Malaya Journal of Matematik, Vol. 9, No. 1, 112-117, 2021

https://doi.org/10.26637/MJM0901/0019

Existence and uniqueness of solution of nonlinear boundary value problems for ψ-Caputo fractional differential equations

D.B. Dhaigude [1]*, V.S. Gore[2] and P.D. Kundgar [3]

Abstract

The aim of the paper is to develop monotone iterative technique and its associated iterative scheme and apply it to prove existence and uniqueness of solution of nonlinear boundary value problem for ψ-Caputo fractional differential equation.First we consider initial value problem and choose suitable initial iterations and construct two monotone sequences.It is shown that these two sequences converge monotonically from above and below to maximal and minimal solutions of initial value problem. Further we show that maximal and minimal solutions are quasi solutions of nonlinear boundary value problem for ψ-Caputo fractional differential equation which leads to existence and uniqueness of solution of nonlinear boundary value problem for ψ-Caputo fractional differential equation.

Keywords

ψ-Caputo fractional differential equation,nonlinear boundary conditions, monotone iterative technique, upper lower and quasi solutions, existence uniqueness results.

AMS Subject Classification

26A33, 34A08, 34A12, 34B15.

[1] *Emeritus Professor of Mathematics, Marathwada Mathematical Society, Aurangabad-431001, India.*

[2,3] *Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad–431004, India.*

***Corresponding author**: [1] dnyanraja@gmail.com

Article History: Received **12** October **2020**; Accepted **19** December **2020**

Contents

1. Introduction

Different types of real world phenomena with anomalous dynamics in the field of science and engineering are modeled adequately by fractional differential equations (see [17],[21] and references therein).In the literature the qualitative theory of various fractional differential equations are studied by many researchers by different methods such as fixed point technique,monotone technique, method of successive approximation etc and obtained existence uniqueness results (see [1, 2], [6]-[20],[22]-[30] and references therein).Recently,Almeida et.al.[3–5], Derbazi et,al.[8]and Samet et.al.[28]have studied fractional differential equations involving ψ-Caputo fractional derivative.More recently Dhaigude et.al.[10]have proved existence and uniqueness of solution of nonlinear boundary value problems for Riemann - Liouville fractional differential equations by applying monotone technique.In this paper, we develop monotone iterative technique combined with coupled upper, lower and quasi solutions in which we construct two monotone sequences of iterate and obtain the existence and uniqueness of solution of nonlinear boundary value problem (BVP) for ψ-Caputo fractional differential equation.

The rest of the paper is organized as follows. In section 2 some basic definitions,assumptions and useful lemmas are given.Further upper,lower and quasi solutions of nonlinear boundary value problem for ψ-Caputo fractional differential equation are introduced. Section 3 is devoted for the development of monotone iterative scheme and monotone property.The existence and uniqueness of solution of nonlinear BVP for ψ-Caputo fractional differential equation are proved.The concluding remarks are given in the last section.

2. Preliminaries and Assumptions

In this section, we introduce some basic definitions,assumptions and important lemmas which are useful for further discussion.Let $J=[0,T]$ be a finite interval on the real axis $\mathbb{R}$ and $\alpha>0$. Fractional integrals and fractional derivatives of a function $x(t)$ with respect to another function ψ are defined as follows [3, 21].We begin with the definition of left sided ψ- Riemann - Liouville fractional integral.

Definition 2.1. *The left sided ψ- Riemann - Liouville fractional integral of order $\alpha>0$ for an integrable function $x:J\to\mathbb{R}$ with respect to another function $\psi:J\to\mathbb{R}$ which is an increasing on J and $\psi(t)\in C^1(J)$ such that $\psi'(t)\neq 0$, for all $t\in J$, is defined by*

$$I_{0^+}^{\alpha;\psi}x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(t)(\psi(t)-\psi(s))^{\alpha-1}x(s)ds, \quad (2.1)$$

where Γ is the Gamma function.

Definition 2.2. *[3] Let $n\in\mathbb{N}$ and let $\psi,x\in C^n(J,\mathbb{R})$ be two functions such that ψ is an increasing on J and $\psi'(t)\neq 0$, for all $t\in J$. The lift ψ-Riemann - Liouville fractional derivative of a function $x(t)$ of order $\alpha>0$ is defined by*

$$\begin{aligned}D_{0^+}^{\alpha;\psi}x(t)&=\left(\frac{1}{\psi'(t)}\frac{d}{dx}\right)^n I_{0^+}^{n-\alpha;\psi}x(t)\\&=\frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dx}\right)^n\\&\qquad\int_0^t \psi'(t)(\psi(t)-\psi(s))^{n-\alpha-1}x(s)ds,\end{aligned}$$

where $n=[\alpha]+1$.

Definition 2.3. *[3] Let $n\in\mathbb{N}$ and let $\psi,x\in C^n(J,\mathbb{R})$ be two functions such that ψ is an increasing on J and $\psi'(t)\neq 0$, for all $t\in J$. The lift ψ-Caputo fractional derivative of a function $x(t)$ of order $\alpha>0$ is defined by*

$$^cD_{0^+}^{\alpha;\psi}x(t)=I_{0^+}^{n-\alpha;\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dx}\right)^n x(t), \quad (2.2)$$

where $n=[\alpha]+1$ for $\alpha\notin\mathbb{N}, n=\alpha$ for $\alpha\in\mathbb{N}$.

If we denote symbolically $\left(\frac{1}{\psi'(t)}\frac{d}{dx}\right)^n x(t)$ by $x_\psi^{[n]}(t)$ then the left ψ-Caputo fractional derivative of order α of $x(t)$ can be written as

$$^cD_{0^+}^{\alpha;\psi}x(t)=I_{0^+}^{n-\alpha;\psi}x_\psi^{[n]}(t).$$

Summing up we observe that the left ψ-Caputo fractional derivative of order α of a function $x(t)$ can be expressed as

$$^cD_{0^+}^{\alpha;\psi}x(t)=\begin{cases}\int_0^t \frac{\psi'(t)(\psi(t)-\psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)}x_\psi^{[n]}(s)ds, & \text{if } \alpha\notin\mathbb{N},\\ x_\psi^{[n]}(s), & \text{if } \alpha\in\mathbb{N}.\end{cases}$$

Similarly ,we can define right ψ-Riemann - Liouville and right ψ-Caputo fractional derivatives. The relation between the ψ-Riemann - Liouville fractional derivative of order α and the ψ-Caputo fractional derivative of order α of a function $x(t)$ can be expressed as

$$^cD_{0^+}^{\alpha;\psi}x(t)=D_{0^+}^{\alpha;\psi}\left[x(t)-\sum_{k=0}^{n-1}\frac{x_\psi^{[n]}(0)}{k!}(\psi(t)-\psi(0))^k\right],$$

(See for details [3]).

Consider the ψ-Caputo fractional differential equation

$$^cD_t^{\alpha;\psi}x(t)=F(t,x),\ \ 0<t\leq T, \quad (2.3)$$

with nonlinear boundary conditions

$$G(x(0),x(T))=0, \quad (2.4)$$

where $^cD_t^{\alpha;\psi}$ is the ψ-Caputo fractional derivative of order $\alpha\in(0,1]$ and $F(t,x):[0,T]\times\mathbb{R}\to\mathbb{R}$ is real valued continuous function.

Definition 2.4. *A function $x(t)\in C([0,T])$ is called classical solution of nonlinear BVP (2.3)-(2.4) if (i) $x(t)$ satisfies the nonlinear fractional differential equation (2.3) and (ii) $x(t)$ satisfies the nonlinear boundary conditions (2.4).*

Now we introduce the notion of upper,lower and quasi solutions.

Definition 2.5. *A function $\xi\in C([0,T])$ is called an upper solution of equation (2.3) if it satisfies differential inequality,*

$$^cD_t^{\alpha;\psi}\xi\geq F(t,\xi),\quad 0<t\leq T.$$

A function $\eta\in C([0,T])$ is called lower solution if η satisfies above inequality in reverse order.

Definition 2.6. *A pair of functions $(\tilde{u},\hat{u})$ in $C([0,T])$ with $\tilde{u}\geq\hat{u}$ are called coupled upper and lower solutions of the nonlinear BVP (2.3)-(2.4) if they satisfy the differential inequalities,*

$$\begin{aligned}&^cD_t^{\alpha;\psi}\tilde{u}\geq F(t,\tilde{u}),\\&^cD_t^{\alpha;\psi}\hat{u}\leq F(t,\hat{u}),\end{aligned}\quad 0<t\leq T,$$

and nonlinear boundary conditions,

$$G(\hat{u}(0),\tilde{u}(T))\leq 0\leq G(\tilde{u}(0)),\hat{u}(T)). \quad (2.5)$$

In what follows,we assume that

$$\tilde{u}(t)\geq\hat{u}(t),\quad 0<t\leq T. \quad (2.6)$$

Now,for any ordered pair of upper and lower solutions,we define sector.

Definition 2.7. *Let $\tilde{u}$ and $\hat{u}$ be any two functions with $\tilde{u}\geq\hat{u}$ then we denote the (functional interval)sector as $\langle\hat{\boldsymbol{u}},\tilde{\boldsymbol{u}}\rangle$ and defined by*

$$\langle\hat{\boldsymbol{u}},\tilde{\boldsymbol{u}}\rangle=\left\{u\in C([0,T]):\hat{u}(t)\leq u(t)\leq\tilde{u}(t),t\in(0,T]\right\}.$$

Definition 2.8. *A pair of functions p and q in $C([0,T])$ are called coupled quasi solutions of the nonlinear BVP (2.3)-(2.4) if they are solutions of the differential equation (2.3)as well as satisfy the relation*

$$\hat{u} \le q \le p \le \tilde{u}, \quad 0 \le t \le T, \tag{2.7}$$
$$G(q(0),p(T)) = 0 = G(p(0),q(T)), \tag{2.8}$$

where $\tilde{u}$ and $\hat{u}$ in $C([0,T])$ are coupled upper and lower solutions of the nonlinear BVP (2.3)-(2.4).

Suppose that nonlinear functions $F(t,u)$ and $G(x,y)$ satisfy following assumptions:
(D_1) A function $F(t,u)$ satisfies Lipschitz condition in u :

$$\begin{aligned} &F(t,u)-F(t,v) \ge -\underline{c}(u-v) \text{ for } \hat{u} \le v \le u \le \tilde{u},\\ &F(t,u)-F(t,v) \le \bar{c}(u-v) \text{ for } \hat{u} \le v \le u \le \tilde{u}. \end{aligned} \tag{2.9}$$

In view of Lipschitz condition (2.9) the function $H(t,u)$ given by

$$H(t,u) = \underline{c}u + F(t,u),$$

is monotone nondecreasing in u for $u \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$.
(D_2) A function $G(x,y)$ satisfies condition:

$$\begin{aligned} &G(x,.) \text{ is nonincreasing } \forall x \in \mathbb{R}\\ \text{and } \ &G(.,y) \text{ is nondecreasing } \forall y \in \mathbb{R}. \end{aligned} \tag{2.10}$$

The following lemmas play an important role in further developments.

Lemma 2.9. *[8] The initial value problem for ψ-Caputo fractional differential equation*

$$\begin{aligned} {}^{c}D_t^{\alpha;\psi}x(t) + rx(t) = h(t), \quad t \in [0,T],\\ x(0) = x_0, \end{aligned}$$

has unique solution

$$\begin{aligned} x(t) =& x_0 \mathbb{E}_{\alpha,1}(-r(\psi(t)-\psi(0))^\alpha) +\\ &\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\\ &\mathbb{E}_{\alpha,\alpha}(-r(\psi(t)-\psi(0))^\alpha)h(s)ds,\ t \in [0,T], \end{aligned}$$

where $\mathbb{E}_{\alpha,\beta}$ is the two-parametric Mittag-Leffler function defined in [21].

Lemma 2.10. *[8]Suppose that $x(t) \in ((0,T],\mathbb{R})$ satisfies*

$$\begin{aligned} {}^{c}D_t^{\alpha;\psi}x(t) + dx(t) \ge 0, \quad 0 < \alpha \le 1, \quad t \in [0,T],\\ x(0) \ge 0, \end{aligned}$$

where $d \in \mathbb{R}$, then $x(t) \ge 0$ for any $t \in [0,T]$.

3. Existence Uniqueness of Solution

In this section,we develop monotone iterative scheme and prove the existence and uniqueness of solution of the nonlinear BVP (2.3)-(2.4).

Theorem 3.1. . *Suppose that the functions $\tilde{u}$ and $\hat{u}$ are coupled upper and lower solutions of the nonlinear BVP (2.3)-(2.4),such that (2.6) holds.Further assume that the function $F(t,u)$ satisfies condition (2.9) and the function $G(x,y)$ satisfies condition (2.10).Then solution of the nonlinear BVP (2.3)-(2.4) exists and is unique in $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$.*

Proof. We complete the proof in the following two steps.
Step I: First we find the maximal solution p and minimal solution q of the following IVP for nonlinear ψ-Caputo fractional differential equation

$${}^{c}D_t^{\alpha;\psi}u(t) = F(t,u),\ t \in (0,T]; \quad u(0) = \gamma, \tag{3.1}$$

where $\hat{u}(0) \le \gamma \le \tilde{u}(0)$.
Step II: Secondly, we show that the maximal solution p and minimal solution q are truly coupled quasi solutions of the nonlinear BVP (2.3)-(2.4).Therefore,first we introduce the notion of upper and lower solutions of the IVP (3.1) and then propose the monotone iterative scheme and develop monotone technique.Observe that the IVP (3.1) is equivalent to the following IVP

$${}^{c}D_t^{\alpha;\psi}p(t) + \underline{c}p = \underline{c}p + F(t,p), t \in (0,T] \quad \text{and} \quad p(0) = \gamma.$$

Definition: Functions w and v in $C([0,T])$ are called ordered upper and lower solutions of the IVP (3.1) if they satisfy the inequalities,

$$\begin{aligned} {}^{c}D_t^{\alpha;\psi}w \ge F(t,w) \quad \text{in} \quad (0,T] \quad \text{and} \quad w(0) \ge \gamma,\\ {}^{c}D_t^{\alpha;\psi}v \le F(t,v) \quad \text{in} \quad (0,T] \quad \text{and} \quad v(0) \le \gamma. \end{aligned}$$

Monotone Iterative Scheme: We introduce the following monotone iterative scheme

$$\begin{aligned} {}^{c}D_t^{\alpha;\psi}p^{(k)} + \underline{c}p^{(k)} = \underline{c}p^{(k-1)} + F(t,p^{(k-1)}),\\ p^{(k)}(0) = \gamma. \end{aligned} \tag{3.2}$$

Since for each k, we have linear IVP for ψ-Caputo fractional differential equation. The existence theory of linear IVP for ψ-Caputo fractional differential equation is studied in [8].Therefore, existence of solution $\{p^{(k)}\}$ of the linear problem (3.2) follows immediately.Choose $p^{(0)}$ as an initial iteration, we construct a sequence $\{p^{(k)}\}$ from above iterative scheme (3.2).
Monotone Property: We claim that the sequences $\{\overline{p}^{(k)}\}$ and $\{\underline{p}^{(k)}\}$ possess the monotone property

$$\hat{u} \le \underline{p}^{(k)} \le \underline{p}^{(k+1)} \le \overline{p}^{(k+1)} \le \overline{p}^{(k)} \le \tilde{u}, \quad \text{in} \quad [0,T]. \tag{3.3}$$

We prove our claim by applying well known principle of mathematical induction: Define $r^{(0)} = \underline{p}^{(1)} - \underline{p}^{(0)}$ where $\underline{p}^{(0)} = \hat{u}$.

By the definition of a lower solution,the monotone iterative scheme (3.2) and the condition (2.6),we have

$$\begin{aligned}{}^{c}D_t^{\alpha;\psi}r^{(0)}+\underline{c}r^{(0)} &= {}^{c}D_t^{\alpha;\psi}(\overline{p}^{(1)}-\hat{u})+\underline{c}(\overline{p}^{(1)}-\hat{u})\\ &= -{}^{c}D_t^{\alpha;\psi}\hat{u}+F(t,\hat{u})\geq 0, t\in(0,T],\\ r^{(0)}(0) &= \underline{p}^{(1)}(0)-\underline{p}^{(0)}(0)=\gamma-\hat{u}(0)\geq 0.\end{aligned}$$

Applying Lemma 2.10,we get $r^{(0)}(t)\geq 0$ implies that $\underline{p}^{(1)}\geq\hat{u}$ in [0,T]. On similar lines, using the definition of an upper solution,the monotone iterative scheme (3.2) and the condition (2.6),we get $\tilde{u}\geq\overline{p}^{(1)}$. Define $r^{(1)}=\overline{p}^{(1)}-\underline{p}^{(1)}$. By the iterative scheme (3.2) and the monotone property of H, we have

$$\begin{aligned}{}^{c}D_t^{\alpha;\psi}r^{(1)}+\underline{c}r^{(1)} &= H(t,\overline{p}^{(0)})-H(t,\underline{p}^{(0)})\geq 0, t\in(0,T],\\ r^{(1)}(0) &= \overline{p}^{(1)}(0)-\underline{p}^{(1)}(0)=0.\end{aligned}$$

Applying Lemma 2.10, we get $r^{(1)}(t)\geq 0$, implies that $\overline{p}^{(1)}\geq\underline{p}^{(1)}$ in [0,T]. Thus we have $\underline{u}^{(0)}\leq\underline{p}^{(1)}\leq\overline{p}^{(1)}\leq\overline{u}^{(0)}, in\ [0,T]$. The result is true for k = 1.Assume that the result $\underline{p}^{(k-1)}\leq\underline{p}^{(k)}\leq\overline{p}^{(k)}\leq\overline{p}^{(k-1)}, in\ [0,T]$, is true for k. The monotone property (3.3) follows by induction for all k.

We easily conclude that the sequence $\{\overline{p}^{(k)}\}$ is monotone nonincreasing and bounded from below hence converges to some limit function.Also the sequence $\{\underline{p}^{(k)}\}$ is monotone nondecreasing and bounded from above hence converges to some limit function.So the point wise limits

$$\lim_{k\to\infty}\overline{p}^{(k)}(t)=p(t),\quad \lim_{k\to\infty}\underline{p}^{(k)}(t)=q(t), \tag{3.4}$$

exist and they are called maximal and minimum solutions respectively.These limits also satisfy the monotone property

$$\hat{u}\leq\underline{p}^{(k)}\leq\underline{p}^{(k+1)}\leq q\leq p\leq\overline{p}^{(k+1)}\leq\overline{p}^{(k)}\leq\tilde{u}. \tag{3.5}$$

Now we prove that the functions $p(t)$ and $q(t)$ are solutions of the IVP(3.1) for nonlinear ψ-Caputo fractional differential equation.Let $p^{(k)}$ be either $\overline{p}^{(k)}$ or $\underline{p}^{(k)}$. The integral representation of solution of the linear problem (3.2) is

$$\begin{aligned}p^{(k)}(t)=&\gamma E_{\alpha,1}\big(-\underline{c}[\psi(t)-\psi(0)]^{\alpha}\big)+\\ &\int_0^t\psi'(s)[\psi(t)-\psi(s)]^{\alpha-1}\\ &E_{\alpha,\alpha}\big(-\underline{c}[\psi(t)-\psi(0)]^{\alpha}\big)H(t,p^{(k-1)})ds,\end{aligned} \tag{3.6}$$

where $\quad H(t,p^{(k-1)})=\underline{c}p^{(k-1)}(t)+F(t,p^{(k-1)})$.

The function H is continuous and monotone nondecreasing.The monotone convergence of $p^{(k)}(t)$ to $p(t)$ implies that $H(t,p^{(k-1)})$ converges to $H(t,p)$. Taking limit as $k\to\infty$, in the equation (3.6)and applying the dominated convergence theorem, we observe that the function $p(t)$ satisfies the integral equation

$$\begin{aligned}p(t)=&\gamma E_{\alpha,1}\big(-\underline{c}[\psi(t)-\psi(0)]^{\alpha}\big)+\\ &\int_0^t\psi'(s)[\psi(t)-\psi(s)]^{\alpha-1}E_{\alpha,\alpha}\big(-\underline{c}[\psi(t)-\psi(0)]^{\alpha}\big)\\ &H(t,p)ds,\end{aligned}$$

We conclude that $p(t)\in C([0,T])$ is an integral representation of solution of the IVP (3.1)for nonlinear ψ-Caputo fractional differential equation.On similar lines we prove $q(t)\in C([0,T])$ is an integral representation of solution of the IVP (3.1).In this way we obtained the maximal solution p and minimal solution q of the IVP (3.1)for nonlinear ψ-Caputo fractional differential equation and satisfy the relation $p(t)\geq q(t),\ t\in[0,T]$.Now we prove that functions $p(t)$ and $q(t)$ satisfy the nonlinear boundary conditions (2.8). Since the function $G(x,.)$ is nonincreasing in x, we get

$$G\big(p(0),q(T)\big)\leq G\big(\hat{u}(0),q(T)\big),\quad \text{for}\quad \hat{u}(0)\leq p(0), \tag{3.7}$$

$$G\big(p(0),q(T)\big)\geq G\big(\tilde{u}(0),q(T)\big)\quad \text{for}\quad p(0)\leq\tilde{u}(0). \tag{3.8}$$

Also we know that the function $G(.,y)$ is nondecreasing in y, we have

$$G\big(\hat{u}(0),q(T)\big)\leq G\big(\hat{u}(0),\tilde{u}(T)\big),\quad \text{for}\quad q(T)\leq\tilde{u}(T), \tag{3.9}$$

$$G\big(\tilde{u}(0),q(T)\big)\geq G\big(\tilde{u}(0),\hat{u}(T)\big),\quad \text{for}\quad \hat{u}(T)\leq q(T). \tag{3.10}$$

From equations (3.7),(3.9) and equations (3.8),(3.10),we obtain

$$G\big(p(0),q(T)\big)\leq G\big(\hat{u}(0),q(T)\big)\leq G\big(\hat{u}(0),\tilde{u}(T)\big)\leq 0, \tag{3.11}$$

$$G\big(p(0),q(T)\big)\geq G\big(\tilde{u}(0),q(T)\big)\geq G\big(\tilde{u}(0),\hat{u}(T)\big)\geq 0. \tag{3.12}$$

From inequalities (3.11) and (3.12),we get $G(p(0),q(T))=0$. On similar lines we can prove $G(q(0),p(T))=0$. We get

$$G\big(p(0),q(T)\big)=0=G\big(q(0),p(T)\big). \tag{3.13}$$

Thus p and q are coupled quasi solutions of the nonlinear BVP (2.3)-(2.4). Lastly,we claim that $p(t)=q(t), t\in[0,T]$. To prove our claim it is sufficient to show that $p(t)\leq q(t)$. Define $s(t)=q(t)-p(t)$. From the Lipschitz condition (2.9),we obtain

$$\begin{aligned}{}^{c}D_t^{\alpha;\psi}s(t) &= {}^{c}D_t^{\alpha;\psi}q(t)-{}^{c}D_t^{\alpha;\psi}p(t)\\ &= -[f(t,p)-f(t,q)]\geq\overline{c}p(t),\quad t\in(0,T],\\ \text{and}\quad s(0) &= q(0)-p(0)\geq 0.\end{aligned}$$

In view of Lemma 2.10, $s(t)\geq 0,\ t\in[0,T]$ implies that $p(t)\leq q(t)$, which ensures that $p(t)=q(t)$ is the unique solution of IVP (3.1) in the sector $\langle\hat{\mathbf{u}},\tilde{\mathbf{u}}\rangle$, i.e. $p(t)=q(t)$ is the unique solution of the nonlinear BVP (2.3)- (2.4) and proof is complete. □

4. Conclusion

In this paper, an elegant technique commonly known as monotone iterative technique has been developed together with its monotone iterative scheme by introducing the new notion of quasi solutions for proving some results such as an existence and uniqueness of solution for a class of nonlinear BVP for nonlinear ψ-Caputo fractional differential equations.Firstly,we have obtained maximal and minimal solutions of IVP for nonlinear ψ-Caputo fractional differential equations.Further, it is proved that these maximal and minimal solutions are truly quasi solutions of nonlinear BVP for nonlinear ψ-Caputo fractional differential equations.The existence and uniqueness results have been obtained under more general boundary conditions.

References

[1] M.S. Abdo,S.K.Panchal and A.M. Saeed,Fractional boundary value problem with ψ-Caputo fractional derivative, *Proc. Math. Sci.*, 129:65(2019), 1– 14.

[2] S,Ali,K.Shah and F.Jarad,On stable iterative solutions for a class of boundary value problem of nonlinear fractional order differential equations,*Math. Methods Appl. Sci.*, 42(2019), 969–981.

[3] R. Almeida,A Caputo fractional derivative of a function with respect to another function, *Commun. Nonlinear Sci.Numer. Simul.*,44 (2017), 460–481.

[4] R.Almeida, M. Jleli and B. Samet, A numerical study of fractional relaxation-oscillation equations involving ψ-Caputo fractional derivative,*Rev. R. Acad. Cienc.Exactas Fís. Nat. Ser. A Mat. RACSAM.*, 113(2019), 1873–1891.

[5] R.Almeida,A.B.Malinowska and M.T. Monteiro, Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications,*Math. Meth. Appl. Sci.*, 41(2018), 336–352.

[6] C.P.Chitalkar Dhaigude, S.P.Bhairat and D.B. Dhaigude,Basic theory of generalized Riemann- Liouville fractional differential equations,*Bull.Marathwada Math. Soc.*, 19(2) (2018), 1–12.

[7] C.P.Chitalkar-Dhaigude and D.B.Dhaigude,Existence and uniqueness results for fractional differential equations with periodic boundary conditions, *Bull.Marathwada Math. Soc.*,20(1)(2019), 1–10.

[8] C.Derbazi, Z.Baitiche,M.Benchohra and A. Cabada, Initial value problem for nonlinear fractional differential equations with $\psi-$Caputo derivative via monotone iterative technique, *Axioms.* , 9(57)(2020), 1–13.

[9] D.B.Dhaigude and C.P.Chitalkar-Dhaigude,Method of upper lower solutions for fractional integro - diffusion equation,*Nonlinear Studies.*,22(3)(2015), 495–502.

[10] D.B.Dhaigude and C.P.Chitalkar-Dhaigude,Monotone iterative technique for fractional differential equations with nonlinear boundary conditions,*Dyn. Contin.Discrete Impuls.Syst.Ser.A: Math.Anal.*, 27(4)(2020), 265–273.

[11] D.B. Dhaigude,N.B.Jadhav and J.A. Nanware,Method of upper lower solutions for nonlinear system of fractional differential equations and applications,*Malaya J. Mat.*,6(3)(2018), 1–6.

[12] D.B. Dhaigude,J.A.Nanware and V.R.Nikam,Monotone technique for weakly coupled system of Caputo fractional differential equations with periodic boundary conditions, *Dyn. Contin. Discrete Impuls. Syst. Ser. A : Math. Anal.*,19(2012), 575–584.

[13] D. B. Dhaigude and B. H. Rizqan,Existence and uniqueness of solutions for fractional differential equations with advanced arguments,*Adv. Math. Models. Appl.*, 2(3)(2017), 240–250.

[14] D. B. Dhaigude and B. H. Rizqan,Monotone iterative technique for Caputo fractional differential equations with deviating arguments,*Annal. Pure Appl. Math.*, 16(1)(2018), 181–191.

[15] D.B.Dhaigude and B.H.Rizqan,Existence results for nonlinear fractional differential equations with deviating arguments under integral boundary conditions,*Far East J.Math.Sci.*,108(2)(2018), 273–284.

[16] D. B. Dhaigude and B. H. Rizqan,Existence and uniqueness of solutions of fractional differential equations with deviating arguments under integral boundary conditions, *Kyungpook Math. J.*, 59(2019), 191–202.

[17] R. Hilfer, Applications of Fractional Calculus in Physics, *World Scientific, Singapore*, 2000.

[18] T.Jankowski,Fractional problems with advanced arguments,*Appl.Math.Comput.*,230(2014), 371–382.

[19] V. V. Kharat,D. R. Hasabe and D. B. Dhaigude,On existence of solution to mixed nonlinear fractional integro differential equations,*Appl.Math. Sci.*,11(45)(2017), 2237 – 2248.

[20] V. V. Kharat, D. B. Dhaigude and D. R. Hasabe, On nonlinear mixed fractional integro differential inclusion with four-piont nonlocal Riemann-Liouville integral boundary conditions,*Indian J.Pure Appl.Math.*,50(4)(2019), 937–951.

[21] A.A.Kilbas,H.M.Srivastava and J.J.Trujillo, Theory and Applications of Fractional Differential Equations, *Elsevier, Amsterdam*,2006.

[22] J.A.Nanware and D.B. Dhaigude; Monotone technique for finite weakly coupled system of Caputo fractional differential equations with periodic boundary conditions,*Dyn.Contin. Discrete Impuls.Syst.Ser.A Math.Anal.*,22(1)(2015),13 – 23.

[23] J.A.Nanware and D.B. Dhaigude,System of initial value problems involving Riemann - Liouville sequential fractional derivative,*Comm.Appl.Anal.*,22(3)(2018), 353–368.

[24] J.A.Nanware,N.B.Jadhav and D.B. Dhaigude,Initial value problems for fractional differential equations involving Riemann-Liouville derivative,*Malaya J. Mat.*,5(2)(2017),337–345.

[25] J.A.Nanware,N.B. Jadhav and D.B.Dhaigude,Existence and uniqueness of solutions of nonlinear implicit fractional differential equations,*Dyn,Contin.Discrete Impuls.Syst.Ser.A: Math.Anal.*, 27(4)(2020), 275–282.

[26] B. H. Rizqan and D. B. Dhaigude,Nonlinear boundary value problem of fractional differential equations

with advanced arguments under integral boundary conditions,*Tamkang J.Math.*, 51(2)(2020), 101–112.

[27] B. H. Rizqan and D. B. Dhaigude,Positive solutions of nonlinear fractional differential equations with advanced arguments under integral boundary conditions,*Indian J. Math.*, 60(3)(2018), 491–507.

[28] B.Samet and H. Aydi,Lyapunov-type inequalities for an anti-periodic fractional boundary value problem involving $\psi-$ Caputo fractional derivative,*J. Inequal. Appl.,* 2018(2018), 286.

[29] S.Song,H.Li and Y.Zou, Monotone iterative method for fractional differential equations with integral boundary conditions, *J. Function Spaces.,* 2020(2020), Article ID 7319098, 7 pages

[30] W.Yang,Monotone iterative technique for a coupled system of nonlinear Hadamard fractional differential equations,*J. Appl. Math. Comput.,* 59(2019), 585–596.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆