\mathrm{t}_1}^{\mathrm{t}-1} \frac{\mu_2(\mathrm{s})}{\mathrm{c}_1(\mathrm{s})}$$

$$\psi_1(t)=\sum_{s=t}^{\infty} \frac{1}{c_1(s)}, \quad \psi_2(t)=\sum_{s=t}^{\infty} \frac{1}{c_2(s)}, \quad \psi(t)=\sum_{s=t}^{\infty} \frac{\psi_1(s)}{c_2(s)}$$

$$\mu(t,t_1)=\sum_{s=t_1}^{t-1} \frac{1}{c_1(s)} \sum_{u=s}^{t-1} \frac{1}{c_2(u)}, \quad \tilde{\mu}(t,t_1)=\sum_{s=t_1}^{t-1} \frac{1}{c_1(s)} \sum_{u=s}^{t-1} \frac{1}{c_2(u)u^\beta}$$

where β is a constant satisfying

$$0 \leq \frac{q_0 \beta}{q_0-1} \leq \frac{t p(t) \mu(t, t+\xi-\sigma)}{q(t+\xi-\sigma)} \tag{2.1}$$

Lemma 2.1. *Suppose that* $(\mathrm{LH}_1)-(\mathrm{LH}_3)$ *satisfy & $x(t)$ an positive eventually remediation for (1.1).*

$$y(t)>x(t) \geq \frac{1}{q(t+\xi)}\left[y(t+\xi)-\frac{y(t+2\xi)}{q(t+2\xi)}\right] \tag{2.2}$$

& the corresponding sequence $y(t)$ belongs to one of following cases;

$$\begin{aligned}
& y(t) \in G_1 \Leftrightarrow y>0, E_1 y<0, E_2 y<0 \\
& y(t) \in G_2 \Leftrightarrow y>0, E_1 y<0, E_2 y>0 \\
& y(t) \in G_3 \Leftrightarrow y>0, E_1 y>0, E_2 y>0 \\
& y(t) \in G_4 \Leftrightarrow y>0, E_1 y>0, E_2 y<0
\end{aligned}$$

Is eventually.

Proof. Choose $t_1>t_0$ suchlike $x(t-\sigma)>0$ and $x(t-\xi)>0$. From the definition of y, $y(t)>x(t)>0$ and

$$\begin{aligned}
x(t) &= \frac{y(t+\xi)-x(t+\xi)}{q(t+\xi)} \\
&\geq \frac{1}{q(t+\xi)}\left(y(t+\xi)-\frac{y(t+2\xi)}{q(t+2\xi)}\right)
\end{aligned}$$

for $t \geq t_1$. Obviously, $E_3 y(t)$ non-increasing, since $E_3 y(t)=-p(t) x(t-\sigma) \leq 0$. Hence $E_1 y(t)$ and $E_2 y(t)$ eventually one sign, implied 4 cases G_1-G_4 possibility $y(t)$.

Next state the nonexistence for non negative non-decrease remediation for (1.1). That state is included eliminating remediation that class G_1. In proof, take the useful truth

$$\lim_{t \rightarrow \infty} \frac{\mu(t+\xi)}{\mu(t)}=\lim_{t \rightarrow \infty} \frac{\mu_1(t+\xi)}{\mu_1(t)}=1 \tag{2.3}$$

which comes from equation (1.2). □

Lemma 2.2. *Presume that* $(\mathrm{LH}_1)-(\mathrm{LH}_3)$ *are satisfied. If*

$$\sum_{s=t_0}^{\infty} \frac{\psi_2(s) p(s)}{q(s+\xi-\sigma)}=\infty, \tag{2.4}$$

then $G_3=G_4=\varphi$.

Proof. Sake for contravention, lets (2.4) satisfy $y \in G_3 \cup G_4$. Choose $t_1>t_0$ such like $x(t)>0, x(t-\sigma)>0$ and $x(t-\xi)>0$. Assume that $y \in G_3$. Since $E_2 y$ is decreasing,

$$E_1 y(t) \geq \sum_{s=t_1}^{t-1} \frac{1}{c_2(s)} E_2 y(s) \geq E_2 y(t) \mu_2(t)$$

Thus,

$$\Delta\left(\frac{E_1 y(t)}{\mu_2(t)}\right)=\frac{E_2 y(t) \mu_2(t)-E_1 y(t)}{c_2(t) \mu_2^2(t+1)} \leq 0 .$$

Therefore, $\frac{E_1 y(t)}{\mu_2(t+1)}$ is non-increasing

$$y(t) \geq \sum_{s=t_1}^{t-1} \frac{\mu_2(t)}{c_1(s) \mu_2(t)} E_1 y(s) \geq \frac{E_1 y(t) \mu(t)}{\mu_2(t)} \text{ for } t \geq t_1$$

Consequently, $\frac{y(t)}{\mu(t)}$ is non-increasing,

$$\Delta\left(\frac{y(t)}{\mu(t)}\right)=\frac{E_1 y(t) \mu(t)-y(t) \mu_2(t)}{c_1(t) \mu^2(t+1)} \leq 0$$

From $t+2\xi \geq t+\xi$

$$y(t+2\xi) \leq \frac{\mu(t+2\xi)}{\mu(t+\xi)} y(t+\xi) \tag{2.5}$$

Using this in (2.2),

$$x(t) \geq \frac{y(t+\xi)}{q(t+\xi)}\left[1-\frac{\mu(t+2\xi)}{\mu(t+\xi) q(t+2\xi)}\right], \quad t \geq t_1$$

By virtue of (LH_2) and (2.3), there is $t_2 \geq t_1$ such that for any constant $\varepsilon \in(0, q_0-1)$ and $t \geq t_2$

$$\frac{\mu(t+2\xi)}{\mu(t+\xi) q(t+2\xi)} \leq \frac{1+\varepsilon}{q_0}$$

which implies,

$$x(t) \geq \frac{y(t+\xi)}{q(t+\xi)}\left[1-\frac{1+\varepsilon}{q_0}\right]>0 \tag{2.6}$$

Combining (2.6) with (1.1) we have

$$\begin{aligned}0 \geq & E_3y(t) + \left(1 - \frac{1+\varepsilon}{q_0}\right)\frac{p(t)}{q(t+\xi-\sigma)}y(t+\xi-\sigma)\\ \geq & E_3y(t) + k\left(1-\frac{1+\varepsilon}{q_0}\right)\frac{p(t)}{q(t+\xi-\sigma)} \qquad (2.7)\end{aligned}$$

where we uses y is non-decreases, & set $k = y(t_2+\xi-\sigma) < y(t+\xi-\sigma)$. Summing (2.7) t_2 to $t-1$

$$E_2y(t) \leq E_2y(t_2) - k\left(1-\frac{1+\varepsilon}{q_0}\right)\sum_{s=t_2}^{t-1}\frac{p(s)}{q(s+\xi-\sigma)} \qquad (2.8)$$

On the other hand, from (1.2) and (2.4), it follows that

$$\sum_{s=t_0}^{\infty}\frac{p(s)}{q(s+\xi-\sigma)} = \infty$$

visible for (2.8), contravention non-negativity for E_2y. Assuming $y \in G_4$ of $t \geq t_1$. Uses monotonicity for E_1y

$$y(t) \geq \sum_{s=t_1}^{t-1}\frac{1}{c_1(s)}E_1y(s) \geq E_1y(t)\mu_1(t).$$

Thus, one visible that

$$\Delta\left(\frac{y(t)}{\mu_1(t)}\right) = \frac{E_1y(t)\mu_1(t) - y(t)}{c_1(t)\mu_1^2(t+1)} \leq 0$$

which implies that $\frac{y(t)}{\mu_1(t)}$ is non-increasing. Hence,

$$y(t+2\xi) \leq \frac{\mu_1(t+2\xi)}{\mu_1(t+\xi)}y(t+\xi)$$

uses (2.3) arrive (2.7), holds of anyone $\varepsilon > 0$ and $t \geq t_2$ for $t_2 \geq t_1$ sufficiently large. Summing (2.7) from t_2 to $t-1$, we have

$$-\Delta(E_1y(t)) \geq k\left(1-\frac{1+\varepsilon}{q_0}\right)\frac{1}{c_2(t)}\sum_{s=t_2}^{t-1}\frac{p(s)}{q(s+\xi-\sigma)}$$

Summation above in-equality again t_2 to $t-1$

$$E_1y(t) \leq E_1y(t_2) - k\left(1-\frac{1+\varepsilon}{q_0}\right)\sum_{u=t_2}^{t-1}\frac{1}{c_2(u)}\sum_{s=t_2}^{u-1}\frac{p(s)}{q(s+\xi-\sigma)}$$

Letting t to ∞ changing the summation and using (2.4) we obtain

$$\begin{aligned}0 \leq & E_1y(\infty) \leq E_1y(t_2) - k\left(1-\frac{1+\varepsilon}{q_0}\right)\sum_{u=t_2}^{\infty}\frac{1}{c_2(u)}\sum_{s=t_2}^{u-1}\frac{p(s)}{q(s+\xi-\sigma)}\\ = & E_1y(t_2) - k\left(1-\frac{1+\varepsilon}{q_0}\right)\sum_{u=t_2}^{\infty}\frac{p(s)\Psi_2(s)}{q(s+\xi-\sigma)} = -\infty\end{aligned}$$

a contravention. Proof was intact. □

Theorem 2.3. *Presume that* $(\mathrm{LH}_1) - (\mathrm{LH}_3)$ *are satisfied. If*

$$\sum_{s=t_0}^{\infty}\frac{\psi(s)p(s)}{q(s+\xi-\sigma)} = \infty \qquad (2.9)$$

that (1.1) have characteristic V_2.

Proof. Assuming that $x(t)$ non-oscillatory remediation for (1.1). Generality, create it positive eventually. Presume $x(t) > 0, x(t-\sigma) > 0$ and $x(t-\xi) > 0$. By decision Lemma 2.1, $y \in G_i, i = 1,2,3,\ldots$ for $t \geq t_1$. Visible for (1.2), state (2.9) implied

$$\sum_{s=t_0}^{\infty}\frac{\psi_2(s)p(s)}{q(s+\xi-\sigma)} = \sum_{s=t_0}^{\infty}\frac{p(s)}{q(s+\xi-\sigma)} = \infty$$

Thus by Lemma 2.2, $G_3 = G_4 = \varphi$ and so either $y \in G_1$ or $y \in G_2$. Using (LH_2) and the fact that y is non-increasing in (2.2),

$$x(t) \geq \frac{y(t+\xi)}{q(t+\xi)}\left[1 - \frac{1}{q(t+2\xi)}\right] \geq \left(1-\frac{1}{q_0}\right)\frac{y(t+\xi)}{q(t+\xi)} \qquad (2.10)$$

Onwards $\Delta y < 0$ & $l > 0$ suchlike

$$\lim_{t\to\infty} y(t) = l < \infty$$

If $l > 0$, occurs $t_2 \geq t_1$ suchlike $y(t) \geq l$ for $t \geq t_2$. Hence, from (2.10),

$$x(t) \geq \frac{l(q_0-1)}{q_0}\frac{1}{q(t+\xi)}, \quad t \geq t_2$$

Using this in (1.1), we find

$$E_3y(t) + \frac{l(q_0-1)}{q_0}\frac{p(t)}{q(t+\xi-\sigma)} \leq 0, \quad t \geq t_2 \qquad (2.11)$$

we assume that $y \in G_1$, then by summing (2.11) from t_2 to $t-1$

$$-\Delta(E_1y(t)) \geq \frac{l(q_0-1)}{q_0}\frac{1}{c_2(t)}\sum_{s=t_2}^{t-1}\frac{p(s)}{q(s+\xi-\sigma)}$$

Summation above in-equality t_2 to $t-1$

$$-\Delta y(t) \geq \frac{l(q_0-1)}{q_0}\frac{1}{c_1(t)}\sum_{u=t_2}^{t-1}\frac{1}{c_2(u)}\sum_{s=t_2}^{u-1}\frac{p(s)}{q(s+\xi-\sigma)} \qquad (2.12)$$

Summing (2.12) from t_2 to $t-1$, letting t to infinity & changed in-equality, & takes (2.9),

$$\begin{aligned}l = & y(\infty) \leq y(t_2) - \frac{l(q_0-1)}{q_0}\sum_{v=t_2}^{\infty}\frac{1}{c_1(v)}\\ & \sum_{u=t_2}^{v-1}\frac{1}{c_2(u)}\sum_{s=t_2}^{u-1}\frac{p(s)}{q(s+\xi-\sigma)} \qquad (2.13)\\ = & y(t_2) - \frac{l(q_0-1)}{q_0}\sum_{s=n_2}^{\infty}\frac{\psi(s)p(s)}{q(s+\xi-\sigma)} = -\infty\end{aligned}$$

is contravention. Thence, $l = 0$.
Takes $y \in G_2$, summation (2.11) t_2 to $t-1$& uses (2.9)

$$E_2y(t) \leq E_2y(t_2) - \frac{l(q_0-1)}{q_0} \sum_{s=t_2}^{t-1} \frac{p(s)}{q(s+\xi-\sigma)} \to -\infty \text{ as } t \to \infty \tag{2.14}$$

which contradicts the positivity of E_2y and so $l = 0$. Since $y(t) \geq x(t)$, we find $\lim_{t\to\infty} x(t) = 0$. Proof was intact.
Following outcomes, For nonexistence G_1 type remediation, comparability for studious Equation (1.1) connected delay first-order difference in-equality. Given criteria excludes remediation G_3 and G_4. □

Lemma 2.4. *Presume that* $(\mathrm{LH}_1)-(\mathrm{LH}_4)$ *are satisfied. If*

$$\liminf_{t\to\infty} \sum_{s=t+\xi-\sigma}^{t-1} \frac{p(s)\psi(s)}{q(s+\xi-\sigma)} > \frac{q_0}{q_0-1} \tag{2.15}$$

then $G_1 = G_3 = G_4 = \varphi$.

Proof. Sake for contravention, lets (2.15) satisfy $y \in G_1 \cup G_3 \cup G_4$. Choose $t_1 > t_0$ suclike $x(t) > 0, x(t-\sigma) > 0$ and $x(t-\xi) > 0$. Assume first that $y \in G_1$. Proof for Theorem 2.3 arrive (2.10), visible for (1.1) provide

$$E_3y(t) + \frac{q_0-1}{q_0} \frac{p(t)}{q(t+\xi-\sigma)} y(t+\xi-\sigma) \leq 0 \tag{2.16}$$

Define the function

$$w(t) = \psi_1(t)E_1y(t) + y(t) \tag{2.17}$$

From

$$y(t) \geq -\sum_{s=t}^{\infty} \frac{1}{c_1(s)} E_1y(s) \geq -E_1y(t)\psi_1(t) = -E_1y(t+1)\psi_1(t) \tag{2.18}$$

and

$$\Delta w(t) = \psi_1(t)\Delta(E_1y(t)) = \frac{\psi_1(t)}{c_2(t)} E_2y(t) < 0$$

$w(t)$ strictly non-increase positive eventually sequence. Using the definition of w in (2.16), we have

$$\Delta\left(\frac{c_2(t)}{\psi_1(t)}\Delta w(t)\right) + \frac{q_0-1}{q_0} \frac{p(t)y(t+\xi-\sigma)}{q(t+\xi-\sigma)} \leq 0$$

Hence w is remediation for second-order difference delay in-equality

$$\Delta\left(\frac{c_2(t)\Delta w(t)}{\psi_1(t)}\right) + \frac{q_0-1}{q_0} \frac{p(t)w(t+\xi-\sigma)}{q(t+\xi-\sigma)} \leq 0 \tag{2.19}$$

Similarity before, defined function u by

$$u(t) = \frac{\psi(t)c_2(t)}{\psi_1(t)} \Delta w(t) + w(t)$$

From

$$\Delta u(t) = \Delta\left(\frac{c_2(t)\Delta w(t)}{\psi_1(t)}\right)\Psi(t) = E_3y(t)\Psi(t) \leq 0$$

and

$$w(t) \geq -\sum_{s=t}^{\infty} \frac{\psi_1(s)c_2(s)}{c_2(s)\Psi_1(s)} \Delta w(s) \geq -\frac{c_2(t)}{\psi_1(t)} \Delta w(t)\psi(t) = -\frac{c_2(t+1)}{\psi_1(t+1)} \Delta w(t+1)\psi(t) \tag{2.20}$$

Come to end u positive eventually & non-increasing. Uses definition for u on (2.19), visible that u satisfy delay first-order difference in-equality

$$\Delta u(t) + \frac{q_0-1}{q_0} \frac{p(t)\psi(t)}{q(t+\xi-\sigma)} u(t+\xi-\sigma) \leq 0 \tag{2.21}$$

However, by [1] (Theorem 6.20.5), state (2.15) make sure that above in-equality doesn't possess a non-negative remediation, which was contravention.

Showing also $G_3 = G_4 = \varphi$, it enough (2.9) is required for validity for (2.15) onwards otherwise, left side for (2.15) equal be zero. Come to an end suddenly from Theorem 2.3. Proof was intact. □

Lemma 2.5. *Presume that* $(\mathrm{LH}_1)-(\mathrm{LH}_4)$ *are satisfied and (2.4) holds. If for any* $t_1 \geq t_0$ *large enough,*

$$\limsup_{t\to\infty} \sum_{s=t_1}^{t-1} \left(\frac{\psi(s)p(s)}{q(s+\xi-\sigma)} - \left(\frac{q_0}{q_0-1}\right) \frac{\psi_1(s+1)}{4\psi(s)c_2(s+1)}\right) > \frac{q_0}{q_0-1} \tag{2.22}$$

then $G_1 = G_3 = G_4 = \varphi$.

Proof. Sake for contravention, lets (2.15) satisfy $y \in G_1 \cup G_3 \cup G_4$. Choose $t_1 > t_0$ such like $x(t) > 0, x(t-\sigma) > 0$ and $x(t-\xi) > 0$. Assume that $y \in G_1$. Proof for Lemma 2.4 come by (2.19), here w given (2.17). Then ρ defines by

$$\rho(t) = \frac{c_2(t)\Delta w(t)}{\psi_1(t)w(t)} \tag{2.23}$$

Clearly, $\rho < 0$, from (2.20),

$$-1 \leq \psi(t)\rho(t) < 0 \tag{2.24}$$

Using (2.19) together with (2.23), we have

$$\begin{aligned}\Delta\rho(t) &= \Delta\left(\frac{c_2(t)\Delta w(t)}{\psi_1(t)}\right)\frac{1}{w(t)} - \frac{c_2(t+1)[\Delta w(t+1)]^2}{\psi_1(t+1)w^2(t+1)} \\ &\leq -\left(\frac{q_0-1}{q_0}\right)\frac{p(t)}{q(t+\xi-\sigma)}\frac{w(t+\xi-\sigma)}{w(t)} \\ &\quad -\frac{\psi_1(t+1)\rho^2(t+1)}{c_2(t+1)} \\ &\leq -\left(\frac{q_0-1}{q_0}\right)\frac{p(t)}{q(t+\xi-\sigma)} - \frac{\psi_1(t+1)\rho^2(t+1)}{c_2(t+1)}\end{aligned} \tag{2.25}$$

Multiplied both side for (2.25) at $\psi(t)$& summing in-equality t_1 to $t-1$

$$\begin{aligned}\psi(t)\rho(t) \leq& \psi(t_1)\rho(t_1) + \sum_{s=t_1}^{t-1}\frac{\rho(s+1)\psi_1(s+1)}{c_2(s+1)} \\ &- \frac{q_0-1}{q_0}\sum_{s=t_1}^{t-1}\frac{\psi(s)p(s)}{q(s+\xi-\sigma)} \\ &- \sum_{s=t_1}^{t-1}\frac{\psi_1(s+1)\rho^2(s+1)\psi(s)}{c_2(s+1)} \\ =& \psi(t_1)\rho(t_1) - \frac{q_0-1}{q_0}\sum_{s=t_1}^{t-1}\frac{\psi(s)p(s)}{q(s+\xi-\sigma)} \\ &+ \sum_{s=t_1}^{t-1}\frac{\psi_1(s+1)\psi(s)}{c_2(s+1)}\left[\frac{\rho(s+1)}{\psi(s)} - \rho^2(s+1)\right] \\ \leq& -\left(\frac{q_0-1}{q_0}\right)\sum_{s=t_1}^{t-1}\left[\frac{\psi(s)p(s)}{q(s+\xi-\sigma)}\right. \\ &\left. - \left(\frac{q_0}{q_0-1}\right)\frac{\psi_1(s+1)}{4\psi(s)c_2(s+1)}\right]\end{aligned}$$

visible for (27), in-equality contravention (2.22). Thence $G_1 = \varphi$. At Lemma 2.2, $G_3 = G_4 = \varphi$ caused by (2.4). Proof was intact. □

Corollary 2.6. *Presume that* $(\mathrm{LH}_1) - (\mathrm{LH}_3)$ *satisfy* &(2.4) *holds. Occurs constant* C_k *suchlike*

$$\frac{\psi^2(t)p(t)c_2(t)}{q(t+\xi-\sigma)\psi_1(t)} \geq C_k > \frac{q_0}{4(q_0-1)} \tag{2.26}$$

then $G_1 = G_3 = G_4 = \varphi$.
Achieve oscillatory for all remediation, remains eliminates remediation for G_2 *type.*

Lemma 2.7. *Presume that* $(\mathrm{LH}_1) - (\mathrm{LH}_4)$ *are satisfied. If*

$$\limsup_{t\to\infty}\sum_{s=t+\xi-\sigma}^{t-1}\frac{p(s)\mu(t+\xi-\sigma,s+\xi-\sigma)}{q(s+\xi-\sigma)} > \frac{q_0}{q_0-1} \tag{2.27}$$

then $G_2 = \varphi$.

Proof. Sake for contravention, lets (2.27) satisfy $y \in G_2$. Choose $t_1 > t_0$ suchlike $x(t) > 0, x(t-\sigma) > 0$ and $x(t-\xi) > 0$. Using (2.10) in (1.1), we obtain

$$E_3y(t) + \frac{q_o-1}{q_0}\frac{p(t)}{q(t+\xi-\sigma)}y(t+\xi-\sigma) \leq 0 \tag{2.28}$$

Uses monotonicity of E_2y

$$-E_1y(u) \geq E_1y(v) - E_1y(u) = \sum_{s=u}^{v-1}\frac{E_2y(s)}{c_2(s)} \geq E_2y(v)\sum_{s=u}^{v-1}\frac{1}{c_2(s)} \tag{2.29}$$

for $v \geq u \geq t_1$. Summation latter in-equality u to $v-1$,

$$y(u) \geq E_2y(v)\sum_{s=u}^{v-1}\frac{1}{c_1(s)}\sum_{x=s}^{v-1}\frac{1}{c_2(x)} = E_2y(v)\mu(v,u). \tag{2.30}$$

Setting $u = s+\xi-\sigma$ and $v = t+\xi-\sigma$ in (2.30), we find

$$y(s+\xi-\sigma) \geq E_2y(t+\xi-\sigma)\mu(t+\xi-\sigma,s+\xi-\sigma) \tag{2.31}$$

Summation (2.28) $t+\xi-\sigma$ to $t-1$& using (2.31), we see that

$$\begin{aligned}E_2y(t+\xi-\sigma) \geq& E_2y(t+\xi-\sigma) - E_2y(t) \\ \geq& \frac{q_0-1}{q_0}\sum_{s=t+\xi-\sigma}^{t-1}\frac{p(s)y(s+\xi-\sigma)}{q(s+\xi-\sigma)} \\ \geq& \frac{q_0-1}{q_0}E_2y(t+\xi-\sigma) \\ &\sum_{s=t+\xi-\sigma}^{t-1}\frac{p(s)\mu(t+\xi-\sigma,s+\xi-\sigma)}{q(s+\xi-\sigma)}\end{aligned}$$

Dividing the above inequality by $E_2y(t+\xi-\sigma)$& takes the limsup on two sides for in- equality $t\to\infty$, get contravention in (2.27). Proof was intact. □

Lemma 2.8. *Presume that* $(\mathrm{LH}_1) - (\mathrm{LH}_4)$ *satisfy & lets* β *was constant satisfy (2.1) eventually. If*

$$\begin{aligned}\mathrm{limitsup}_{t\to\infty}(t+\xi-\sigma)^\beta &\sum_{s=t+\xi-\sigma}^{t-1}\frac{p(s)\tilde{\mu}(t+\xi-\sigma,s+\xi-\sigma)}{q(s+\xi-\sigma)} \\ &> \frac{q_0}{q_0-1}\end{aligned} \tag{2.32}$$

then $G_2 = \varphi$.

Proof. Setting $u = t+\xi-\sigma$ and $v = t$ in (2.30),

$$\begin{aligned}y(t+\xi-\sigma) &\geq E_2y(t)\mu(t,t+\xi-\sigma) \\ &= E_2y(t+1)\mu(t,t+\xi-\sigma)\end{aligned} \tag{2.33}$$

By (2.1), (2.28) and (2.33), we have

$$\Delta\left(t^{\beta}E_2y(t)\right) = \beta t^{\beta-1}E_2y(t+1) + t^{\beta}E_3y(t) \le \beta t^{\beta-1}E_2y(t+1)$$
$$-\left(\frac{q_0-1}{q_0}\right)\frac{t^{\beta}p(t)y(t+\xi-\sigma)}{q(t+\xi-\sigma)}$$
$$\le \beta t^{\beta-1}E_2y(t+1) - \left(\frac{q_0-1}{q_0}\right)\frac{t^{\beta}p(t)E_2y(t+1)\mu(t,t+\xi-\sigma)}{q(t+\xi-\sigma)}$$
$$= t^{\beta-1}E_2y(t+1)\left[\beta - \left(\frac{q_0-1}{q_0}\right)\frac{tp(t)\mu(t,t+\xi-\sigma)}{q(t+\xi-\sigma)}\right]$$
$$\le 0$$

That is $t^{\beta}E_2y(t+1)$ is eventually non-increasing. From here we obtain that

$$-E_1y(u) \ge E_1y(v) - E_1y(u) = \sum_{s=u}^{v-1}\frac{E_2y(s)s^{\beta}}{s^{\beta}c_2(s)}$$
$$\ge E_2y(v)v^{\beta}\sum_{s=u}^{v-1}\frac{1}{s^{\beta}c_2(s)} \quad (2.34)$$

for $v \ge u \ge t_1$. Proof for Lemma 2.7 in (2.29) replaces (2.34), at contravention in (2.32). Proof was intact. □

Theorem 2.9. *Suppose that* $(\mathrm{LH}_1)-(\mathrm{LH}_4)$ *satisfy. Whether* $(2.15)($ *or* $(2.22))\&(2.27)($ *or* $(2.32))$ *hold, that then (1.1) was oscillatory.*

3. Example

Example 3.1. *Observe third order delay difference equation*

$$\Delta\left(\frac{1}{2}\Delta\left(\frac{1}{6}\Delta(x(t)+2x(t-2))\right)\right)+2x(t-4)=0. \quad (3.1)$$

Hence $\xi=2,\quad \sigma=4,\quad q(t)=2,\quad c_1(t)=\frac{1}{6},\quad c_2(t)=\frac{1}{2},$ *and* $p(t)=2$. *Verify that the states for Theorem 2.3 satisfy. Here all remediation for (3.1) has characteristic* V_2, *one such solution is* $x_n=(-1)^t$.

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