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Nonstandard χ-hulls of uniform spaces

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Abstract

A Nonstandard hull of a metric space is constructed by working in a κ-saturated superstructure with $\kappa > \chi_0$. The hypothesis $\kappa > \chi_0$ accounts for the fact that we work with sequences in metric spaces. In uniform spaces, we naturally replace sequences by nets $(x_\alpha)_{\alpha \in D}$. Hence the need for focusing on the cardinality of D arises here and hence the need for χ-hulls for various cardinalities χ. In this article we obtain a χ-hull of a uniform space X by considering κ-saturated superstructures $V(^*X)$ with $\kappa > \chi$.

Keywords

Standard, Nonstandard, Uniform Structure, Uniform spaces, Filter, Nets, Cauchy filter, Cauchy net, Completeness.

AMS Subject Classification

54J05, 54E15, 54E50.

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Contents

1. Introduction

Non-standard analysis is a branch of Mathematics introduced by Abraham Robinson in 1966[1]. Abraham Robinson constructed a superstructure to work in any given structure like the Euclidean spaces, topological spaces, algebraic structures (rings, fields etc.,.), graphs and so on. The basic idea is not necessarily to study the superstructure but to study the classical spaces by getting on to a higher platform, namely a superstructure, and get a microscopic view of the classical space below.

The notion of nonstandard hull of a metric space was introduced by Luxemburg([5],[2]). The study has been carried out by considering a κ- saturated enlargement, $\kappa > \chi_0$, where χ_0 is the countably infinite cardinal. This motivates our present study of χ hulls of uniform spaces. Since convergence of sequences characterizes the topology of a metric space, one is able to define the nonstandard hull of a metric space by considering a κ-saturated enlargement with $\kappa > \chi_0$. However this is not so in topological spaces, in general and hence we get into χ-hulls for uniform spaces.

2. Preliminaries

We assume preliminaries and notations like $V(X), V(^*X)$ for superstructures, as in [1] and [2]. For preliminaries on uniform spaces we refer to [3],[4],[6].

Definition 2.1. *A binary relation P is said to be concurrent on $A \subseteq domP$ if for each finite set $\{x_1, x_2 \ldots . x_n\}$ in A there is a $y \in rangeP$ so that $\langle x_i, y\rangle \in P, 1 \leq i \leq n$. P is said to be concurrent if it is concurrent on domP.*

Definition 2.2. *Let κ be a cardinal number. We say $V(^*X)$ is κ-saturated if, for each internal binary relation $P \in V(^*X)$ which is concurrent on some (not necessarily internal) set A in $V(^*X)$ with card$A < \kappa$, there exists an element $y \in$ rangeP so that $(x, y) \in P$ for all $x \in A$.*

We enumerate the following two theorems from [2].

Theorem 2.3. *Given any superstructure $V(X)$ and cardinal κ, there is a κ-saturated superstructure $V(^*X)$.*

Theorem 2.4. *Let $V(^*X)$ be a κ-saturated extension of $V(X)$. Let C be a (not necessarily internal) set of entities in $V(^*X)$ with card $C < \kappa$ and D be an internal set in $V(^*X)$. For any mapping $\phi : C \to D$, there is an internal extension $\tilde{\phi} : \tilde{C} \to D$ of ϕ, where $\tilde{C}$ is internal and contains C.*

Now we start with our following two definitions.

Definition 2.5. *Let χ be a cardinal. A uniform space X with uniform structure Ψ is said to be χ-complete if every Cauchy net $\langle x_\alpha \rangle_{\alpha\in D}$ in X, with card$D \le \chi$, converges.*

Let $V(^*X)$ be an enlargement of $V(X)$.

Definition 2.6. *$y \in {}^*X$ is said to be finite if $\exists\, x \in X$ and $U \in \Psi$ such that $x, y \in {}^*U$.*

Let $G = \{y \in {}^*X : y \text{ is finite}\}$.
On G, define $\simeq$ by $y_1 \simeq y_2$ if for $U \in \Psi$, $y_1 \in {}^*U \Leftrightarrow y_2 \in {}^*U$
Clearly $\simeq$ is an equivalence relation on G. Denote by $m(x')$, the equivalence class of x'. Let $\widehat{X}$ denote the set of equivalence classes.
For $U \in \Psi$, define $\widehat{U} = \{(m(x'), m(y')) : (x', y') \in {}^*U\}$
The following is an easy observation:
$\widehat{X}$ is a uniform space with uniform structure $\widehat{\Psi} = \left\{\widehat{U} : U \in \Psi\right\}$.

3. Main Result

In this section, we present and prove the main result of this paper.

Proposition 3.1. *Let χ be any cardinal. Let $^*X$ lie in a κ-saturated superstructure with $\kappa > \chi$. Then $\left(\widehat{X}, \widehat{\Psi}\right)$ is a χ-complete uniform space.*

Proof. Let $\langle m(a_\alpha)\rangle_{\alpha\in D}$ be a Cauchy net in $\left(\widehat{X}, \widehat{\Psi}\right)$, where $a_\alpha \in G\ \forall\ \alpha \in D$ and card$D \le \chi$
Now we may place our argument by assuming $D \subseteq X$. Otherwise one works with $X \cup D$ in this segment.
For each $V \in \Psi$, $\exists\ \alpha_V \in D$ such that $\alpha, \beta \in D$ and $\alpha, \beta > \alpha_V \Rightarrow \left(m(a_\alpha), m\left(a_\beta\right)\right) \in \widehat{V} \Rightarrow \left(a_\alpha, a_\beta\right) \in {}^*V$
Define $\phi : D \to {}^*X$ by $\phi(\alpha) = a_\alpha$
Then, by Theorem 2.4, ϕ gets extended to an internal map $\tilde{\phi} : \tilde{D} \to {}^*X$, where $\tilde{D}$ is an internal subset of $^*D$ and $\tilde{D} \supseteq D$
Claim : $\exists\ \delta' \in \tilde{D} - D$ such that $V \in \Psi,\ \alpha \in D,\ \alpha > \alpha_V \Rightarrow (a_\alpha,\ a_{\delta'}) \in {}^*V$, where $a_{\delta'} = \tilde{\phi}(\delta'$. This will imply $a_{\delta'} \in G$ and $\alpha \in D, \alpha > \alpha_V \Rightarrow (m(a_\alpha), m(a_{\delta'})) \in \widehat{V}$
This means $\langle m(a_\alpha)\rangle \to m(a_{\delta'})$, completing the proof.
Define $E(V) = \left\{\tilde{\delta} \in \tilde{D} : \left(a_\alpha,\ a_\beta\right) \in {}^*V\right\}$ for $\alpha,\ \beta \in \tilde{D},\ \alpha_V < \alpha \le \tilde{\delta},\ \alpha_V < \beta \le \tilde{\delta}$. Then $E(V)$ is internal, contains D and hence contains $\left\{\tilde{\delta} \in \tilde{D} : \tilde{\delta} \le \delta'_V\right\}$ for some $\delta'_V \in \tilde{D} - D$
By Transfinite recursion, we may take $\delta'_{V_2} \le \delta'_{V_1}$ if $V_2 \subseteq V_1$
Define $\mu : \Psi \to \tilde{D}$ by $\mu(V) = \delta'_V$
Again by Theorem 2.4, this gets extended to an internal map $\tilde{\mu} : \tilde{\Psi} \to \tilde{D}$, where $\tilde{\Psi}$ is an internal set containing Ψ.
Since $\delta'_V > \alpha_V\ \forall\ V \in \Psi,\ \exists\ \tilde{U} \in \tilde{\Psi} - \Psi$ such that $\tilde{\mu}\left(\tilde{U}\right) \ge \tilde{\phi}\left(\tilde{U}\right)$ and $\tilde{\mu}\left(\tilde{U}\right) \in E(V)\ \forall\ V \in \Psi$
Let $\delta' = \tilde{\mu}\left(\tilde{U}\right)$
Thus $V \in \Psi,\ \alpha \in D, \alpha > \alpha_V \Rightarrow (a_\alpha, a_{\delta'}) \in {}^*V$
This completes the proof. ◻

Definition 3.2. *We call the above $\left(\widehat{X}, \widehat{\Psi}\right)$ as the nonstandard χ-hull of the uniform space (X, Ψ)*

By Theorem 2.3, for every cardinal κ, there is a κ-saturated superstructure $V(^*X)$ for $V(X)$. Hence we get the following.

Proposition 3.3. *Let (X, Ψ) be a uniform space. Given any cardinal χ, there exists a nonstandard χ-hull $\left(\widehat{X}, \widehat{\Psi}\right)$ of (X, Ψ).*

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