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Perfect domination number of path graph P_n and its Corona product with another path graph P_{n-1}

Tushar J Bhatt[1*] and G. C. Bhimani[2]

Abstract
According to the research paper on Perfect Dominating Sets by Marilynn Livingston and Quentin F. Stout[1]they have been constructed the PDSs in families of graphs arising from the interconnected networks of parallel computers also contained perfect domination numbers of trees, dags, series-parallel graphs, meshes, tori, hypercubes, cube connected cycles and de Bruijin graphs and give linear algorithms for determining if a PDS exist, and generate a PDS when one does.They also proved that 2 and 3-dimensional hypercube graph having infinitely many PDSs.In this paper We are trying to apply their concept on path graphs and obtained their perfect domination number we also trying to find a corona product of path graph P_n with path graph P_{n-1} and as a conclusion we give such applications of it.

Keywords
Dominating set, Minimal dominating set, Minimum dominating set.

AMS Subject Classification
05C10.

[1] *Department Mathematics, Atmiya University, Rajkot-360005, Gujarat, India.*
[2] *Department Statistics, Saurashtra University, Rajkot-360005, Gujarat, India.*
***Corresponding author**: [1] tjbhatt2002@gmail.com; [2] gcbhimani@yahoo.co.in
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Contents

1. Introduction

Here first we define the necessary terms those are mentioned as keywords, also give an example of each if necessary.

Definition 1.1. *Let G be a graph and $V(G)$ be the set of all vertices of graph G and S be the subset of $V(G)$.Then the set is said to be dominating set of the graph G if and only if for any $w \in V(G) - S$ i.e.we can find at least one vertex $v \in S$ such that w is adjacent to v. For example, consider the following graph G, according to the definition of dominating set; the set $S = \{v_1, v_4\}$ is a dominating set.[7][67-72]*

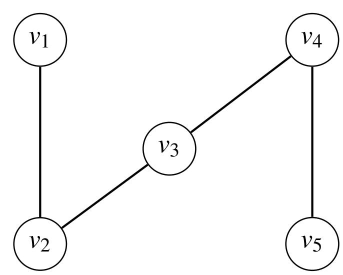

Figure 1. Dominating Set

Definition 1.2. *Assume that G be a graph. A dominating set S in the graph G is called minimal dominating set in the graph G if and only if for any $v \in S$, $S-v$ is not a dominating Set in the graph G.[7] For example, consider the following graph G, in which according to the definition of minimal dominating set, the set $S = \{v_1, v_4\}$ is a minimal dominating set. [7][67-72]*

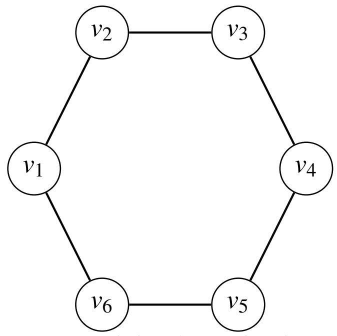

Figure 2. Minimal Dominating Set

Definition 1.3. *Let G be any graph. A dominating set S in G with minimum cardinality is called a minimum dominating set of the graph G.A minimum dominating set in G is also called a $\gamma-$set in the graph G. [8][51-55] For example, consider the following graph G, in which according to the definition of minimum dominating set, the set $S = \{v_1\}$ is a minimum dominating set.*

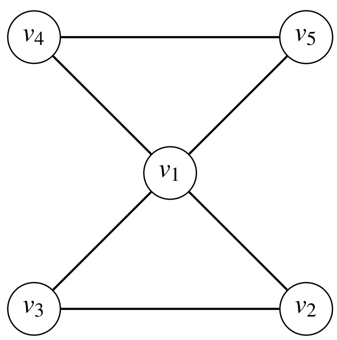

Figure 3. Minimum Dominating Set

Definition 1.4. *Consider G be any graph and S being a minimum dominating set in G. Then $|S|$ means cardinality of set S, is called the domination number of the graph G and it is denoted by $\gamma(G)$. [7][67-72] For example, consider the following graph G, in which according to the definition of domination number, the set $S = \{v_1\}$ is a minimum dominating set and that cardinality is one therefore the domination number of graph G is equal to 1 means $\gamma(G) = 1$*

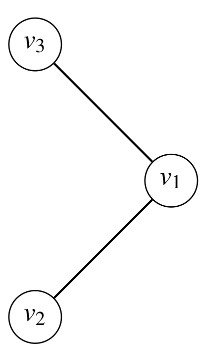

Figure 4. Domination Number

Remark 1.5.

(a) If S is a dominating set in graph G, then $\gamma(G) \leq |S|$.

(b) Every minimum dominating set in G is a minimal dominating set in G.

Definition 1.6. *A copy graph $G = (V,E)$ is denoted by G' with set of vertices and edges are as it is.For example, Let us observe graph $G = C_5$, cycle with 5 vertices then their copy of a graph $G' = C_5$ showing in the following figures 5,and 6.*

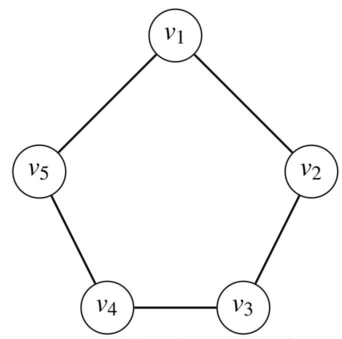

Figure 5. Graph $G = cycle\ C_5$

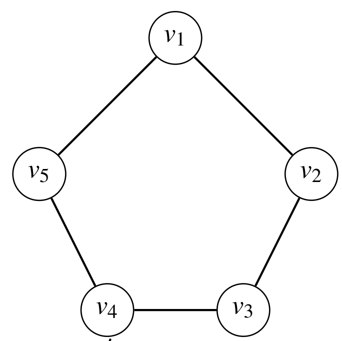

Figure 6. Graph G' = Copy of a Graph $G, cycle\ C_5$

Definition 1.7. *A subset S of V(G) is said to be a perfect dominating set if for each vertex v not in S, v is adjacent to exactly one vertex of S. For example consider the following figure 7, a graph $G = P_4$ a path graph with four vertices say v_1, v_2, v_3 and v_4 in which the set $S_1 = \{v_2, v_3\}$ and $S_2 = \{v_1, v_4\}$ are perfect dominating sets in this graph. It may be noted that if G is a graph then V(G) is always a perfect dominating set of G.*

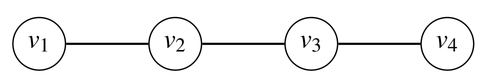

Figure 7. Graph $G = P_4$, a path graph with 4- vertices

Definition 1.8. *A perfect dominating set S of the graph G is said to be minimal perfect dominating set if each vertex v in S, $S - \{v\}$ is not a perfect dominating set.It may be noted that it is not necessary that a proper subset of minimal perfect* *dominating set is not a perfect dominating set. For example consider a following figure 8, a graph $G = C_6$, cycle graph with 6- vertices say $v_1, v_2, ..., v_6$ then obviously V(G) is a minimal perfect dominating set of graph $G = C_6$. However, the set $\{v_1, v_4\}$ is a proper subset of V(G) and is a perfect dominating set in the graph $G = C_6$.*

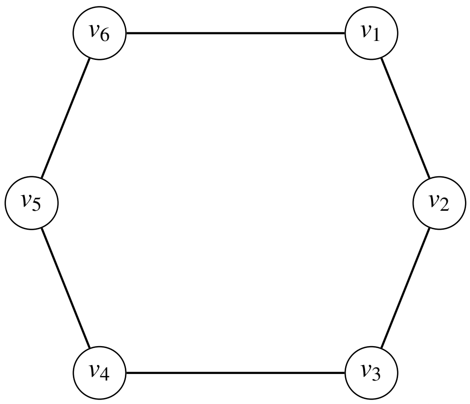

Figure 8. Graph $G = C_6$, a cycle graph with 6- vertices

Definition 1.9. *A perfect dominating set with smallest cardinality is called minimum perfect dominating set and it is also called $\gamma_{pf} - set$.*

Definition 1.10. *A cardinality of minimum perfect dominating set is called perfect domination number of the graph G and it is denoted by $\gamma_{pf}(G)$.In the above figure 8 we observe that the perfect domination number of cycle $\gamma_{pf}(C_6) = 2$ and in the following figure 9, we observe that the perfect domination number of path graph $\gamma_{pf}(P_3) = 1$. [2]*

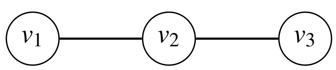

Figure 9. Graph $G = P_3$, a path graph with 3- vertices

Definition 1.11. *The Corona Product of two graphs G_1 and G_2 is the graph denoted by $G_1 \odot G_2$, is the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 such that i^{th} vertex of the copy of G_1 is adjacent to each vertex of the i^{th} copy of G_2. For example consider the above graphs figure 5, $G_1 = C_5$ a cycle graph with 5- vertices and figure 9, $G_2 = P_3$ a path graph with 3- vertices now we obtain the corona product of these two graphs $C_5 \odot P_3$ showing in the following figure 10.*

2. Main Results

Let $G = (V, E)$ be any graph where V indicates set of vertices and E indicates set of edges. Through out the result we are consider a path graph P_n with n number of vertices and $n-1$ number of edges in which we invented some new results regarding the perfect domination number of path graph P_n and its corona product with the path graph P_{n-1}.

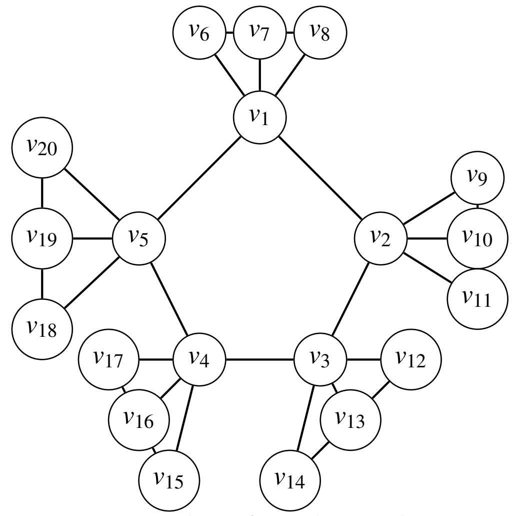

Figure 10. Corona product of Graph C_5 and P_3,means $C_5 \odot P_3$

2.1 Perfect Domination Number of Path graph P_n

Theorem 2.1. *Let P_n be any path graph with n number of vertices and $n-1$ number of edges the perfect domination number of P_n is given by*

$$\gamma_{pf}(P_n) = \begin{cases} \frac{n}{3}, \text{ where } n = 3m, m \in N \\ \frac{n+1}{3}, \text{ where } n = 3m-1, m \in N \\ \frac{n+2}{3}, \text{ where } n = 3m-2, m \in N \end{cases}$$

where N indicates set of natural numbers.

Proof. Here we are proving this theorem by using Principle of Mathematical Induction theory . First we are trying to prove that the given result in the hypothesis is true for $n = 1$.

Now for $n = 1$ is suitable in the order of $3m-2$ so that we take $m = 1$.

$$\therefore \gamma_{pf}(P_1) = \frac{1+2}{3} = \frac{3}{3} = 1,$$

since the perfect domination number of isolated vertex is must be 1 and hence it is true for $n = 1$.

Now we assume that above result is true for $n = k$, where k be any positive integer then we get

$$\gamma_{pf}(P_k) = \begin{cases} \frac{k}{3}, \text{ where } k = 3m, m \in N \\ \frac{k+1}{3}, \text{ where } k = 3m-1, m \in N \\ \frac{k+2}{3}, \text{ where } k = 3m-2, m \in N \end{cases}$$

Now next we trying to prove that above result given in the hypothesis is true for $n = k+1$, where k be any positive integer.

To prove :

$$\gamma_{pf}(P_{k+1}) = \begin{cases} \frac{k+1}{3}, \ where\ k+1 = 3m, m \in N \\ \frac{k+2}{3}, \ where\ k+1 = 3m-1, m \in N \\ \frac{k+3}{3}, \ where\ k+1 = 3m-2, m \in N \end{cases}$$

Case-I: If $k+1 = 3m$, means $k+1$ is a multiple of 3, here k be any positive integer therefore $k+1 = \lambda$ is must be a positive integer for some $m \in N$ follows $\lambda = 3m$ so according to above result $\Rightarrow \gamma_{pf}(P_\lambda) = \frac{\lambda}{3}$ but $\lambda = k+1$ then $\gamma_{pf}(P_{k+1}) = \frac{k+1}{3}$.

Case-II: If $k+1 = 3m-1$, means $k+1$ is a multiple of 3 minus 1(one), here k be any positive integer therefore $k+1 = \alpha$ is must be a positive integer for some $m \in N$ follows $\alpha = 3m-1$ so according to above result $\Rightarrow \gamma_{pf}(P_\alpha) = \frac{\alpha+1}{3}$ but $\alpha = k+1$ then $\gamma_{pf}(P_{k+1}) = \frac{k+2}{3}$.

Case-III: If $k+1 = 3m-2$, means $k+1$ is a multiple of 3 minus 2(two), here k be any positive integer therefore $k+1 = \beta$ is must be a positive integer for some $m \in N$ follows $\beta = 3m-2$ so according to above result $\Rightarrow \gamma_{pf}(P_\beta) = \frac{\beta+2}{3}$ but $\beta = k+1$ then $\gamma_{pf}(P_{k+1}) = \frac{k+3}{3}$.

Hence proved the result.

For example consider the following path graph P_{10}, showing in the figure 11 then $10 = 3m-2$, *where* $m = 4 \in N$ then it is clear that as per above theorem 2.1 we say that the perfect domination number of path graph P_{10} is

$$\gamma_{pf}(P_{10}) = \frac{n+2}{3} = \frac{10+2}{3} = \frac{12}{3} = 4.$$

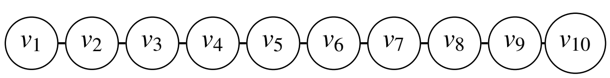

Figure 11. Graph $G = P_{10}$, a path graph with 10- vertices

□

Corollary 2.2. *Let P_{n-1} be any path graph with $n-1$ number of vertices and $n-2$ number of edges the perfect domination number of P_{n-1} is given by*

$$\gamma_{pf}(P_{n-1}) = \begin{cases} \frac{n-1}{3}, \ where\ n = 3m+1\ or\ 3m-2, m \in N \\ \frac{n}{3}, \ where\ n = 3m, m \in N \\ \frac{n+1}{3}, \ where\ n = 3m-1, m \in N \end{cases}$$

where N indicates set of natural numbers.

Proof. By using mathematical induction we can also prove the corollary 2.2. □

2.2 Bounds of Perfect Domination Number of a Path Graph $-P_n$

Theorem 2.3. *Let P_n be any path graph with n- vertices and $\gamma_{pf}(P_n)$ indicates the perfect domination number of path graph P_n then $n-2 < \gamma_{pf}(P_n) < n$, Where $n \geq 2 \in N$.*

Proof. Here we can try to prove the above result by contradictory way.

To prove : $n-2 < \gamma_{pf}(P_n) < n$, Where $n \in N$.

now we partitioned an interval (n-2, n) into two parts

(1) $n-2 < \gamma_{pf}(P_n)$ and

(2) $\gamma_{pf}(P_n) < n$

Now Conversely we assume that $n-2 > \gamma_{pf}(P_n)$ and

$$\gamma_{pf}(P_n) > n \tag{2.1}$$

As per the above theorem 2.1, we divide the hypothesis into three cases as follows: □

[**Case-I:** If $n = 3m$, for some $m \in N$ then $\gamma_{pf}(P_n) = \frac{n}{3}$.] Now let $n-2 > \gamma_{pf}(P_n)$

$\Rightarrow n-2 > \frac{n}{3}$

$\Rightarrow 3n-6 > n$

$\Rightarrow 2n > 6$

$\Rightarrow n > 3$

is contradict with our assumption because if we take $m = 1$ follows $n = 3$ it gives $3 > 3$ this is conflict. So our assumption is wrong therefore $n-2 < \gamma_{pf}(P_n)$.

Let $\gamma_{pf}(P_n) > n$,

$\Rightarrow \frac{n}{3} > n$

$\Rightarrow \frac{1}{3} > 1$

This is contradiction, therefore our assumption is wrong and hence $\gamma_{pf}(P_n) < n$.

So in the case - I the original result is true for all $n \in N$.

[**Case-II:** If $n = 3m-1$, for some $m \in N$ then $\gamma_{pf}(P_n) = \frac{n+1}{3}$.] Now let $n-2 > \gamma_{pf}(P_n)$

$\Rightarrow n-2 > \frac{n+1}{3}$

$\Rightarrow 3n-6 > n+1$

$\Rightarrow 2n > 7$

is contradict with our assumption because if we take $m = 1$ follows $n = 2$ it gives $4 > 7$ this is conflict. So our assumption is wrong therefore $n-2 < \gamma_{pf}(P_n)$.

Let $\gamma_{pf}(P_n) > n$,

$\Rightarrow \frac{n+1}{3} > n$

$\Rightarrow n+1 > 3n$

$\Rightarrow 1 > 2n$

It is true for all $n = 3m-1, m \in N$. Since $m = 1$ follows $n = 2$ this gives $1 > 4$ is conflict, therefore our assumption is wrong and hence $\gamma_{pf}(P_n) < n$.

So in the case - II the original result is true for all $n \in N$.

[**Case-III:** If $n = 3m-2$, for some $m \in N$ then $\gamma_{pf}(P_n) = \frac{n+2}{3}$.] Now let $n-2 > \gamma_{pf}(P_n)$

$\Rightarrow n-2 > \frac{n+2}{3}$
$\Rightarrow 3n-6 > n+2$
$\Rightarrow 2n > 4$

is contradict with our assumption because if we take $m = 1$ follows $n = 2$ it gives $4 > 4$ this is conflict. So our assumption is wrong therefore $n-2 < \gamma_{pf}(P_n)$.

Let $\gamma_{pf}(P_n) > n$,
$\Rightarrow \frac{n+2}{3} > n$
$\Rightarrow n+2 > 3n$
$\Rightarrow 2 > 2n$
$\Rightarrow 1 > n$
i.e $n < 1$

It is true for all $n = 3m-1, m \in N$. Since $m = 1$ follows $n = 2$ this gives $2 < 1$ is conflict, therefore our assumption is wrong and hence $\gamma_{pf}(P_n) < n$.

So in the case - III the original result is true for all $n \in N$.

Corollary 2.4. *Let P_{n-1} be any path graph with $n-1$ vertices and $\gamma_{pf}(P_{n-1})$ indicates the perfect domination number of path graph P_{n-1} then $n-3 < \gamma_{pf}(P_{n-1}) < n-1$, Where $n \geq 3 \in N$.*

Proof. In the above theorem - 2.1 just replace n by $n-1$ we can prove the corollary 2.3. □

2.3 Bounds of the perfect domination number of corona products of two path graphs P_n and P_{n-1}

Lemma 2.5. *The perfect domination number of corona product of two path graphs P_n and P_{n-1} is given by $\gamma_{pf}(P_n \bigodot P_{n-1})$ and it is bounded by $0 < \gamma_{pf}(P_n \bigodot P_{n-1}) < n^2$, where $n \geq 2$.*

Proof. Here we want to prove that

$0 < \gamma_{pf}(P_n \bigodot P_{n-1}) < n^2$, *where* $n \geq 2$.

That can be rewritten as $0 < \gamma_{pf}(P_n \bigodot P_{n-1})$ ......(1) and $\gamma_{pf}(P_n \bigodot P_{n-1}) < n^2$ ......(2)

Since the perfect domination number is always grater than zero so the above inequality .....(1) is but obvious.

For the further proof of result ....(2) observe the following figure 12.

Let us consider path graph P_n with n - vertices say $v_1, v_2, \dots v_{n-1}, v_n$ and another path graph P_{n-1} with $n-1$ vertices namely $w_1, w_2, \dots, w_{n-2}, w_{n-1}$. Now thake a corona product of them we get at most n^2 vertices.

Since each vertex of graph P_n is adjacent with every vertex of graph P_{n-1}. So as per the definition of minimum perfect dominating set it contains at least one vertex and most n^2 vertices otherwise the set becomes a maximal dominating set or lost their dominating property.

In above figure 12, the perfect domination number of corona product of P_n and P_{n-1} is at most n^2 and always greater than zero that is $0 < \gamma_{pf}(P_n \bigodot P_{n-1}) < n^2$, *where* $n \geq 2$. Hence proved the result. □

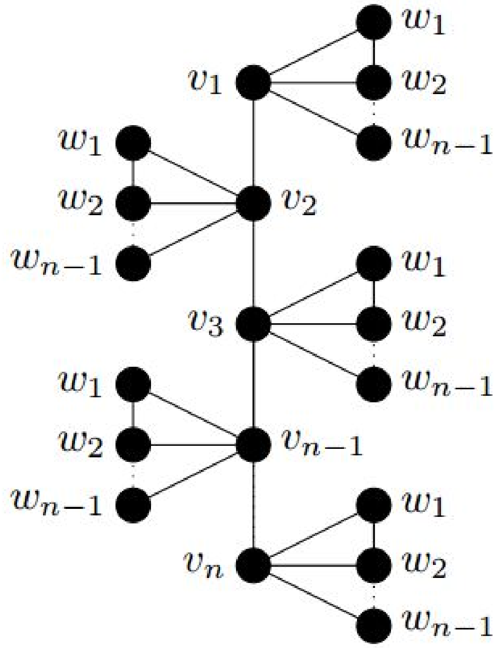

Figure 12. Corona Product of $P_n \bigodot P_{n-1}$

2.4 The Perfect Domination Number of Corona Product of two path graphs P_n and P_{n-1}

Theorem 2.6. *Let P_n, path graph with n vertices and P_{n-1}, path graph with $n-1$ vertices where $n \geq 2 \in N$ then the perfect domination number of corona product of two path graphs P_n and P_{n-1} is equal to the sum of perfect domination number of P_n and perfect domination number of P_{n-1} and natural number 1 (one).*

i.e $\gamma_{pf}(P_n \bigodot P_{n-1}) \geq \gamma_{pf}(P_n) + \gamma_{pf}(P_{n-1}) + 1$ *for some* $n \geq 2$.

Proof. The given hypothesis is proving by using contradictory method as follows.

Now as per theorem - 2, corollary - 2 and lemma - 1 we have

(1) $n-2 < \gamma_{pf}(P_n) < n$, Where $n \geq 2 \in N$.
(2) $n-3 < \gamma_{pf}(P_{n-1}) < n-1$, Where $n \geq 3 \in N$.
(3) $0 < \gamma_{pf}(P_n \bigodot P_{n-1}) \leq n^2$, *where* $n \geq 2$.

Now construct (1) + (2), we get

$2n-5 < \gamma_{pf}(P_n) + \gamma_{pf}(P_{n-1}) < n+n-1$
$\Rightarrow 2n-5+1 < \gamma_{pf}(P_n) + \gamma_{pf}(P_{n-1}) + 1 < n+n-1+1$
$\Rightarrow 2n-4 < \gamma_{pf}(P_n) + \gamma_{pf}(P_{n-1}) + 1 < 2n$ where $2n \geq 5 \in N$ .....(4)

Conversely assume that

$$\gamma_{pf}(P_n \bigodot P_{n-1}) < \gamma_{pf}(P_n) + \gamma_{pf}(P_{n-1}) + 1$$

The above statement can be rewritten as

$0 < \gamma_{pf}(P_n \bigodot P_{n-1}) < \gamma_{pf}(P_n) + \gamma_{pf}(P_{n-1}) + 1$
$\Rightarrow 0 < \gamma_{pf}(P_n \bigodot P_{n-1}) < n^2 < \gamma_{pf}(P_n) + \gamma_{pf}(P_{n-1}) + 1 < 2n$

$\Rightarrow n^2 < 2n$
$\Rightarrow n^2 - 2n < 0$
$\Rightarrow n(n-2) < 0$
$\Rightarrow$ Either $n < 0$ and $n-2 > 0$ or $n > 0$ and $n-2 < 0$
$\Rightarrow$ Either $n < 0$ and $n > 2$ or $n > 0$ and $n < 2$
Is conflict with values of 'n', $n \geq 2$

Therefore our assumption is wrong and hence proved the theorem.

i.e $\gamma_{pf}(P_n \bigodot P_{n-1}) \geq \gamma_{pf}(P_n) + \gamma_{pf}(P_{n-1}) + 1$ for some $n \geq 2, n \in N$. □

3. Conclusion

In computer science, the main problem is wireless connectivity in complex network. By using graph theory specially a theory of domination concept we will reduce it.Here our aim is to find minimum perfect dominating set in path graph using it we recognize the special nodes which are working as a principal controller or server.

Those vertices who are in minimum perfect dominating set in path graph working as a special node or server or controller. The corona product of two path graphs create a complex network and their perfect domination number indicates How many vertices working as a server? so according to that we can easily find out our best server from the complex network.

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